Manual for SOA Exam MLC.

Chapter 2. Survival models.

Section 2.4. Expectation of life.

Extract from:
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Expectation of life of a newborn

Definition 1

The **life expectancy** *of a newborn is* $\overset{\circ}{e}_0 = E[X]$.

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Theorem 1

Let X be an age–at–death r.v. with density f_X and survival function S_X. Then,

(i)

$$\overset{\circ}{e}_0 = \int_0^\infty x f_X(x)\,dx = \int_0^\infty S_X(x)\,dx = \int_0^\infty {}_tp_0\,dt.$$

(ii)

$$E[X^2] = \int_0^\infty x^2 f_X(x)\,dx = \int_0^\infty 2x S_X(x)\,dx = \int_0^\infty 2t \cdot {}_tp_0\,dt.$$

Example 1

An actuary models the lifetime in years of a random selected person as a r.v. X *with survival function* $S_X(x) = \frac{90^6 - x^6}{90^6}$, *for* $0 < x < 90$. *Find* $\overset{\circ}{e}_0$ *and* $\mathrm{Var}(X)$.

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An actuary models the lifetime in years of a random selected person as a r.v. X with survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, for $0 < x < 90$. Find $\overset{\circ}{e}_0$ and $\mathrm{Var}(X)$.

Solution: (using the density of X) (i) The density of X is

$$f_X(x) = -\frac{d}{dx} S_X(x) = -\frac{d}{dx} \frac{90^6 - x^6}{90^6} = \frac{6x^5}{90^6}, 0 < x < 90.$$

$$\overset{\circ}{e}_0 = \int_0^\infty x f_X(x) dx = \int_0^{90} x \frac{6x^5}{90^6} dx = \frac{6x^7}{(7)90^6} \Big|_0^{90} = \frac{(6)(90)}{7}$$

$=77.142857$,

$$E[X^2] = \int_0^{90} x^2 \frac{6x^5}{90^6} dx = \frac{6x^8}{(8)90^6} \Big|_0^{90} = \frac{(3)(90)^2}{4} = 6075,$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 6075 - (77.142857)^2 = 1123.9796139.$$

Example 1

An actuary models the lifetime in years of a random selected person as a r.v. X with survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, for $0 < x < 90$. Find $\overset{\circ}{e}_0$ and $\mathrm{Var}(X)$.

(ii) (using the survival function of X) We have that

$$\overset{\circ}{e}_0 = \int_0^\infty S_X(x)dx = \int_0^{90} \frac{90^6 - x^6}{90^6} dx = x - \frac{x^7}{(7)90^6} \Big|_0^{90}$$

$$= 90 - \frac{90^7}{(7)90^6} = \frac{(6)(90)}{7} = 77.142857,$$

$$E[X^2] = \int_0^\infty 2xS_X(x)dx = \int_0^{90} 2x\frac{90^6 - x^6}{90^6} dx = x^2 - \frac{x^8}{(4)90^6} \Big|_0^{90}$$

$$= 90^2 - \frac{90^8}{(4)90^6} = \frac{(3)(90)^2}{4} = 6075,$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 6075 - (77.142857)^2$$

$$= 1123.9796139.$$

expected future lifetime of (x)

Definition 2

The **expected future lifetime at age** x *is*

$$\overset{\circ}{e}_x = E[T(x)].$$

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The **expected future lifetime at age** x *is*

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Theorem 2

(i)

$$\overset{\circ}{e}_x = E[X - x | X > x] = \frac{E[(X-x)I(X > x)]}{\mathbb{P}\{X > x\}} = \frac{\int_x^\infty (t-x) f_X(t) dt}{\mathbb{P}\{X > x\}}.$$

(ii)

$$\overset{\circ}{e}_x = \int_0^\infty {}_t p_x dt = \int_0^\infty \frac{S_X(x+t)}{S_X(x)} dt.$$

Example 2

An actuary models the lifetime in years of a random selected person as a r.v. X *with survival function* $S_X(x) = \frac{90^6 - x^6}{90^6}$, *for* $0 < x < 90$. *Find:*

Example 2

An actuary models the lifetime in years of a random selected person as a r.v. X *with survival function* $S_X(x) = \frac{90^6 - x^6}{90^6}$*, for* $0 < x < 90$*. Find:*

(i) The density of $T(x)$.

Example 2

An actuary models the lifetime in years of a random selected person as a r.v. X with survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, for $0 < x < 90$. Find:

(i) The density of $T(x)$.

Solution: (i) The density of X is

$$f_X(x) = -\frac{d}{dx}S_X(x) = -\frac{d}{dx}\frac{90^6 - x^6}{90^6} = \frac{6x^5}{90^6}, 0 < x < 90.$$

The density of $T(x)$ is

$$f_{T(x)}(t) = \frac{f_X(x+t)}{S_X(x)} = \frac{\frac{6(x+t)^5}{90^6}}{\frac{90^6 - x^6}{90^6}} = \frac{6(x+t)^5}{90^6 - x^6}, 0 < t < 90 - x.$$

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(ii) $\overset{\circ}{e}_x$ using the density of $T(x)$.

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An actuary models the lifetime in years of a random selected person as a r.v. X with survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, for $0 < x < 90$. Find:

(ii) $\overset{\circ}{e}_x$ using the density of $T(x)$.

Solution: (ii) By the change of variables $s = x + t$

$$\overset{\circ}{e}_x = \int_0^\infty t f_{T(x)}(t)\, dt = \int_0^{90-x} t \frac{6(x+t)^5}{90^6 - x^6}\, dt$$

$$= \int_x^{90} (s - x) \frac{6s^5}{90^6 - x^6}\, ds = \frac{6s^7}{(7)(90^6 - x^6)} - \frac{xs^6}{90^6 - x^6}\Big|_x^{90}$$

$$= \frac{6(90)^7}{(7)(90^6 - x^6)} - \frac{x(90)^6}{90^6 - x^6} - \frac{6x^7}{(7)(90^6 - x^6)} + \frac{x^7}{90^6 - x^6}$$

$$= \frac{x^7 - 7(90)^6 + 6(90)^7}{(7)(90^6 - x^6)}$$

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An actuary models the lifetime in years of a random selected person as a r.v. X *with survival function* $S_X(x) = \frac{90^6 - x^6}{90^6}$, *for* $0 < x < 90$. *Find:*

(iii) The survival function of $T(x)$.

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An actuary models the lifetime in years of a random selected person as a r.v. X with survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, for $0 < x < 90$. Find:

(iii) The survival function of $T(x)$.

Solution: (iii) The survival function of $T(x)$ is

$$ {}_t p_x = \frac{S_X(x+t)}{S_X(x)} = \frac{90^6 - (x+t)^6}{90^6 - x^6}, 0 \le t \le 90 - x. $$

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(iv) $\overset{\circ}{e}_x$ using the survival function of $T(x)$.

Solution: (iv)

$$\overset{\circ}{e}_x = \int_0^\infty {}_tp_x \, dt = \int_0^{90-x} \frac{90^6 - (x+t)^6}{90^6 - x^6} \, dt$$

$$= \frac{90^6 t}{90^6 - x^6} - \frac{(x+t)^7}{(7)(90^6 - x^6)} \Big|_0^{90-x}$$

$$= \frac{90^6(90 - x)}{90^6 - x^6} - \frac{(90)^7}{(7)(90^6 - x^6)} + \frac{x^7}{(7)(90^6 - x^6)}$$

$$= \frac{x^7 - 7(90)^6 + 6(90)^7}{(7)(90^6 - x^6)}$$

n–th year temporary complete life expectancy

Definition 3

The n–**th year temporary complete life expectancy** *is the expected number of years lived between age* x *and age* $x + n$ *by a survivor aged* x.

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The n–th year temporary complete life expectancy is denoted by $\overset{\circ}{e}_{x:\overline{n}|}$.

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The n–th year temporary complete life expectancy is denoted by $\overset{\circ}{e}_{x:\overline{n}|}$.

Theorem 3

(i) $\overset{\circ}{e}_{x:\overline{n}|} = E[\min(T(x), n)]$.

(ii) $\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n {}_tp_x \, dt$.

Example 3

An actuary models the lifetime in years of a random selected person as a r.v. X *with survival function* $S_X(x) = \frac{90^6 - x^6}{90^6}$*, for* $0 < x < 90$*. Find* $\overset{\circ}{e}_{x:\overline{n}|}$*, for* $0 < x < 90$*.*

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An actuary models the lifetime in years of a random selected person as a r.v. X with survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, for $0 < x < 90$. Find $\overset{\circ}{e}_{x:\overline{n}|}$, for $0 < x < 90$.

Solution: For $0 \le x, t$ and $x + t < 90$

$$ {}_tp_x = \frac{s(x+t)}{s(x)} = \frac{90^6 - (x+t)^6}{90^6 - x^6}; $$

For $0 \le x, t$ and $x + t > 90$, ${}_tp_x = 0$. If $0 < x < 90 - n$,

$$ \overset{\circ}{e}_{x:\overline{n}|} = \int_0^n {}_tp_x \, dt = \int_0^n \frac{90^6 - (x+t)^6}{90^6 - x^6} \, dt = \frac{(7)(90^6)n + x^7 - (x+n)^7}{(7)(90^6 - x^6)} $$

If $90 - n < x < 90$,

$$ \overset{\circ}{e}_{x:\overline{n}|} = \int_0^n {}_tp_x \, dt = \int_0^{90-x} \frac{90^6 - (x+t)^6}{90^6 - x^6} \, dt $$

$$ = \frac{(7)(90^6)(90 - x) + x^7 - (90)^7}{(7)(90^6 - x^6)} $$

Theorem 4

$$\overset{\circ}{e}_x = \overset{\circ}{e}_{x:\overline{n}|} + {}_n p_x \overset{\circ}{e}_{x+n}$$

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Proof: We have that

$$\begin{aligned}\overset{\circ}{e}_x &= \int_0^\infty {}_t p_x \, dt = \int_0^n {}_t p_x \, dt + \int_n^\infty {}_t p_x \, dt \\ &= \overset{\circ}{e}_{x:\overline{n}|} + \int_n^\infty {}_n p_x \cdot {}_{t-n} p_{x+n} \, dt \\ &= \overset{\circ}{e}_{x:\overline{n}|} + \int_0^\infty {}_n p_x \cdot {}_t p_{x+n} \, dt = \overset{\circ}{e}_{x:\overline{n}|} + {}_n p_x \overset{\circ}{e}_{x+n}\end{aligned}$$

Example 4

Assume that
(i) The expected future lifetime of a 40–year old is 45 years.
(ii) The expected future lifetime of a 50–year old is 36 years.
(iii) The probability that a 40–year old survives to age 50 is 0.98.
Find the expected number of years lived between age 40 *and age* 50 *by a 40 year old.*

Example 4

Assume that

(i) The expected future lifetime of a 40–year old is 45 years.

(ii) The expected future lifetime of a 50–year old is 36 years.

(iii) The probability that a 40–year old survives to age 50 is 0.98.

Find the expected number of years lived between age 40 *and age* 50 *by a 40 year old.*

Solution: Using that $\overset{\circ}{e}_x = \overset{\circ}{e}_{x:\overline{n}|} + {}_np_x \overset{\circ}{e}_{x+n}$, we get that $45 = \overset{\circ}{e}_{40:\overline{10}|} + (0.98)(36)$ and $\overset{\circ}{e}_{40:\overline{10}|} = 45 - (0.98)(36) = 9.72$.

central rate of failure

Definition 4

The **central rate of failure on the interval from age** x **to age** $x+t$ **is**

$$ {}_t m_x = \frac{\int_x^{x+t} S_X(y)\lambda_X(y)\,dy}{\int_x^{x+t} S_X(y)\,dy} = \frac{\int_0^t {}_y p_x \mu_{x+y}\,dy}{\int_0^t {}_y p_x\,dy}. $$

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The central rate of failure on the interval is an average of the rate of failure weighted using the survival function. We abbreviate $m_x = {}_1 m_x$.

Theorem 5

$$ {}_t m_x = \frac{S_X(x) - S_X(x+t)}{\int_x^{x+t} S_X(y)\,dy}. $$

Proof.

We have that

$$ \int_x^{x+t} S_X(y)\lambda_X(y)\,dy = \int_x^{x+t} f_X(y)\,dy = S_X(x) - S_X(x+t). $$

So,

$$ {}_t m_x = \frac{\int_x^{x+t} S_X(y)\lambda_X(y)\,dy}{\int_x^{x+t} S_X(y)\,dy} = \frac{S_X(x) - S_X(x+t)}{\int_x^{x+t} S_X(y)\,dy}. $$

□

Example 5

For the survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, *for* $0 < x < 90$. *Find* ${}_t m_x$.

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Solution: We have that

$$S_X(x) - S_X(x+t) = \frac{(x+t)^6 - x^6}{90^6},$$

$$\int_x^{x+t} S_X(y)\,dy = \int_x^{x+t} \frac{90^6 - y^6}{90^6}\,dy = \frac{(7)90^6 t - (x+t)^6 + x^6}{(7)90^6},$$

$${}_t m_x = \frac{S_X(x) - S_X(x+t)}{\int_x^{x+t} S_X(y)\,dy} = \frac{7(x+t)^6 - 7x^6}{(7)90^6 t - (x+t)^6 + x^6}.$$

median future lifetime

Definition 5

The **median future lifetime** *of* (x) *is* $m(x)$ *such that* $\mathbb{P}\{T(x) < m(x)\} \leq \frac{1}{2} \leq \mathbb{P}\{T(x) \leq m(x)\}$.

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Usually, $m(x)$ is the solution of

$$\frac{1}{2} = \mathbb{P}\{T(x) > m(x)\} = \frac{S_X(x + m(x))}{S_X(x)}.$$

Example 6

For the survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, *for* $0 < x < 90$. *Find the median future lifetime of* (x).

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For the survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, *for* $0 < x < 90$. *Find the median future lifetime of* (x).

Solution: Let $m(x)$ be the median future lifetime of (x). We have that

$$\frac{1}{2} = \mathbb{P}\{T(x) > m(x)\} = \frac{S_X(x + m(x))}{S_X(x)} = \frac{90^6 - (x + m(x))^6}{90^6 - x^6}$$

and $m(x) = \left(\frac{90^6 + x^6}{2}\right)^{1/6} - x$.

