Manual for SOA Exam MLC.

Chapter 2. Survival models.

Section 2.5. Curtate failure.

Extract from:
"Arcones' Manual for SOA Exam MLC. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Curtate failure

Definition 1

The **time interval of failure** *of a life aged* x *is denoted by* K_x.

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Notice that $K_x = k$ if $k - 1 < T(x) \leq k$, i.e.

$$K_x = \begin{cases} 1 & \text{if } 0 < T(x) \leq 1, \\ 2 & \text{if } 1 < T(x) \leq 2, \\ 3 & \text{if } 2 < T(x) \leq 3, \\ . & \dots \\ . & \dots \end{cases}$$

Curtate duration of failure of a life age x

Definition 2

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$K(x) = K_x - 1.$

We have that

$$
\begin{aligned}
&\mathbb{P}\{K_x = k\} = \mathbb{P}\{k-1 < T(x) \le k\} \\
=&\mathbb{P}\{k-1 < X - x \le k \mid X > x\} \\
=&\frac{s(x+k-1) - s(x+k)}{s(x)} \\
=&{}_{k-1}p_x - {}_kp_x = {}_{k-1|}q_x = {}_{k-1}p_x q_{x+k-1}, k = 1, 2, \dots
\end{aligned}
$$

Hence,

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We have that

$$\mathbb{P}\{K(x) = k\} = \mathbb{P}\{k < T(x) \le k+1\} = {}_kp_x \cdot q_{x+k}, \ k = 0, 1, 2, \dots$$

Hence,

$$\mathbb{P}\{K(x) \ge k\} = \mathbb{P}\{T(x) > k\} = {}_kp_x, \ k = 0, 1, 2, \dots$$

Example 1

Suppose that ${}_tp_x = \frac{80^5-(x+t)^5}{80^5-x^5}$, $0 \le t \le 80 - x$, *where* x *is a positive integer. Find the probability mass function of* $K(x)$*.*

Solution: For $k = 0, 1, 2, \dots, 79 - x$,

$$\mathbb{P}\{K(x) = k\} = \mathbb{P}\{k < T(x) \le k+1\} = {}_kp_x - {}_{k+1}p_x$$

$$= \frac{(x+k+1)^5 - (x+k)^5}{80^5 - x^5}.$$

Curtate duration of failure

Definition 3

The **curtate life expectation of a life age** x *is the expectation of the curtate duration of this life, i.e.* $E[K(x)]$.

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The **curtate life expectation of a life age** x *is the expectation of the curtate duration of this life, i.e.* $E[K(x)]$.

We denote the curtate life expectation by e_x.

Theorem 1

$$e_x = E[K(x)] = \sum_{k=1}^{\infty} k\mathbb{P}\{K(x) = k\} = \sum_{k=1}^{\infty} k \cdot {}_kp_x \cdot q_{x+k},$$

$$e_x = E[K(x)] = \sum_{k=1}^{\infty} \mathbb{P}\{K(x) \geq k\} = \sum_{k=1}^{\infty} {}_kp_x,$$

$$E[(K(x))^2] = \sum_{k=1}^{\infty} k^2\mathbb{P}\{K(x) = k\} = \sum_{k=1}^{\infty} k^2 \cdot {}_kp_x \cdot q_{x+k},$$

$$E[(K(x))^2] = \sum_{k=1}^{\infty}(2k-1)\mathbb{P}\{K(x) \geq k\} = \sum_{k=1}^{\infty}(2k-1) \cdot {}_kp_x.$$

Example 2

Suppose that $s(x) = \frac{100-x}{100}$, *for* $0 \le x \le 100$. *Find* $\overset{\circ}{e}_x$ *and* e_x, *where* $1 \le x \le 99$ *is an integer.*

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Solution: We have that ${}_t p_x = \frac{s(x+t)}{s(x)} = \frac{100-x-t}{100-x}$,

$$\overset{\circ}{e}_x = \int_0^{100-x} {}_t p_x \, dt = \int_0^{100-x} \frac{100-x-t}{100-x} \, dt$$
$$= \frac{(100-x-t)^2}{2(100-x)} \Big|_0^{100-x} = \frac{100-x}{2},$$
$$e_x = \sum_{k=1}^{\infty} {}_k p_x = \sum_{k=1}^{100-x} \frac{100-x-k}{100-x}$$
$$= 100 - x - \frac{1}{100-x} \frac{(100-x)(100-x+1)}{2}$$
$$= 100 - x - \frac{100-x+1}{2} = \frac{99-x}{2}$$

Theorem 2

Suppose that $p_k = p$*, for each* $k \geq 1$*. Show that* $e_x = \frac{p}{1-p}$ *and* $\mathrm{Var}(K(x)) = \frac{p}{(1-p)^2}$.

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Solution: We have that

$$
\begin{aligned}
{}_k p_x &= p_x p_{x+1} \cdots p_{x+k-1} = p^k, \\
e_x &= \sum_{k=1}^{\infty} {}_k p_x = \sum_{k=1}^{\infty} p^k = \frac{p}{1-p}, \\
E[K(x)^2] &= \sum_{k=1}^{\infty} (2k-1)\, {}_k p_x = \sum_{k=1}^{\infty} (2k-1) p^k = \frac{2p}{(1-p)^2} - \frac{p}{1-p}
\end{aligned}
$$

$$
= \frac{p + p^2}{(1-p)^2},
$$

$$
\mathrm{Var}(K(x)) = \frac{p + p^2}{(1-p)^2} = \frac{p}{(1-p)^2}.
$$

Theorem 3

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Proof: We have that

$$e_x = \sum_{k=1}^{\infty} {}_k p_x = p_x + \sum_{k=2}^{\infty} p_x \cdot {}_{k-1} p_{x+1} = p_x + \sum_{k=1}^{\infty} p_x \cdot {}_k p_{x+1}$$
$$= p_x(1 + e_{x+1}).$$

Example 3

Suppose that $e_x = 30$, $p_x = 0.97$ *and* $p_{x+1} = 0.95$. *Find* e_{x+2}.

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Solution: Using that $e_{x+1} = \frac{e_x}{p_x} - 1$, we get that

$$e_{x+1} = \frac{30}{0.97} - 1 = 29.92783505$$

and

$$e_{x+2} = \frac{29.92783505}{0.95} - 1 = 30.50298426.$$

expected whole years lived in the interval $(x, x+n]$

Definition 4

The expected whole years lived in the interval $(x, x+n]$ *by an entity alive at age* x *is* $e_{x:\overline{n}|}$.

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It is easy to see that $e_{x:\overline{n}|} \leq \mathring{e}_{x:\overline{n}|}$.

Theorem 4

$$e_{x:\overline{n}|} = \sum_{k=1}^{n-1} k \cdot {}_k p_x q_k + n \cdot {}_n p_x = \sum_{k=1}^{n} {}_k p_x .$$

Theorem 4

$$e_{x:\overline{n}|} = \sum_{k=1}^{n-1} k \cdot {}_kp_x q_k + n \cdot {}_np_x = \sum_{k=1}^{n} {}_kp_x.$$

Proof: The whole years lived in the interval $(x, x+n]$ by an entity alive at age x is $\min(K(x), n)$. Hence,

$$e_{x:\overline{n}|} = E[\min(K(x), n)] = \sum_{k=1}^{n-1} k\mathbb{P}\{K(x) = k\} + n\mathbb{P}\{K(x) \geq n\}$$

$$= \sum_{k=1}^{n-1} k \cdot {}_kp_x q_k + n \cdot {}_np_x.$$

and

$$e_{x:\overline{n}|} = E[\min(K(x), n)] = \sum_{k=1}^{n} \mathbb{P}\{K(x) \geq k\} = \sum_{k=1}^{n} {}_kp_x.$$

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Solution: We have that

$$_kp_x = p_x p_{x+1} \cdots p_{x+k-1} = p^k,$$

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x = \sum_{k=1}^{n} p^k = \frac{p - p^{n+1}}{1 - p}.$$

Theorem 5

$$e_x = e_{x:\overline{n}|} + {}_np_x e_{x+n}.$$

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$$e_x = e_{x:\overline{n}|} + {}_n p_x e_{x+n}.$$

Proof: We have that

$$e_{x:\overline{n}|} + {}_n p_x e_{x+n} = \sum_{k=1}^{n} {}_k p_x + {}_n p_x \sum_{k=1}^{\infty} {}_k p_{x+n} = \sum_{k=1}^{n} {}_k p_x + \sum_{k=1}^{\infty} {}_{k+n} p_x$$

$$= \sum_{k=1}^{\infty} {}_k p_x = e_x.$$

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Solution: We have that

$$e_{x:\overline{2}|} = p_x + {}_2p_x = 0.97 + (0.97)(0.95) = 1.8915.$$

and

$$30 = e_{x:\overline{2}|} + {}_2p_x e_{32} = 1.8915 + (0.97)(0.95)e_{x+2}$$

So, $e_{x+2} = \frac{30-1.8915}{(0.97)(0.95)} = 30.50298426.$

Definition 5

S_x denotes the period of time lived through the death interval of an entity aged x, i.e. $S_x = T_x - K(x)$.

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S_x is a r.v. taking values in the interval $(0, 1]$. Notice that $E[S_x] = \mathring{e}_x - e_x$.

