Manual for SOA Exam MLC.

Chapter 3. Life tables.

Section 3.3. Continuous computations.

Extract from:
"Arcones' Manual for SOA Exam MLC. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Continuous computations

Although, a life table does not show values of ℓ_x for non integers numbers, we will assume that ℓ_x is known for each $x \geq 0$. In the next section, we will discuss how to estimate ℓ_x for non integers. Knowing ℓ_x, for each $x \geq 0$, we can get

$$s(x) = \frac{\ell_x}{\ell_0},$$

$$\mu(x) = -\frac{d}{dx}(\log(\ell_x)) = -\frac{\ell_x'}{\ell_x},$$

$$\overset{\circ}{e}_0 = \int_0^\infty \frac{\ell_x}{\ell_0}\,dx,$$

$$\overset{\circ}{e}_x = \int_0^\infty \frac{\ell_{x+t}}{\ell_x}\,dt.$$

Definition 1

T_x is the **expected number of years lived beyond age x by the cohort group with l_0 members**.

Theorem 1

(i) $T_x = \int_0^\infty \ell_{x+t}\, dt$ and $\mathring{e}_x = E[T(x)] = \frac{T_x}{\ell_x}$.
(ii)

$$E[(T(x))^2] = \frac{2\int_x^\infty T_y\, dy}{\ell_x}.$$

Notice that the expected number of years lived beyond age x by an individual alive at age x is $\mathring{e}_x$. The expected number of individuals alive at age x is ℓ_x. Hence, $T_x = \ell_x \mathring{e}_x$.

Proof.

(i) We have that

$$T_x = \ell_0 E[(X-x)I(X>x)] = \ell_0 \mathbb{P}\{X>x\}E[X-x|X>x]$$
$$= \ell_x E[T(x)] = \ell_x \int_0^\infty {}_tp_x\,dt = \int_0^\infty \ell_{x+t}\,dt.$$

(ii) Using that $T_x = \int_0^\infty \ell_{x+t}\,dt = \int_x^\infty \ell_t\,dt$, we get that

$$2\int_x^\infty T_y\,dy = 2\int_x^\infty \int_y^\infty \ell_t\,dt\,dy = 2\int_x^\infty \int_x^t \ell_t\,dy\,dt$$
$$= 2\int_x^\infty (t-x)\ell_t\,dt = 2\int_0^\infty u\ell_{x+u}\,du.$$

So,

$$\frac{2\int_x^\infty T_y\,dy}{\ell_x} = \int_0^\infty 2u \cdot {}_up_x\,du = E[(T(x))^2].$$

Example 1

Suppose $\ell_x = \ell_0 \left(1 - \frac{x^2}{\omega^2}\right)$*, for* $0 \le x \le \omega$. *Find* $\overset{\circ}{e}_x$, $E[(T(x))^2]$ *and* $\mathrm{Var}(T(x))$ *using Theorem 1.*

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Suppose $\ell_x = \ell_0 \left(1 - \frac{x^2}{\omega^2}\right)$, *for* $0 \le x \le \omega$. *Find* $\overset{\circ}{e}_x$, $E[(T(x))^2]$ *and* $\mathrm{Var}(T(x))$ *using Theorem 1.*

Solution:

We have that

$$T_x = \int_0^\infty \ell_{x+t}\,dt = \int_0^{\omega-x} \ell_0 \left(1 - \frac{(x+t)^2}{\omega^2}\right) dt$$

$$= \ell_0 \left(\frac{3\omega^2 t - (x+t)^3}{3\omega^2}\right)\Bigg|_0^{\omega-x} = \ell_0 \left(\frac{3\omega^2(\omega - x) - \omega^3 + x^3}{3\omega^2}\right) =$$

$$= \ell_0 \left(\frac{2\omega^3 - 3\omega^2 x + x^3}{3\omega^2}\right), 0 \le x \le \omega$$

Hence,

$$\overset{\circ}{e}_x = \frac{T_x}{\ell_x} = \frac{2\omega^3 - 3\omega^2 x + x^3}{3(\omega^2 - x^2)} = \frac{2\omega^2 - \omega x - x^2}{3(\omega + x)} = \frac{(\omega - x)(2\omega + x)}{3(\omega + x)}.$$

Example 1

Suppose $\ell_x = \ell_0 \left(1 - \frac{x^2}{\omega^2}\right)$, *for* $0 \le x \le \omega$. *Find* $\overset{\circ}{e}_x$, $E[(T(x))^2]$ *and* $\mathrm{Var}(T(x))$ *using Theorem 1.*

Solution:

We also have that

$$\int_x^\infty T_y \, dy = \int_x^\omega \ell_0 \left(\frac{2\omega^3 - 3\omega^2 y + y^3}{3\omega^2}\right) dy$$

$$= \left(\frac{8\omega^3 y - 6\omega^2 y^2 + y^4}{12\omega^2}\right) \Big|_x^\omega$$

$$= \frac{8\omega^4 - 6\omega^4 + \omega^4 - 8\omega^3 x + 6\omega^2 x^2 - x^4}{12\omega^2}$$

$$= \frac{3\omega^4 - 8\omega^3 x + 6\omega^2 x^2 - x^4}{12\omega^2}.$$

Example 1

Suppose $\ell_x = \ell_0 \left(1 - \frac{x^2}{\omega^2}\right)$*, for* $0 \le x \le \omega$*. Find* $\mathring{e}_x$*,* $E[(T(x))^2]$ *and* $\mathrm{Var}(T(x))$ *using Theorem 1.*

Solution:

$$E[(T(x))^2] = \frac{2\int_x^\infty T_y \, dy}{\ell_x} = \frac{3\omega^4 - 8\omega^3 x + 6\omega^2 x^2 - x^4}{6(\omega^2 - x^2)}$$

$$= \frac{3\omega^3 - 5\omega^2 x + \omega x^2 + x^3}{6(\omega + x)} = \frac{(\omega - x)^2(3\omega + x)}{6(\omega + x)},$$

$$\mathrm{Var}(T(x)) = \frac{(\omega - x)^2(3\omega + x)}{6(\omega + x)} - \left(\frac{(\omega - x)(2\omega + x)}{3(\omega + x)}\right)^2$$

$$= \frac{(\omega - x)^2}{18(\omega + x)^2}\left(3(\omega + x)(3\omega + x) - 2(2\omega + x)^2\right)$$

$$= \frac{(\omega - x)^2}{18(\omega + x)^2}\left(\omega^2 + 4\omega x + x^2\right).$$

Definition 2

${}_nL_x$ *is the* **expected number of years lived between age x and age $x+n$ by the ℓ_x survivors at age x.**

Theorem 2

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{}_nL_x = T_x - T_{x+n} = \ell_x \mathring{e}_{x:\overline{n}|} = \int_0^n \ell_{x+t}\, dt.
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Proof: (i) Since the deceased at age x do not live between age x and age $x + n$, ${}_nL_x$ is the expected number of years lived between age x and age $x + n$ by the initial ℓ_0 lives. So,

$$ {}_nL_x = T_x - T_{x+n}. $$

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(ii) Since $\mathring{e}_{x:\overline{n}|}$ is the expected number of years lived between age x and age $x+n$ by a live aged x,

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$${}_nL_x = T_x - T_{x+n}.$$

(ii) Since $\mathring{e}_{x:\overline{n}|}$ is the expected number of years lived between age x and age $x+n$ by a live aged x,

$${}_nL_x = \ell_x \mathring{e}_{x:\overline{n}|}.$$

(iii) $\ell_x \mathring{e}_{x:\overline{n}|} = \ell_x \int_0^n {}_tp_x\,dt = \int_0^n \ell_{x+t}\,dt$.

We will abbreviate $L_x = {}_1L_x$, i.e. $L_x = \int_0^1 \ell_{x+t}\,dt$ is the expected number of years lived between age x and age $x+1$ by the ℓ_x survivors at age x. Previous equation display implies that

$$L_x = \int_0^1 \ell_{x+t}\,dt = T_x - T_{x+1}.$$

Corollary 1

(i) $T_x = \sum_{k=x}^{\infty} L_k$.

(ii) $\overset{\circ}{e}_x = \frac{\sum_{k=x}^{\infty} L_k}{\ell_x}$.

(iii) $\overset{\circ}{e}_{x:\overline{n}|} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x}$.

Corollary 1

(i) $T_x = \sum_{k=x}^{\infty} L_k$.
(ii) $\overset{\circ}{e}_x = \frac{\sum_{k=x}^{\infty} L_k}{\ell_x}$.
(iii) $\overset{\circ}{e}_{x:\overline{n}|} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x}$.

Proof:

(i)

$$T_x = \int_0^{\infty} \ell_{x+t}\, dt = \int_x^{\infty} \ell_t\, dt = \sum_{k=x}^{\infty} \int_k^{k+1} \ell_t\, dt = \sum_{k=x}^{\infty} \int_0^{1} \ell_{k+t}\, dt$$

$$= \sum_{k=x}^{\infty} L_k.$$

(ii) $\overset{\circ}{e}_x = \frac{T_x}{\ell_x} = \frac{\sum_{k=x}^{\infty} L_k}{\ell_x}$.
(iii)

$$\overset{\circ}{e}_x = \frac{\int_0^n \ell_{x+t}\, dt}{\ell_x} = \frac{\int_x^{x+n} \ell_t\, dt}{\ell_x} = \frac{\sum_{k=x}^{x+n-1} \int_k^{k+1} \ell_t\, dt}{\ell_x} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x}.$$

