Manual for SOA Exam MLC.

Chapter 3. Life tables.

Section 3.1. Life tables.

Extract from:
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ℓ_x is also called the **number living** or the **number of lives** at age x.

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A **life table** *is a display of* ℓ_k*, for each nonnegative integer* k*.*

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A **life table** *is a display of* ℓ_k*, for each nonnegative integer* k*.*

Theorem 1

The number of individual alive ℓ_x *satisfies:*

$$\ell_0 \geq \ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0$$

Table: Life table for the total population: United States, 2004. Source: http://www.cdc.gov/

Age	ℓ_x	$1000q_x$	$\overset{\circ}{e}_x$	Age	ℓ_x	$1000q_x$	$\overset{\circ}{e}_x$
0	100000	6.8	77.837	20	98709	0.871	58.7718
1	99320	0.483	77.3665	21	98623	0.923	57.8226
2	99272	0.292	76.4036	22	98532	0.964	56.8755
3	99243	0.232	75.4258	23	98437	0.975	55.93
4	99220	0.181	74.4432	24	98341	0.966	54.9841
5	99202	0.171	73.4566	25	98246	0.957	54.0367
6	99185	0.161	72.4691	26	98152	0.958	53.088
7	99169	0.151	71.4807	27	98058	0.948	52.1384
8	99154	0.141	70.4914	28	97965	0.96	51.1874
9	99140	0.111	69.5013	29	97871	0.971	50.2361
10	99129	0.111	68.509	30	97776	0.992	49.2845
11	99118	0.111	67.5165	31	97679	1.024	48.3329
12	99107	0.151	66.524	32	97579	1.066	47.3819
13	99092	0.222	65.534	33	97475	1.118	46.4319
14	99070	0.343	64.5484	34	97366	1.191	45.4834
15	99036	0.454	63.5704	35	97250	1.275	44.537
16	98991	0.586	62.5991	36	97126	1.369	43.5932
17	98933	0.677	61.6355	37	96993	1.495	42.6523
18	98866	0.769	60.6769	38	96848	1.631	41.7154
19	98790	0.82	59.7232	39	96690	1.789	40.7828

Age	ℓ_x	$1000q_x$	$\overset{\circ}{e}_x$	Age	ℓ_x	$1000q_x$	$\overset{\circ}{e}_x$
40	96517	1.948	39.855	60	88038	9.485	22.5082
41	96329	2.107	38.9318	61	87203	10.458	21.7189
42	96126	2.289	38.013	62	86291	11.438	20.9432
43	95906	2.492	37.099	63	85304	12.426	20.1797
44	95667	2.728	36.1904	64	84244	13.413	19.4273
45	95406	2.977	35.2881	65	83114	14.474	18.6847
46	95122	3.248	34.3919	66	81911	15.7	17.9517
47	94813	3.523	33.5024	67	80625	17.079	17.2301
48	94479	3.8	32.6191	68	79248	18.625	16.5208
49	94120	4.091	31.7416	69	77772	20.329	15.8249
50	93735	4.395	30.8699	70	76191	22.102	15.1429
51	93323	4.758	30.004	71	74507	24.025	14.4738
52	92879	5.114	29.145	72	72717	26.211	13.8178
53	92404	5.487	28.2923	73	70811	28.738	13.1763
54	91897	5.876	27.4456	74	68776	31.566	12.5513
55	91357	6.294	26.6049	75	66605	34.427	11.9442
56	90782	6.752	25.7702	76	64312	37.38	11.3522
57	90169	7.286	24.942	77	61908	40.754	10.7736
58	89512	7.898	24.1214	78	59385	44.759	10.2101
59	88805	8.637	23.3095	79	56727	49.394	9.6651

Age	ℓ_x	$1000q_x$	$\mathring{e}_x$	Age	ℓ_x	$1000q_x$	$\mathring{e}_x$
80	53925	54.483	9.1413	100	2500	291.2	2.4816
81	50987	59.76	8.6392	101	1772	314.898	2.2957
82	47940	65.436	8.1565	102	1214	339.374	2.1211
83	44803	71.602	7.6926	103	802	366.584	1.9539
84	41595	78.519	7.2473	104	508	395.669	1.7953
85	38329	85.888	6.8223	105	307	429.967	1.6433
86	35037	93.901	6.4163	106	175	462.857	1.5057
87	31747	102.561	6.0294	107	94	500	1.3723
88	28491	111.86	5.6613	108	47	531.915	1.2447
89	25304	121.917	5.3114	109	22	590.909	1.0909
90	22219	132.724	4.9795	110	9	666.667	0.9444
91	19270	144.318	4.665	111	3	666.667	0.8333
92	16489	156.711	4.3674	112	1	1000	0.5
93	13905	169.939	4.0861	113	0		
94	11542	183.937	3.8203	114	0		
95	9419	198.96	3.5687	115	0		
96	7545	214.579	3.3309	116	0		
97	5926	231.185	3.1043	117	0		
98	4556	248.683	2.8874	118	0		
99	3423	269.647	2.6776	119	0		

Definition 3

${}_t d_x$ denotes the number of people which died between ages x and $x+t$, i.e., ${}_t d_x = \ell_x - \ell_{x+t}$.

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Notice that $d_x = {}_1 d_x$.

Theorem 2

For $k, n \geq 0$,

$$d_k \leq \ell_k,\ \ell_k = \sum_{j=k}^{\infty} d_j,\ {}_n d_k = \ell_k - \ell_{k+n} = \sum_{j=k}^{k+n-1} d_j.$$

From a life table, we estimate the survival function of the age–at–death random variable X by

$$s(x) = \frac{\ell_x}{\ell_0}, x \geq 0.$$

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Here are several actuarial variables which can be obtained from a life table:

$$ {}_tp_x = \frac{\ell_{x+t}}{\ell_x}, \ {}_tq_x = \frac{\ell_x - \ell_{x+t}}{\ell_x} = \frac{{}_td_x}{\ell_x},$$

$$p_x = \frac{\ell_{x+1}}{\ell_x}, \ q_x = \frac{\ell_x - \ell_{x+1}}{\ell_x} = \frac{d_x}{\ell_x}, \ {}_{n|m}q_x = \frac{\ell_{x+n} - \ell_{x+n+m}}{\ell_x}.$$

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Here are several actuarial variables which can be obtained from a life table:

$$\begin{aligned} {}_tp_x &= \frac{\ell_{x+t}}{\ell_x},\ {}_tq_x = \frac{\ell_x - \ell_{x+t}}{\ell_x} = \frac{{}_td_x}{\ell_x}, \\ p_x &= \frac{\ell_{x+1}}{\ell_x},\ q_x = \frac{\ell_x - \ell_{x+1}}{\ell_x} = \frac{d_x}{\ell_x},\ {}_{n|m}q_x = \frac{\ell_{x+n} - \ell_{x+n+m}}{\ell_x}. \end{aligned}$$

Notice that a life table only contains values for nonnegative numbers. This will make a challenge to estimate quantities which depend on a continuous set of values of the survival function s.

Example 1

Complete the entries in the following table:

Age	ℓ_x	d_x	p_x	q_x
0	100000	.	.	.
1	99523	.	.	.
2	89123	.	.	.
3	87174	.	.	.
4	86234	.	.	.
5	85346	—	—	—

Solution: Using that $d_x = \ell_x - \ell_{x+1}$, $p_x = \frac{\ell_{x+1}}{\ell_x}$ and $q_x = \frac{\ell_x - \ell_{x+1}}{\ell_x}$, we get

Age	ℓ_x	d_x	p_x	q_x
0	10000	2477	0.97523	0.02477
1	97523	3400	0.9651364294	0.03486357064
2	94123	2949	0.968668657	0.03133134303
3	91174	3940	0.9567859258	0.04321407419
4	87234	3940	0.9851434074	0.01485659261
5	85938	–	–	–

For example,

$$d_0 = 10000 - 97523 = 2477, p_0 = \frac{97523}{10000} = 0.97523,$$

$$q_0 = \frac{10000 - 97523}{10000} = 0.02477, d_1 = 97523 - 94123 = 3400,$$

$$p_1 = \frac{94123}{97523} = 0.9651364294, q_1 = \frac{97523 - 94123}{97523} = 0.03486357064.$$

Theorem 3

For $k, n \geq 0$,

$$\ell_{k+n} = \ell_k \cdot p_k \cdot p_{k+1} \cdots p_{k+n-1}.$$

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Proof.

We have that

$$\ell_k \cdot p_k \cdot p_{k+1} \cdots p_{k+n-1} = \ell_k \cdot \frac{\ell_{k+1}}{\ell_k} \cdot \frac{\ell_{k+2}}{\ell_{k+1}} \cdots \frac{\ell_{k+n}}{\ell_{k+n-1}} = \ell_{k+n}$$

□

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

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Example 2

Using the life table in page 8, find:

(i) ℓ_{10}.

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(i) ℓ_{10}.

Solution: (i) $\ell_{10} = 99129$.

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

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Example 2

Using the life table in page 8, find:

(ii) d_{35}.

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(ii) d_{35}.

Solution: (ii) $d_{35} = \ell_{35} - \ell_{36} = 97250 - 97126 = 124.$

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

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Example 2

Using the life table in page 8, find:

(iii) ${}_5d_{35}$.

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(iii) ${}_5d_{35}$.

Solution: (iii) ${}_5d_{35} = \ell_{35} - \ell_{40} = 97250 - 96517 = 733.$

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(iv) The probability that a newborn will die before reaching 50 years.

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(iv) The probability that a newborn will die before reaching 50 years.

Solution: (iv) $\frac{\ell_0 - \ell_{50}}{\ell_0} = \frac{100000 - 93735}{100000} = 0.06265.$

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(v) The probability that a newborn will live more than 60 years.

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(v) The probability that a newborn will live more than 60 years.

Solution: (v) $\frac{\ell_{60}}{\ell_0} = \frac{88038}{100000} = 0.88038.$

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(vi) The probability that a newborn will die when his age is between 45 years and 65 years old.

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(vi) The probability that a newborn will die when his age is between 45 years and 65 years old.

Solution: (vi) $\frac{\ell_{45}-\ell_{65}}{l_0} = \frac{95406-83114}{100000} = 0.12292.$

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(vii) The probability that a 25–year old will die before reaching 50 years.

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(vii) The probability that a 25–year old will die before reaching 50 years.

Solution: (vii) $\frac{\ell_{25}-\ell_{50}}{\ell_{25}} = \frac{98246-93735}{98246} = 0.04591535533.$

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(viii) The probability that a 25–year old will live more than 60 years.

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(viii) The probability that a 25–year old will live more than 60 years.

Solution: (viii) $\frac{\ell_{60}}{\ell_{25}} = \frac{88038}{98246} = 0.896097551.$

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(ix) The probability that a 25–year old will die when his age is between 50 years and 65 years old.

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

$$\ell_{36} = 97126.$$

Example 2

Using the life table in page 8, find:

(ix) The probability that a 25–year old will die when his age is between 50 years and 65 years old.

Solution: (ix) $\frac{\ell_{50}-\ell_{65}}{\ell_{25}} = \frac{93735-83114}{98246} = 0.1081061824.$

Age	ℓ_x	Age	ℓ_x	Age	ℓ_x	Age	ℓ_x
0	100000	25	98246	50	93735	75	66605
5	99202	30	97776	55	91357	80	53925
10	99129	35	97250	60	88038	85	38329
15	99036	40	96517	65	83114	90	22219
20	98709	45	95406	70	76191	95	9419

Example 3

For the life table in page 8, find ${}_5p_{20}$, ${}_5p_{40}$, ${}_5p_{60}$, ${}_5p_{80}$; *check whether* ${}_5p_{20} > {}_5p_{60} > {}_5p_{80}$.

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For the life table in page 8, find ${}_5p_{20}$, ${}_5p_{40}$, ${}_5p_{60}$, ${}_5p_{80}$; *check whether* ${}_5p_{20} > {}_5p_{60} > {}_5p_{80}$.

Solution: We have that

$$
\begin{aligned}
{}_5p_{20} &= \frac{\ell_{25}}{\ell_{20}} = \frac{98246}{98709} = 0.9953094449, \\
{}_5p_{60} &= \frac{\ell_{65}}{\ell_{60}} = \frac{83114}{88038} = 0.9440696063, \\
{}_5p_{80} &= \frac{\ell_{85}}{\ell_{80}} = \frac{38329}{53925} = 0.7107834956.
\end{aligned}
$$

From a life table, we can find the distribution of the curtate life $K(x)$ and of the time interval of death K_x. We have that

$$\mathbb{P}\{K(x) = k\} = P\{k < T(x) \leq k+1\} = \frac{\ell_{x+k} - \ell_{x+k+1}}{\ell_x}$$
$$= \frac{d_{x+k}}{\ell_x}, k = 0, 1, \ldots,$$

and

$$\mathbb{P}\{K_x = k\} = P\{k-1 < T(x) \leq k\} = \frac{\ell_{x+k-1} - \ell_{x+k}}{\ell_x}$$
$$= \frac{d_{x+k-1}}{\ell_x}, k = 1, 2, \ldots$$

Example 4

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Calculate d_x, $x = 80, 81, \dots, 86$.
(ii) Calculate the p.m.f. of the curtate life $K(80)$.
(iii) Calculate the expected curtate life e_{80}.

Example 4

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Calculate d_x, $x = 80, 81, \dots, 86$.
(ii) Calculate the p.m.f. of the curtate life $K(80)$.
(iii) Calculate the expected curtate life e_{80}.

Solution: (i) Using that $d_x = \ell_x - \ell_{x+1}$, we get that

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
d_x	33	56	54	45	34	28	0

Example 4

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Calculate d_x, $x = 80, 81, \dots, 86$.
(ii) Calculate the p.m.f. of the curtate life $K(80)$.
(iii) Calculate the expected curtate life e_{80}.

Solution: (ii) Using that $\mathbb{P}\{K(x) = k\} = \frac{d_{x+k}}{\ell_x}$ and

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
d_x	33	56	54	45	34	28	0

we get that

k	0	1	2	3	4	5
$\mathbb{P}\{K(80) = k\}$	$\frac{33}{250}$	$\frac{56}{250}$	$\frac{54}{250}$	$\frac{45}{250}$	$\frac{34}{250}$	$\frac{28}{250}$

Example 4

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Calculate d_x, $x = 80, 81, \dots, 86$.
(ii) Calculate the p.m.f. of the curtate life $K(80)$.
(iii) Calculate the expected curtate life e_{80}.

Solution: (iii) Using that

k	0	1	2	3	4	5
$\mathbb{P}\{K(80) = k\}$	$\frac{33}{250}$	$\frac{56}{250}$	$\frac{54}{250}$	$\frac{45}{250}$	$\frac{34}{250}$	$\frac{28}{250}$

we get that

$$e_{80} = E[K(80)]$$
$$=(0)\frac{33}{250} + (1)\frac{56}{250} + (2)\frac{54}{250} + (3)\frac{45}{250} + (4)\frac{34}{250} + (5)\frac{28}{250} = 2.3.$$

We know that

$$e_x = E[K(x)] = \sum_{k=1}^{\infty} {}_k p_x \quad \text{and} \quad E[(K(x))^2] = \sum_{k=1}^{\infty} (2k-1)\, {}_k p_x.$$

Using the number of living, we have that

$$e_x = \sum_{k=1}^{\infty} \frac{\ell_{x+k}}{\ell_x} \quad \text{and} \quad E[(K(x))^2] = \sum_{k=1}^{\infty} (2k-1) \frac{\ell_{x+k}}{\ell_x}. \qquad (1)$$

The expected whole years lived in the interval $(x, x+n]$ by an entity alive at age x is

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_k p_x.$$

Using the number of living, we have that

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} \frac{\ell_{x+k}}{\ell_x}. \qquad (2)$$

Example 5

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Using (1), *find* e_{80} *and* $\mathrm{Var}(K(80))$.
(ii) Using (2), *find* $e_{80:\overline{3}|}$.

Example 5

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Using (1), *find* e_{80} *and* $\mathrm{Var}(K(80))$.
(ii) Using (2), *find* $e_{80:\overline{3}|}$.

Solution: (i)

$$e_{80} = \sum_{k=1}^{\infty} \frac{\ell_{80+k}}{\ell_{80}} = \frac{217 + 161 + 107 + 62 + 28}{250} = 2.3,$$

$$E[(K(80))^2] = \sum_{k=1}^{\infty} (2k-1) \frac{\ell_{x+k}}{\ell_x}$$

$$=(1)\frac{217}{250} + (3)\frac{161}{250} + (5)\frac{107}{250} + (7)\frac{62}{250} + (9)\frac{28}{250} = 7.684,$$

$$\mathrm{Var}(K(80)) = 7.684 = (2.3)^2 = 2.394.$$

Example 5

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Using (1), *find* e_{80} *and* $\mathrm{Var}(K(80))$.
(ii) Using (2), *find* $e_{80:\overline{3}|}$.

(ii)

$$e_{80:\overline{3}|} = \sum_{k=1}^{3} \frac{\ell_{80+k}}{\ell_{80}} = \frac{217}{250} + \frac{161}{250} + \frac{107}{250} = \frac{217+161+107}{250} = 1.94.$$

