# Manual for SOA Exam MLC.

## Chapter 2. Survival models.
## Section 2.7 Common Analytical Survival Models



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# Common Analytical Survival Models

Sometimes it is of interest to assume that the survival function follows a **parametric model**, i.e. it is of the form $S(x, \theta)$, where $\theta$ is an unknown parameter. There are several reasons to make this assumption:

1. Data supports this assumption. Actuaries realized that the models observed in real life follow this assumption.
2. Computations are simpler using a parametric model.
3. There are valid scientific reasons to justify the use of a particular parametric model.



 

The most common approach is not to use parametric models due to the following reasons:
1. Modern computers allow to handle the computations needed using the collected data.
2. It is difficult to justify that a parametric model applies.
3. Knowing that a particular model applies we can get more accurate estimates. But, this increase in accuracy is not much. If a parametric model does not apply, using the parametric approach we can get much worse estimates than the nonparametric estimates.


 

# De Moivre model.

**De Moivre's law** (1729) assumes that deaths happen uniformly over the interval of deaths, i.e. the density of the age–at–failure is $f_X(x) = \frac{1}{\omega}$, for $0 \le x \le \omega$. Therefore,

$$S_X(x) = \frac{\omega - x}{\omega}, \text{ for } 0 \le x < \omega,$$

$$F_X(x) = \frac{x}{\omega}, \text{ for } 0 \le x < \omega,$$

$$\mu(x) = \frac{1}{\omega - x}, \text{ for } 0 \le x < \omega,$$

$${}_t p_x = \frac{s(x+t)}{s(x)} = \frac{\omega - x - t}{\omega - x}, \text{ for } 0 \le t \le \omega - x,$$

$${}_t q_x = \frac{t}{\omega - x}, \text{ for } 0 \le t \le \omega - x.$$



 

## Example 1

*Find the median of an age–at–death subject to de Moivre's law if the probability that a life aged 20 years survives 40 years is* $\frac{1}{3}$.



 

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*Find the median of an age–at–death subject to de Moivre's law if the probability that a life aged 20 years survives 40 years is $\frac{1}{3}$.*

**Solution:** We know that ${}_{40}p_{20} = \frac{2}{3}$. For the uniform distribution, ${}_tp_x = \frac{\omega-x-t}{\omega-x}$. Hence, $\frac{1}{3} = \frac{\omega-20-40}{\omega-20}$, $\omega - 20 = 3\omega - 180$ and $\omega = 80$. Let $m$ be the median of the age–at–death. Then,

$$\frac{1}{2} = S_X(m) = \frac{\omega - m}{\omega} = \frac{80 - m}{80}$$

and $m = 40$.



 

Under the De Moivre's law, $T(x)$ has a uniform distribution on the interval $[0, \omega - x]$. Hence,

### Theorem 1

*For the De Moivre's law,*

$$\overset{\circ}{e}_x = \frac{\omega - x}{2} \text{ and } \mathrm{Var}(T(x)) = \frac{(\omega - x)^2}{12}.$$



 

## Example 2

*Suppose that the survival of a cohort follows the De Moivre's law. Suppose that the expected age–at–death of a new born is 70 years. Find the expected future lifetime of a 50–year old.*

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**Solution:** Since $70 = e_0 = \frac{\omega}{2}$, $\omega = 140$. The expected future lifetime of a 50–year old is

$$\overset{\circ}{e}_{50} = \frac{140 - 50}{2} = 45.$$



 

## Theorem 2

*For the de Moivre's law, for $0 \leq x \leq \omega$, $x$, $\omega$ integers,*

$$e_x = \frac{\omega - x - 1}{2}.$$



 

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$$e_x = \frac{\omega - x - 1}{2}.$$

**Proof:** We have that

$$\begin{aligned} e_x &= \sum_{k=1}^{\omega-x} \frac{\omega - x - k}{\omega - x} = \sum_{k=1}^{\omega-x} 1 - \sum_{k=1}^{\omega-x} \frac{k}{\omega - x} \\ &= (\omega - x) - \frac{\omega - x)((\omega - x) + 1)}{2(\omega - x)} = (\omega - x) - \frac{\omega - x) + 1}{2} \\ &= \frac{\omega - x - 1}{2}. \end{aligned}$$



 

## Theorem 3

*For the de Moivre's law with terminal age $\omega$, where $\omega$ is a positive integer, and each for $0 \leq x \leq \omega$, where $x$ is an integer,*

$$\mathbb{P}\{K(x) = k\} = \frac{1}{\omega - x}, k = 0, 1, 2, \dots, \omega - x.$$

*and*

$$\mathbb{P}\{K_x = k\} = p_1^{k-1}(1 - p_1), k = 1, 2, \dots, \omega - x - 1.$$

**Proof:**

$$\mathbb{P}\{K(x) = k\} = \mathbb{P}\{k < T_x \leq k + 1\} = \int_k^{k+1} \frac{1}{\omega - x}\, dt = \frac{1}{\omega - x}$$



 

# Exponential model.

An **exponential model** assumes that the age–at–death has an exponential distribution, i.e.

$$S_X(x) = e^{-\mu x}, x \geq 0$$

where $\mu > 0$. In this case,

$$\begin{aligned} F_X(x) &= 1 - e^{-\mu x}, \text{ for } 0 \leq x, \\ f_X(x) &= \mu e^{-\mu x}, \text{ for } 0 \leq x, \\ \mu(x) &= \mu, \text{ for } 0 \leq x, \\ {}_t p_x &= \frac{s(x+t)}{s(x)} = e^{-\mu t} = p_x^t. \end{aligned}$$

For an exponential model, the force of mortality is constant. The exponential model is also called the **constant force model**.



 

## Example 3

*Suppose that:*
*(i) the force of mortality is constant.*
*(ii) the probability that a 30–year–old will survive to age 40 is* 0.95.
*Calculate:*
*(i) the probability that a 40–year–old will survive to age 50.*
*(ii) the probability that a 30–year–old will survive to age 50.*
*(iii) the probability that a 30–year–old will die between ages 40 and 50.*


 

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**Solution:** (i) Since the force of mortality is constant,

$_{10}p_{40} = {_{10}p_{30}} = 0.95.$



 

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**Solution:** (i) Since the force of mortality is constant,
${}_{10}p_{40} = {}_{10}p_{30} = 0.95$.
(ii) ${}_{20}p_{30} = {}_{10}p_{30} \cdot {}_{10}p_{40} = (0.95)(0.95) = 0.9025$.



 

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**Solution:** (i) Since the force of mortality is constant,
${}_{10}p_{40} = {}_{10}p_{30} = 0.95$.
(ii) ${}_{20}p_{30} = {}_{10}p_{30} \cdot {}_{10}p_{40} = (0.95)(0.95) = 0.9025$.
(iii) We can do either

$${}_{10|10}p_{30} = {}_{10}p_{30} - {}_{20}p_{30} = 0.95 - 0.9025 = 0.0475.$$

or

$${}_{10|10}p_{30} = {}_{10}p_{30} \cdot {}_{10}q_{40} = (0.95)(1 - 0.95) = 0.0475.$$

## Theorem 4

*Suppose that the lifetime random variable of a new born has constant mortality force $\mu$. Then,*

$$\overset{\circ}{e}_x = \frac{1}{\mu} \text{ and } \mathrm{Var}(X) = \frac{1}{\mu^2}.$$



 

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*Suppose that the lifetime random variable of a new born has constant mortality force $\mu$. Then,*

$$\overset{\circ}{e}_x = \frac{1}{\mu} \text{ and } \mathrm{Var}(X) = \frac{1}{\mu^2}.$$

**Proof:** Since ${}_tp_x = e^{-\mu t}$, $T(x)$ has an exponential distribution with mean $\frac{1}{\mu}$.



 

## Theorem 5

*Suppose that the lifetime random variable of a new born has constant mortality force $\mu$. Then, the curtate future lifetime of $(x)$ is*

$$e_x = \frac{1}{e^{\mu} - 1}.$$



 

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**Proof:** We have that

$$e_x = \sum_{k=1}^{\infty} {}_k p_x = \sum_{k=1}^{\infty} e^{-k\mu} = \frac{e^{-\mu}}{1 - e^{-\mu}} = \frac{1}{e^{\mu} - 1}.$$



 

## Example 4

*Suppose that:*

*(i) the force of mortality is constant.*

*(ii) the probability that a 30–year–old will survive to age 40 is* 0.95.

*Calculate:*

*(i) the future lifetime of a 40–year–old.*

*(ii) the future curtate lifetime of a 40–year–old.*



 

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*Calculate:*
*(i) the future lifetime of a 40–year–old.*
*(ii) the future curtate lifetime of a 40–year–old.*

**Solution:** (i) We know that ${}_{10}p_{30} = 0.95 = e^{-(10)\mu}$. Hence, $\mu = \frac{-\ln(0.95)}{10}$ and $\overset{\circ}{e}_{40} = \frac{1}{\mu} = \frac{10}{-\ln(0.95)} = 194.9572575$.



 

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*Calculate:*
*(i) the future lifetime of a 40–year–old.*
*(ii) the future curtate lifetime of a 40–year–old.*

**Solution:** (i) We know that ${}_{10}p_{30} = 0.95 = e^{-(10)\mu}$. Hence, $\mu = \frac{-\ln(0.95)}{10}$ and $\overset{\circ}{e}_{40} = \frac{1}{\mu} = \frac{10}{-\ln(0.95)} = 194.9572575$.
(ii) Since $e^{-\mu} = (0.95)^{0.1}$,
$e_{40} = \frac{1}{e^{\mu}-1} = \frac{1}{(0.95)^{-0.1}-1} = 194.4576849$.


 

# Gompertz model.

**Gompertz's model** (1825) says that $\mu_x = Bc^x$, where $B > 0$ and $c > 1$. Hence, $s(x) = e^{-m(c^x-1)}$ for $x \geq 0$, where $m = \frac{B}{\log c}$.



 

# Makeham model.

Makehan (1860) introduced the model $\mu_x = A + Bc^x$, where $A \geq -B$, $B > 0$ and $c > 1$. Hence, $s(x) = e^{-Ax-m(c^x-1)}$ for $x \geq 0$, where $m = \frac{B}{\log c}$. We also have that

$$f_X(x) = s(x)\mu(x) = (A + Bc^x)e^{-Ax-m(c^x-1)}, x \geq 0.$$



 

# Weibull model.

**Weibull** (1939) introduced the model $\mu(x) = kx^n$, for $x \geq 0$, where $k > 0$ and $n > -1$. Then,

$$s(x) = e^{-\frac{kx^{n+1}}{n+1}}, x \geq 0$$

$$f_X(x) = s(x)\mu(x) = kx^n e^{-\frac{kx^{n+1}}{n+1}}, x \geq 0.$$



 