**Force of Mortality**

**Definition 1**

The **force of mortality** of the survival function $S_X$ is defined as

$$
\mu(x) = -\frac{d}{dx} \ln S_X(x) = \frac{f_X(x)}{S_X(x)}.
$$

The force of mortality is also called the **force of failure**. An alternative notation for the force of mortality is $\mu_x$. Force of mortality and force of failure are names used in actuarial sciences.
Definition 2

The **hazard rate function** of the survival function $S_X$ is defined as

$$\lambda_X(x) = -\frac{d}{dx} \ln S_X(x) = \frac{f_X(x)}{S_X(x)},$$

where $S_X$ is the survival function and $f_X$ is the probability density function of the distribution.

The hazard rate is also called the **failure rate**. Hazard rate and failure rate are names used in reliability theory.
Example 1

Find the force of mortality of the survival function \( S_X(x) = \frac{90^6 - x^6}{90^6} \), for \( 0 < x < 90 \).
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Solution: We have that

\[
\mu(x) = -\frac{d}{dx} \ln S_X(x) = -\frac{d}{dx} \ln \left( \frac{90^6 - x^6}{90^6} \right)
\]

\[
= - \frac{d}{dx} \ln(90^6 - x^6) + \frac{d}{dx} \ln(90^6) = \frac{6x^5}{90^6 - x^6}.
\]
Theorem 1

The hazard function of $T(x)$ is

$$
\lambda_{T(x)}(t|X > x) = \mu_{x+t}, \quad t \geq 0.
$$

The hazard rate is the rate of death for lives aged $x$. For a life aged $x$, the force of mortality $t$ years later is the force of mortality for a ($x+t$)–year old.
Theorem 1
The hazard function of $T(x)$ is

$$
\lambda_{T(x)}(t|X > x) = \mu_{x+t}, \ t \geq 0.
$$

Proof: The survival function of $T(x)$ is $S_X(x+t)/S_X(x)$, $t \geq 0$. Hence,

$$
\lambda_{T(x)}(t|X > x) = - \frac{d}{dt} \log \left( \frac{S_X(x+t)}{S_X(x)} \right) = - \frac{d}{dt} \log(S_X(x+t)) + \frac{d}{dt} \log(S_X(x)) = \frac{f_X(x+t)}{S_X(x+t)} = \mu_{x+t}.
$$
Theorem 1

The hazard function of $T(x)$ is

$$\lambda_{T(x)}(t|X > x) = \mu_{x+t}, \, t \geq 0.$$  

Proof: The survival function of $T(x)$ is $\frac{S_{X(x+t)}}{S_{X(x)}}, \, t \geq 0$. Hence,

$$\lambda_{T(x)}(t|X > x) = -\frac{d}{dt} \log \left( \frac{S_{X(x+t)}}{S_{X(x)}} \right)$$

$$= -\frac{d}{dt} \log(S_{X(x+t)}) + \frac{d}{dt} \log(S_{X(x)}) = \frac{f_{X(x+t)}}{S_{X(x+t)}} = \mu_{x+t}.$$  

The hazard rate is the rate of death for lives aged $x$. For a life aged $x$, the force of mortality $t$ years later is the force of mortality for a $(x+t)$–year old.
Example 2

Suppose that the survival function of a new born is
\[ S_X(t) = \frac{85^4 - t^4}{85^4}, \text{ for } 0 < t < 85. \]
(i) Find the force of mortality of a new born.
(ii) Find the force of mortality of a life aged \( x \).
Example 2

Suppose that the survival function of a new born is
\[ S_X(t) = \frac{85^4 - t^4}{85^4}, \text{ for } 0 < t < 85. \]

(i) Find the force of mortality of a new born.
(ii) Find the force of mortality of a life aged \( x \).

Solution: (i) We have that
\[
\mu(t) = -\frac{d}{dt} \ln S_X(t) = -\frac{d}{dt} \ln \left( \frac{85^4 - t^4}{85^4} \right) 
\]
\[
= -\frac{d}{dt} \ln(85^4 - t^4) + \frac{d}{dt} \ln(85^4) = \frac{4t^3}{85^4 - t^4}, 0 < t < 85. 
\]
Example 2

Suppose that the survival function of a new born is
\( S_X(t) = \frac{85^4 - t^4}{85^4} \), for \( 0 < t < 85 \).

(i) Find the force of mortality of a new born.
(ii) Find the force of mortality of a life aged \( x \).

Solution: (i) We have that

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\mu(t) = -\frac{d}{dt} \ln S_X(t) = -\frac{d}{dt} \ln \left( \frac{85^4 - t^4}{85^4} \right)
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\[
= -\frac{d}{dt} \ln(85^4 - t^4) + \frac{d}{dt} \ln(85^4) = \frac{4t^3}{85^4 - t^4}, \quad 0 < t < 85.
\]

(ii)

\[
\mu(x + t) = \frac{4(x + t)^3}{85^4 - (x + t)^4}, \quad 0 < t < 85 - x.
\]
Definition 3

The cumulative hazard rate function is defined as

\[ \Lambda(x) = \int_0^x \lambda_X(t) \, dt, \quad x \geq 0, \]

where \( \lambda_X \) is the hazard function of the r.v. \( X \).

Notice that

\[ \Lambda(x) = \int_0^x \lambda_X(t) \, dt = \int_0^x \frac{d}{dt}(-\log S_X(t)) \, dt \]

\[ = \left[ -\log S_X(t) \right]_0^x = -\log S_X(x), \quad x \geq 0, \]

where \( S_X \) is the survival function of the r.v. \( X \).
Theorem 2
Let $\mu(x) : [0, \infty] \to \mathbb{R}$ be a function which is continuous except at finitely many points. We have that $\mu$ is the force of mortality of an age–at–death r.v. if and only if the following two conditions hold:

(i) For each $x \geq 0$, $\mu(x) \geq 0$.
(ii) $\int_0^\infty \mu(t) \, dt = \infty$.

Proof: If $\mu$ is a force of mortality, then $\mu(x) = \frac{f_X(x)}{1 - F_X(x)} \geq 0$ and

$$
\int_0^\infty \mu(t) \, dt = \int_0^\infty - \frac{d}{dx} \left( \ln(S_X(x)) \right) \, dt = - \ln(S_X(x)) \bigg|_0^\infty = \infty.
$$
Suppose $\mu$ satisfies conditions

(i) For each $x \geq 0$, $\mu(x) \geq 0$.

(ii) $\int_0^\infty \mu(t) \, dt = \infty$.

Since $\mu$ is continuous everywhere except finitely many points

$$S_X(x) = \exp \left( - \int_0^x \mu(t) \, dt \right), \ x \geq 0$$

defines a survival function. By the fundamental theorem of calculus,

$$- \frac{d}{dx} (\ln(S_X(x))) = - \frac{d}{dx} \left( \int_0^x \mu(t) \, dt \right) = \mu(x),$$

for all points $x$ of continuity of $\mu$. 
Example 3

Determine which of the following functions is a legitime force of mortality of an age–at–death:

(i) $\mu(x) = \frac{1}{(x+1)^2}$, for $x \geq 0$.
(ii) $\mu(x) = x \sin x$, for $x \geq 0$.
(iii) $\mu(x) = 35$, for $x \geq 0$. 
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Determine which of the following functions is a legitime force of mortality of an age–at–death:

(i) \( \mu(x) = \frac{1}{(x+1)^2} \), for \( x \geq 0 \).
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(iii) \( \mu(x) = 35 \), for \( x \geq 0 \).

Solution: (i) \( \mu(x) = \frac{1}{(x+1)^2} \) is not a force of mortality because

\[
\int_0^\infty \frac{1}{(x+1)^2} \, dx = - \frac{1}{x+1} \bigg|_0^\infty = 1.
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Solution: (i) $\mu(x) = \frac{1}{(x+1)^2}$ is not a force of mortality because

$$\int_0^\infty \frac{1}{(x+1)^2} \, dx = - \frac{1}{x + 1} \bigg|_0^\infty = 1.$$

(ii) $\mu(x) = x \sin x$ is not a force of mortality because $x \sin x < 0$, for $x \in (\pi, 2\pi)$. 
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Solution: (i) $\mu(x) = \frac{1}{(x+1)^2}$ is not a force of mortality because

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\int_0^\infty \frac{1}{(x+1)^2} \, dx = -\frac{1}{x+1} \bigg|_0^\infty = 1.
$$

(ii) $\mu(x) = x \sin x$ is not a force of mortality because $x \sin x < 0$, for $x \in (\pi, 2\pi)$.

(iii) $\mu(x) = 35$ is a legitime force of mortality.
Theorem 3

Suppose that $\mu(\cdot)$ is the force of mortality of the age–at–death r.v. $X$. Then,

(i) The survival function of $X$ is

$$S_X(t) = \exp \left( - \int_0^t \mu(s) \, ds \right), \quad t \geq 0.$$ 

(ii) The density of $X$ is

$$f_X(t) = S_X(t) \mu(t) = \exp \left( - \int_0^t \mu(s) \, ds \right) \mu(t), \quad t \geq 0.$$
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Proof: (i) By the fundamental theorem of calculus,

$$\exp \left( - \int_0^t \mu(s) \, ds \right) = \exp \left( - \int_0^x - \frac{d}{ds}(\ln(S_X(s))) \, ds \right)$$

$$= \exp \left( \ln(S_X(s)) \bigg|_{s=0}^t \right) = \exp (\ln(S_X(t))) = S_X(t)$$
Theorem 3

Suppose that \( \mu(\cdot) \) is the force of mortality of the age–at–death r.v. \( X \). Then,

(i) The survival function of \( X \) is

\[
S_X(t) = \exp \left( - \int_0^t \mu(s) \, ds \right), \quad t \geq 0.
\]

(ii) The density of \( X \) is

\[
f_X(t) = S_X(t) \mu(t) = \exp \left( - \int_0^t \mu(s) \, ds \right) \mu(t), \quad t \geq 0
\]

Proof: (ii) follows noticing that \( \mu(t) = \frac{f_X(t)}{S(t)} \).
Theorem 4

Suppose that $\mu(\cdot)$ is the force of mortality of the age–at–death r.v. $X$. Then,

(i) The survival function of $T(x)$ is

$$S_{T(x)}(t) = t p_x = \exp \left( - \int_x^{x+t} \mu(s) \, ds \right), \quad t \geq 0.$$  

(ii) The density of $T(x)$ is

$$f_{T(x)}(t) = S_{T(x)}(t) \mu_{T(x)}(t) = e^{-\int_x^{x+t} \mu(y) \, dy} \mu(x + t), \quad t \geq 0.$$
Theorem 4

Suppose that $\mu(\cdot)$ is the force of mortality of the age–at–death r.v. $X$. Then,

(i) The survival function of $T(x)$ is

$$S_{T(x)}(t) = t p_x = \exp \left( - \int_{x}^{x+t} \mu(s) \, ds \right), \quad t \geq 0.$$ 

(ii) The density of $T(x)$ is

$$f_{T(x)}(t) = S_{T(x)}(t) \mu_{T(x)}(t) = e^{- \int_{x}^{x+t} \mu(y) \, dy} \mu(x + t), \quad t \geq 0.$$ 

Proof: (i) The survival function of $T(x)$ is

$$tp_x = \frac{S_X(x + t)}{S_X(x)} = \frac{\exp \left( - \int_{0}^{x+t} \mu(s) \, ds \right)}{\exp \left( - \int_{x}^{x} \mu(s) \, ds \right)} = \exp \left( - \int_{x}^{x+t} \mu(s) \, ds \right).$$
Theorem 4
Suppose that $\mu(\cdot)$ is the force of mortality of the age–at–death r.v. $X$. Then,
(i) The survival function of $T(x)$ is

$$S_{T(x)}(t) = t p_x = \exp \left( - \int_x^{x+t} \mu(s) \, ds \right), \quad t \geq 0.$$ 

(ii) The density of $T(x)$ is

$$f_{T(x)}(t) = S_{T(x)}(t) \mu_{T(x)}(t) = e^{-\int_x^{x+t} \mu(y) \, dy} \mu(x + t), \quad t \geq 0.$$ 

Proof: (i) The survival function of $T(x)$ is

$$t p_x = \frac{S_X(x + t)}{S_X(x)} = \frac{\exp \left( - \int_0^{x+t} \mu(s) \, ds \right)}{\exp \left( - \int_0^{x} \mu(s) \, ds \right)} = \exp \left( - \int_x^{x+t} \mu(s) \, ds \right).$$

(ii) follows from $\mu_{T(x)}(t) = \frac{f_{T(x)}(t)}{S_{T(x)}(t)}.$
The cumulative distribution function of $X$ is

$$F_X(t) = 1 - e^{-\int_0^t \mu(y)dy}, \quad t \geq 0.$$  

The cumulative distribution function of $T_x$ is

$$t q_x = 1 - e^{-\int_{x}^{x+t} \mu(y)dy}.$$
Example 4

For the force of mortality $\mu(x) = \frac{1}{x+1}$ for $x \geq 0$, find $S_X$, $f_X$, $t p_x$ and $f_T(x)$.
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Solution: We have that

\[
\int_{0}^{x} \mu(t) \, dt = \int_{0}^{x} \frac{1}{t + 1} \, dt = \log(1 + t) \bigg|_{0}^{x} = \log(1 + x), \quad x \geq 0.
\]
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$$\int_0^x \mu(t) \, dt = \int_0^x \frac{1}{t+1} \, dt = \log(1 + t) \bigg|_0^x = \log(1 + x), \, x \geq 0.$$

$$s(x) = \exp \left( - \int_0^x \mu(t) \, dt \right) = \exp(- \log(1 + x)) = \frac{1}{x + 1}, \, x \geq 0.$$
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$$\int_0^x \mu(t) \, dt = \int_0^x \frac{1}{t+1} \, dt = \log(1 + t) \bigg|_0^x = \log(1 + x), \quad x \geq 0.$$

$$s(x) = \exp \left( - \int_0^x \mu(t) \, dt \right) = \exp(- \log(1 + x)) = \frac{1}{x + 1}, \quad x \geq 0.$$  

$$f_X(x) = \mu(x) s(x) = \frac{1}{(x + 1)^2}, \quad x \geq 0.$$
Example 4

For the force of mortality \( \mu(x) = \frac{1}{x+1} \) for \( x \geq 0 \), find \( S_X \), \( f_X \), \( t p_x \) and \( f_T(x) \).

**Solution:** We have that

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\int_0^x \mu(t) \, dt = \int_0^x \frac{1}{t+1} \, dt = \log(1 + t) \bigg|_0^x = \log(1 + x), \ x \geq 0.
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s(x) = \exp \left( - \int_0^x \mu(t) \, dt \right) = \exp(- \log(1 + x)) = \frac{1}{x+1}, \ x \geq 0.
\]

\[
f_X(x) = \mu(x) s(x) = \frac{1}{(x+1)^2}, \ x \geq 0.
\]

\[
t p_x = \frac{s(x+t)}{s(x)} = \frac{x+1}{x+t+1}, \ x, t \geq 0.
\]
Example 4

For the force of mortality \( \mu(x) = \frac{1}{x+1} \) for \( x \geq 0 \), find \( S_X \), \( f_X \), \( tpx \) and \( f_{T(x)} \).

**Solution:** We have that

\[
\int_0^x \mu(t) \, dt = \int_0^x \frac{1}{t+1} \, dt = \log(1 + t) \bigg|_0^x = \log(1 + x), \quad x \geq 0.
\]

\[
s(x) = \exp \left( - \int_0^x \mu(t) \, dt \right) = \exp(- \log(1 + x)) = \frac{1}{x + 1}, \quad x \geq 0.
\]

\[
f_X(x) = \mu(x)s(x) = \frac{1}{(x + 1)^2}, \quad x \geq 0.
\]

\[
Tp_x = \frac{s(x + t)}{s(x)} = \frac{x + 1}{x + t + 1}, \quad x, \, t \geq 0.
\]

\[
f_{T(x)}(t) = tpx\mu(x + t) = \frac{x + 1}{(x + t + 1)^2}, \quad t \geq 0.
\]