Manual for SOA Exam MLC.

Chapter 2. Survival models.
Section 2.3. Force of Mortality.

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Force of Mortality

Definition 1

The **force of mortality** *of the survival function* S_X *is defined as*

$$\mu(x) = -\frac{d}{dx} \ln S_X(x) = \frac{f_X(x)}{S_X(x)}.$$

The force of mortality is also called the **force of failure**.
An alternative notation for the force of mortality is μ_x. Force of mortality and force of failure are names used in actuarial sciences.

Definition 2

The **hazard rate function** *of the survival function* S_X *is defined as*

$$\lambda_X(x) = -\frac{d}{dx} \ln S_X(x) = \frac{f_X(x)}{S_X(x)},$$

where S_X *is the survival function and* f_X *is the probability density function of the distribution.*

The hazard rate is also called the **failure rate**. Hazard rate and failure rate are names used in reliability theory.

Example 1

Find the force of mortality of the survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, *for* $0 < x < 90$.

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Solution: We have that

$$\mu(x) = -\frac{d}{dx} \ln S_X(x) = -\frac{d}{dx} \ln \left(\frac{90^6 - x^6}{90^6} \right)$$
$$= -\frac{d}{dx} \ln(90^6 - x^6) + \frac{d}{dx} \ln(90^6) = \frac{6x^5}{90^6 - x^6}.$$

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$$\lambda_{T(x)}(t|X > x) = \mu_{x+t}, t \geq 0.$$

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Proof: The survival function of $T(x)$ is $\frac{S_X(x+t)}{S_X(x)}$, $t \geq 0$. Hence,

$$\lambda_{T(x)}(t|X > x) = -\frac{d}{dt}\log\left(\frac{S_X(x+t)}{S_X(x)}\right)$$

$$= -\frac{d}{dt}\log(S_X(x+t)) + \frac{d}{dt}\log(S_X(x)) = \frac{f_X(x+t)}{S_X(x+t)} = \mu_{x+t}.$$

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The hazard rate is the rate of death for lives aged x. For a life aged x, the force of mortality t years later is the force of mortality for a $(x+t)$–year old.

Example 2

Suppose that the survival function of a new born is
$S_X(t) = \frac{85^4 - t^4}{85^4}$, *for* $0 < t < 85$.
(i) Find the force of mortality of a new born.
(ii) Find the force of mortality of a life aged x.

Example 2

Suppose that the survival function of a new born is $S_X(t) = \frac{85^4 - t^4}{85^4}$, *for* $0 < t < 85$.
(i) Find the force of mortality of a new born.
(ii) Find the force of mortality of a life aged x.

Solution: (i) We have that

$$\mu(t) = -\frac{d}{dt}\ln S_X(t) = -\frac{d}{dt}\ln\left(\frac{85^4 - t^4}{85^4}\right)$$
$$= -\frac{d}{dt}\ln(85^4 - t^4) + \frac{d}{dt}\ln(85^4) = \frac{4t^3}{85^4 - t^4}, 0 < t < 85.$$

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(ii)

$$\mu(x+t) = \frac{4(x+t)^3}{85^4 - (x+t)^4}, 0 < t < 85 - x.$$

Definition 3

The **cumulative hazard rate function** *is defined as*

$$\Lambda(x) = \int_0^x \lambda_X(t)\,dt, x \geq 0,$$

where λ_X *is the hazard function of the r.v.* X.

Notice that

$$\Lambda(x) = \int_0^x \lambda_X(t)\,dt = \int_0^x \frac{d}{dt}(-\log S_X(t))\,dt$$
$$=(-\log S_X(t))\Big|_0^x = -\log S_X(x), x \geq 0,$$

where S_X is the survival function of the r.v. X.

Theorem 2

Let $\mu(x) : [0, \infty] \to \mathbb{R}$ be a function which is continuous except at finitely many points. We have that μ is the force of mortality of an age–at–death r.v. if and only if the following two conditions hold:
(i) For each $x \geq 0$, $\mu(x) \geq 0$.
(ii) $\int_0^\infty \mu(t)\, dt = \infty$.

Proof: If μ is a force of mortality, then $\mu(x) = \frac{f_X(x)}{1-F_X(x)} \geq 0$ and

$$\int_0^\infty \mu(t)dt = \int_0^\infty -\frac{d}{dx}(\ln(S_X(x))dt = -\ln(S_X(x)) \Big|_0^\infty = \infty.$$

Suppose μ satisfies conditions
(i) For each $x \geq 0$, $\mu(x) \geq 0$.
(ii) $\int_0^\infty \mu(t)\,dt = \infty$.
Since μ is continuous everywhere except finitely many points

$$S_X(x) = \exp\left(-\int_0^x \mu(t)\,dt\right), x \geq 0$$

defines a survival function. By the fundamental theorem of calculus,

$$-\frac{d}{dx}(\ln(S_X(x)) = -\frac{d}{dx}\left(\int_0^x \mu(t)\,dt\right) = \mu(x),$$

for all points x of continuity of μ.

Example 3

Determine which of the following functions is a legitime force of mortality of an age–at–death:

(i) $\mu(x) = \frac{1}{(x+1)^2}$, *for* $x \geq 0$.

(ii) $\mu(x) = x \sin x$, *for* $x \geq 0$.

(iii) $\mu(x) = 35$, *for* $x \geq 0$.

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(iii) $\mu(x) = 35$, *for* $x \geq 0$.

Solution: (i) $\mu(x) = \frac{1}{(x+1)^2}$ is not a force of mortality because

$$\int_0^\infty \frac{1}{(x+1)^2}\, dx = -\frac{1}{x+1} \Big|_0^\infty = 1.$$

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(ii) $\mu(x) = x \sin x$ is not a force of mortality because $x \sin x < 0$, for $x \in (\pi, 2\pi)$.

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(ii) $\mu(x) = x \sin x$ is not a force of mortality because $x \sin x < 0$, for $x \in (\pi, 2\pi)$.
(iii) $\mu(x) = 35$ is a legitime force of mortality.

Theorem 3

Suppose that $\mu(\cdot)$ is the force of mortality of the age–at–death r.v. X. Then,

(i) The survival function of X is

$$S_X(t) = \exp\left(-\int_0^t \mu(s)\,ds\right), t \geq 0.$$

(ii) The density of X is

$$f_X(t) = S_X(t)\mu(t) = \exp\left(-\int_0^t \mu(s)\,ds\right)\mu(t), t \geq 0$$

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Proof: (i) By the fundamental theorem of calculus,

$$\exp\left(-\int_0^t \mu(s)\,ds\right) = \exp\left(-\int_0^x -\frac{d}{ds}(\ln(S_X(s))\,ds\right)$$
$$= \exp\left(\ln(S_X(s))\,\Big|_{s=0}^{t}\right) = \exp\left(\ln(S_X(t)\right) = S_X(t)$$

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Proof: (ii) follows noticing that $\mu(t) = \frac{f_X(t)}{s(t)}$.

Theorem 4

Suppose that $\mu(\cdot)$ is the force of mortality of the age–at–death r.v. X. Then,

(i) The survival function of $T(x)$ is

$$S_{T(x)}(t) = {}_t p_x = \exp\left(-\int_x^{x+t} \mu(s)\, ds\right), t \geq 0.$$

(ii) The density of $T(x)$ is

$$f_{T(x)}(t) = S_{T(x)}(t)\mu_{T(x)}(t) = e^{-\int_x^{x+t} \mu(y) dy} \mu(x+t), t \geq 0.$$

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$$f_{T(x)}(t) = S_{T(x)}(t)\mu_{T(x)}(t) = e^{-\int_x^{x+t} \mu(y) dy}\mu(x+t), t \geq 0.$$

Proof: (i) The survival function of $T(x)$ is

$${}_t p_x = \frac{S_X(x+t)}{S_X(x)} = \frac{\exp\left(-\int_0^{x+t} \mu(s)\, ds\right)}{\exp\left(-\int_0^{x} \mu(s)\, ds\right)} = \exp\left(-\int_x^{x+t} \mu(s)\, ds\right).$$

Theorem 4

Suppose that $\mu(\cdot)$ is the force of mortality of the age–at–death r.v. X. Then,

(i) The survival function of $T(x)$ is

$$S_{T(x)}(t) = {}_tp_x = \exp\left(-\int_x^{x+t} \mu(s)\,ds\right), t \geq 0.$$

(ii) The density of $T(x)$ is

$$f_{T(x)}(t) = S_{T(x)}(t)\mu_{T(x)}(t) = e^{-\int_x^{x+t}\mu(y)dy}\mu(x+t), t \geq 0.$$

Proof: (i) The survival function of $T(x)$ is

$${}_tp_x = \frac{S_X(x+t)}{S_X(x)} = \frac{\exp\left(-\int_0^{x+t}\mu(s)\,ds\right)}{\exp\left(-\int_0^x \mu(s)\,ds\right)} = \exp\left(-\int_x^{x+t}\mu(s)\,ds\right).$$

(ii) follows from $\mu_{T(x)}(t) = \frac{f_{T(x)}(t)}{S_{T(x)}(t)}$.

The cumulative distribution function of X is

$$F_X(t) = 1 - e^{-\int_0^t \mu(y)dy}, t \geq 0.$$

The cumulative distribution function of T_x is

$${}_tq_x = 1 - e^{-\int_x^{x+t} \mu(y)dy}.$$

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For the force of mortality $\mu(x) = \frac{1}{x+1}$ *for* $x \geq 0$*, find* S_X*,* f_X*,* ${}_t p_x$ *and* $f_{T(x)}$.

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Solution: We have that

$$\int_0^x \mu(t)\,dt = \int_0^x \frac{1}{t+1}\,dt = \log(1+t)\Big|_0^x = \log(1+x), x \geq 0.$$

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$${}_tp_x = \frac{s(x+t)}{s(x)} = \frac{x+1}{x+t+1}, x, t \geq 0.$$

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$$f_{T(x)}(t) = {}_tp_x\mu(x+t) = \frac{x+1}{(x+t+1)^2}, t \geq 0.$$

