Manual for SOA Exam MLC.

Chapter 3. Life tables.
Section 3.4. Continuous computations.

Extract from:
"Arcones' Manual for SOA Exam MLC. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Continuous computations

Although, a life table does not show values of ℓ_x for non integers numbers, we will assume that ℓ_x is known for each $x \geq 0$. In the next section, we will discuss how to estimate ℓ_x for non integers. Knowing ℓ_x, for each $x \geq 0$, we can get

$$s(x) = \frac{\ell_x}{\ell_0},$$

$$\mu(x) = -\frac{d}{dx}(\log(\ell_x)) = -\frac{\ell_x'}{\ell_x},$$

$$\mathring{e}_0 = \int_0^\infty \frac{\ell_x}{\ell_0}\,dx,$$

$$\mathring{e}_x = \int_0^\infty \frac{\ell_{x+t}}{\ell_x}\,dt.$$

Definition 1

T_x is the **expected number of years lived beyond age x by the cohort group with l_0 members**.

Theorem 1

(i) $T_x = \int_0^\infty \ell_{x+t}\,dt$ *and* $\overset{\circ}{e}_x = E[T(x)] = \frac{T_x}{\ell_x}$.
(ii)

$$E[(T(x))^2] = \frac{2\int_x^\infty T_y\,dy}{\ell_x}.$$

Notice that the expected number of years lived beyond age x by an individual alive at age x is $\overset{\circ}{e}_x$. The expected number of individuals alive at age x is ℓ_x. Hence, $T_x = \ell_x \overset{\circ}{e}_x$.

Proof.

(i) We have that

$$T_x = \ell_0 E[(X-x)I(X>x)] = \ell_0 \mathbb{P}\{X>x\}E[X-x|X>x]$$
$$=\ell_x E[T(x)] = \ell_x \int_0^\infty {}_tp_x\,dt = \int_0^\infty \ell_{x+t}\,dt.$$

(ii) Using that $T_x = \int_0^\infty \ell_{x+t}\,dt = \int_x^\infty \ell_t\,dt$, we get that

$$2\int_x^\infty T_y\,dy = 2\int_x^\infty \int_y^\infty \ell_t\,dt\,dy = 2\int_x^\infty \int_x^t \ell_t\,dy\,dt$$
$$=2\int_x^\infty (t-x)\ell_t\,dt = 2\int_0^\infty u\ell_{x+u}\,du.$$

So,

$$\frac{2\int_x^\infty T_y\,dy}{\ell_x} = \int_0^\infty 2u\cdot {}_up_x\,du = E[(T(x))^2].$$

Example 1

Suppose $\ell_x = \ell_0 \left(1 - \frac{x^2}{\omega^2}\right)$, *for* $0 \leq x \leq \omega$. *Find* $\overset{\circ}{e}_x$, $E[(T(x))^2]$ *and* $\mathrm{Var}(T(x))$ *using Theorem 1.*

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Solution:

We have that

$$
\begin{aligned}
T_x = &\int_0^\infty \ell_{x+t}\,dt = \int_0^{\omega-x} \ell_0\left(1 - \frac{(x+t)^2}{\omega^2}\right)\,dt \\
=&\ell_0\left(\frac{3\omega^2 t - (x+t)^3}{3\omega^2}\right)\Big|_0^{\omega-x} = \ell_0\left(\frac{3\omega^2(\omega - x) - \omega^3 + x^3}{3\omega^2}\right) = \\
=&\ell_0\left(\frac{2\omega^3 - 3\omega^2 x + x^3}{3\omega^2}\right), 0 \le x \le \omega
\end{aligned}
$$

Hence,

$$
\overset{\circ}{e}_x = \frac{T_x}{\ell_x} = \frac{2\omega^3 - 3\omega^2 x + x^3}{3(\omega^2 - x^2)} = \frac{2\omega^2 - \omega x - x^2}{3(\omega + x)} = \frac{(\omega - x)(2\omega + x)}{3(\omega + x)}.
$$

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Solution:

We also have that

$$\int_x^\infty T_y \, dy = \int_x^\omega \ell_0 \left(\frac{2\omega^3 - 3\omega^2 y + y^3}{3\omega^2} \right) dy$$

$$= \left(\frac{8\omega^3 y - 6\omega^2 y^2 + y^4}{12\omega^2} \right) \Bigg|_x^\omega$$

$$= \frac{8\omega^4 - 6\omega^4 + \omega^4 - 8\omega^3 x + 6\omega^2 x^2 - x^4}{12\omega^2}$$

$$= \frac{3\omega^4 - 8\omega^3 x + 6\omega^2 x^2 - x^4}{12\omega^2}.$$

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Suppose $\ell_x = \ell_0\left(1-\frac{x^2}{\omega^2}\right)$*, for* $0 \le x \le \omega$*. Find* $\overset{\circ}{e}_x$*,* $E[(T(x))^2]$ *and* $\mathrm{Var}(T(x))$ *using Theorem 1.*

Solution:

$$E[(T(x))^2] = \frac{2\int_x^\infty T_y\,dy}{\ell_x} = \frac{3\omega^4 - 8\omega^3 x + 6\omega^2 x^2 - x^4}{6(\omega^2 - x^2)}$$

$$= \frac{3\omega^3 - 5\omega^2 x + \omega x^2 + x^3}{6(\omega + x)} = \frac{(\omega - x)^2(3\omega + x)}{6(\omega + x)},$$

$$\mathrm{Var}(T(x)) = \frac{(\omega - x)^2(3\omega + x)}{6(\omega + x)} - \left(\frac{(\omega - x)(2\omega + x)}{3(\omega + x)}\right)^2$$

$$= \frac{(\omega - x)^2}{18(\omega + x)^2}\left(3(\omega + x)(3\omega + x) - 2(2\omega + x)^2\right)$$

$$= \frac{(\omega - x)^2}{18(\omega + x)^2}\left(\omega^2 + 4\omega x + x^2\right).$$

Definition 2

${}_nL_x$ *is the* **expected number of years lived between age** x **and age** $x+n$ **by the** ℓ_x **survivors at age** x.

Theorem 2

$${}_nL_x = T_x - T_{x+n} = \ell_x \mathring{e}_{x:\overline{n}|} = \int_0^n \ell_{x+t}\, dt.$$

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Proof: (i) Since the deceased at age x do not live between age x and age $x+n$, ${}_nL_x$ is the expected number of years lived between age x and age $x+n$ by the initial ℓ_0 lives. So,

$$ {}_nL_x = T_x - T_{x+n}. $$

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(ii) Since $\mathring{e}_{x:\overline{n}|}$ is the expected number of years lived between age x and age $x+n$ by a live aged x,

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$$ {}_nL_x = \ell_x \mathring{e}_{x:\overline{n}|}. $$

(iii) $\ell_x \mathring{e}_{x:\overline{n}|} = \ell_x \int_0^n {}_tp_x\,dt = \int_0^n \ell_{x+t}\,dt.$

We will abbreviate $L_x = {}_1L_x$, i.e. $L_x = \int_0^1 \ell_{x+t}\,dt$ is the expected number of years lived between age x and age $x+1$ by the ℓ_x survivors at age x. Previous equation display implies that

$$L_x = \int_0^1 \ell_{x+t}\,dt = T_x - T_{x+1}.$$

Corollary 1

(i) $T_x = \sum_{k=x}^{\infty} L_k$.

(ii) $\overset{\circ}{e}_x = \frac{\sum_{k=x}^{\infty} L_k}{\ell_x}$.

(iii) $\overset{\circ}{e}_{x:\overline{n}|} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x}$.

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(ii) $\overset{\circ}{e}_x = \frac{\sum_{k=x}^{\infty} L_k}{\ell_x}$.
(iii) $\overset{\circ}{e}_{x:\overline{n}|} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x}$.

Proof:

(i)

$$T_x = \int_0^\infty \ell_{x+t}\,dt = \int_x^\infty \ell_t\,dt = \sum_{k=x}^{\infty} \int_k^{k+1} \ell_t\,dt = \sum_{k=x}^{\infty} \int_0^1 \ell_{k+t}\,dt$$

$$= \sum_{k=x}^{\infty} L_k.$$

(ii) $\overset{\circ}{e}_x = \frac{T_x}{\ell_x} = \frac{\sum_{k=x}^{\infty} L_k}{\ell_x}$.

(iii)

$$\overset{\circ}{e}_x = \frac{\int_0^n \ell_{x+t}\,dt}{\ell_x} = \frac{\int_x^{x+n} \ell_t\,dt}{\ell_x} = \frac{\sum_{k=x}^{x+n-1} \int_k^{k+1} \ell_t\,dt}{\ell_x} = \frac{\sum_{k=x}^{x+n-1} L_k}{\ell_x}.$$

