Manual for SOA Exam MLC.

Chapter 3. Life tables.
Section 3.5. Interpolating life tables.

Extract from:
"Arcones' Manual for SOA Exam MLC. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Interpolating life tables.

Life tables only show the values of ℓ_x whenever x is a nonnegative integer. In many computations, we need to know ℓ_x for each $x \geq 0$. We can estimate these values ℓ_x in several ways.

Uniform distribution of deaths.

The simplest way is to assume a **uniform distribution of deaths**. That is, assume that between integer–valued years x and $x+1$ the death rate is constant. This implies that the graph of ℓ_{x+t}, $0 \le t \le 1$ is linear. Hence,

$$\ell_{x+t} = (1-t)\ell_x + t\ell_{x+1} = \ell_x + t(\ell_{x+1} - \ell_x) = \ell_x - t \cdot d_x, 0 \le t \le 1. \quad (1)$$

Theorem 1

Under a linear form for the number of living, for each nonnegative integer x and each $0 \le t \le 1$:

(i) ${}_tp_x = 1 - tq_x$.

(ii) ${}_tq_x = tq_x$, $0 \le t \le 1$.

(iii) $f_{T(x)}(t) = q_x$.

(iv) $\mu_{x+t} = \frac{q_x}{1-tq_x}$.

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Proof: (i) By (1),

$$ {}_tp_x = \frac{\ell_{x+t}}{\ell_x} = \frac{\ell_x - t \cdot d_x}{\ell_x} = 1 - t\frac{d_x}{\ell_x} = 1 - tq_x, \qquad 0 \le t \le 1. $$

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Proof: (ii)

$$ {}_tq_x = 1 - {}_tp_x = tq_x, \qquad 0 \le t \le 1. $$

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$$f_{T(x)}(t) = -\frac{d}{dt}\,{}_tp_x = q_x.$$

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Proof: (iv)

$$\mu_{x+t} = -\frac{d}{dt}\log {}_tp_x = -\frac{d}{dt}\log(1 - tq_x) = \frac{q_x}{1 - tq_x}.$$

Example 1

Using the life table in Section 4.1 and assuming a uniform distribution of deaths, find:
(i) ${}_{0.5}p_{35}$
(ii) ${}_{1.5}p_{35}$.

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Solution: (i) We have that $p_{35} = \frac{\ell_{36}}{\ell_{35}} = \frac{97126}{97250} = 0.9987249357$ and

$${}_{0.5}p_{35} = 1 - 0.5q_{35} = 1 - 0.5(1 - 0.9987249357) = 0.9993624678.$$

(ii) We have that $p_{36} = \frac{\ell_{37}}{\ell_{36}} = \frac{96993}{97126} = 0.9986306447$ and

$${}_{0.5}p_{36} = 1 - 0.5q_{36} = 1 - 0.5(1 - 0.9986306447) = 0.9993153224.$$

Hence,

$${}_{1.5}p_{35} = p_{35} \cdot {}_{0.5}p_{36} = (0.9987249357)(0.9993153224) = 0.9980411311.$$

Theorem 2

Given $t \geq 0$, let k be the nonnegative integer such that $k \leq t < k+1$. Under uniform interpolation,

(i) $s(t) = \frac{\ell_k}{\ell_0} - (t-k)\frac{\ell_x}{\ell_0}$.

(ii) $f_X(t) = \frac{d_k}{\ell_0} = {}_{k|}q_0$.

(iii) $f_{T(x)}(t) = \frac{d_{x+k}}{\ell_x} = {}_{k|}q_x$.

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(iii) $f_{T(x)}(t) = \frac{d_{x+k}}{\ell_x} = {}_{k|}q_x$.

Proof: (i) By (1), $\ell_t = \ell_{k+t-k} = \ell_k - (t-k)\cdot d_k$. Hence,
$s(t) = \frac{\ell_t}{\ell_0} = \frac{\ell_k}{\ell_0} - (t-k)\frac{d_k}{\ell_0}$.

Theorem 2

Given $t \geq 0$, let k be the nonnegative integer such that $k \leq t < k+1$. Under uniform interpolation,
(i) $s(t) = \frac{\ell_k}{\ell_0} - (t-k)\frac{\ell_x}{\ell_0}$.
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Proof: (i) By (1), $\ell_t = \ell_{k+t-k} = \ell_k - (t-k)\cdot d_k$. Hence,
$s(t) = \frac{\ell_t}{\ell_0} = \frac{\ell_k}{\ell_0} - (t-k)\frac{d_k}{\ell_0}$.
(ii) $f_X(t) = -\frac{d}{dt}s(t) = \frac{d_k}{\ell_0} = {}_{k|}q_0$.

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(ii) $f_X(t) = \frac{d_k}{\ell_0} = {}_k|q_0$.
(iii) $f_{T(x)}(t) = \frac{d_{x+k}}{\ell_x} = {}_k|q_x$.

Proof: (i) By (1), $\ell_t = \ell_{k+t-k} = \ell_k - (t-k) \cdot d_k$. Hence,
$s(t) = \frac{\ell_t}{\ell_0} = \frac{\ell_k}{\ell_0} - (t-k)\frac{d_k}{\ell_0}$.
(ii) $f_X(t) = -\frac{d}{dt}s(t) = \frac{d_k}{\ell_0} = {}_k|q_0$.
(iii) $f_{T(x)}(t) = \frac{f_X(x+t)}{s(x)} = \frac{d_{x+k}}{\ell_x} = {}_k|q_x$.

Example 2

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

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(i) Calculate d_x, $x = 81, 82, \ldots, 86$.

Example 2

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Calculate d_x, $x = 81, 82, \dots, 86$.

Solution: (i) Using that $d_x = \ell_x - \ell_{x+1}$, we get that

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
d_x	33	56	54	45	34	28	0

Example 2

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(ii) Using linear interpolation, calculate ℓ_{80+t}, $0 \leq t \leq 6$.

Example 2

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(ii) Using linear interpolation, calculate ℓ_{80+t}, $0 \le t \le 6$.

Solution: (ii) Using that $\ell_{x+t} = \ell_x - t \cdot d_x$, $0 \le t \le 1$,

$$\ell_{80+t} = \begin{cases} 250 - 33t & \text{if } 0 \le t \le 1, \\ 217 - 56(t-1) & \text{if } 1 \le t \le 2, \\ 161 - 54(t-2) & \text{if } 2 \le t \le 3, \\ 107 - 45(t-3) & \text{if } 3 \le t \le 4, \\ 62 - 34(t-4) & \text{if } 4 \le t \le 5, \\ 28 - 28(t-5) & \text{if } 5 \le t \le 6. \end{cases}$$

Example 2

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(iii) Using linear interpolation, calculate ${}_tp_{80}$, $0 \le t \le 6$.

Example 2

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(iii) Using linear interpolation, calculate ${}_tp_{80}$, $0 \le t \le 6$.

Solution: (iii) Using that ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$,

$$
{}_tp_{80} = \begin{cases} \frac{250-33t}{250} & \text{if } 0 \le t \le 1, \\ \frac{217-56(t-1)}{250} & \text{if } 1 \le t \le 2, \\ \frac{161-54(t-2)}{250} & \text{if } 2 \le t \le 3, \\ \frac{107-45(t-3)}{250} & \text{if } 3 \le t \le 4, \\ \frac{62-34(t-4)}{250} & \text{if } 4 \le t \le 5, \\ \frac{28-28(t-5)}{250} & \text{if } 5 \le t \le 6. \end{cases}
$$

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Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(iv) Using linear interpolation, calculate the density function of the future life T_{80}.

Example 2

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(iv) Using linear interpolation, calculate the density function of the future life T_{80}.

Solution: (iv) Using that $f_{T_{80}}(t) = -\frac{d({}_tp_x)}{dt}$,

$$f_{T_{80}}(t) = \begin{cases} \frac{33}{250} & \text{if } 0 \leq t \leq 1, \\ \frac{56}{250} & \text{if } 1 \leq t \leq 2, \\ \frac{54}{250} & \text{if } 2 \leq t \leq 3, \\ \frac{45}{250} & \text{if } 3 \leq t \leq 4, \\ \frac{34}{250} & \text{if } 4 \leq t \leq 5, \\ \frac{28}{250} & \text{if } 5 \leq t \leq 6. \end{cases}$$

Theorem 3

Under a linear form for the number of living,

(i) $L_x = \ell_x - \frac{d_x}{2} = \ell_{x+1} + \frac{d_x}{2} = \frac{\ell_x + \ell_{x+1}}{2}$.

(ii) $T_x = \frac{\ell_x}{2} + \sum_{k=x+1}^{\infty} \ell_k$.

(iii) $m_x = \frac{q_x}{1 - \frac{q_x}{2}}$.

(iv) $\overset{\circ}{e}_x = e_x + \frac{1}{2}$.

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Proof:

(i)

$$L_x = \int_0^1 \ell_{x+t}\, dt = \int_0^1 (\ell_x - t \cdot d_x)\, dt = \ell_x - \frac{d_x}{2} = \frac{\ell_x + \ell_{x+1}}{2}$$

$$= \ell_{x+1} + \frac{d_x}{2}.$$

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$$=\ell_{x+1} + \frac{d_x}{2}.$$

(ii) $T_x = \sum_{k=x}^{\infty} L_k = \sum_{k=x}^{\infty}\left(\frac{\ell_x+\ell_{x+1}}{2}\right) = \frac{\ell_x}{2} + \sum_{k=x+1}^{\infty} \ell_k$.

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(iii) $m_x = \frac{d_x}{L_x} = \frac{d_x}{\ell_x - \frac{d_x}{2}} = \frac{q_x}{1 - \frac{q_x}{2}}$.

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(ii) $T_x = \frac{\ell_x}{2} + \sum_{k=x+1}^{\infty} \ell_k$.
(iii) $m_x = \frac{q_x}{1 - \frac{q_x}{2}}$.
(iv) $\overset{\circ}{e}_x = e_x + \frac{1}{2}$.

Proof:

(i)

$$L_x = \int_0^1 \ell_{x+t}\, dt = \int_0^1 (\ell_x - t \cdot d_x)\, dt = \ell_x - \frac{d_x}{2} = \frac{\ell_x + \ell_{x+1}}{2}$$
$$= \ell_{x+1} + \frac{d_x}{2}.$$

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(iii) $m_x = \frac{d_x}{L_x} = \frac{d_x}{\ell_x - \frac{d_x}{2}} = \frac{q_x}{1 - \frac{q_x}{2}}$.

(iv) $\overset{\circ}{e}_x = \frac{T_x}{\ell_x} = \frac{1}{2} + \sum_{k=x+1}^{\infty} \frac{\ell_k}{\ell_x} = e_x + \frac{1}{2}$.

Example 3

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming linear interpolation,

(i) calculate the complete expected life at 80 using that $\overset{\circ}{e}_x = \int_0^\infty t f_{T_{80}}(t)dt$.

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(i) calculate the complete expected life at 80 using that $\overset{\circ}{e}_x = \int_0^\infty t f_{T_{80}}(t)dt$.

Solution: (i) By Example 2,

$$f_{T_{80}}(t) = \begin{cases} \frac{33}{250} & \text{if } 0 \le t \le 1, \\ \frac{56}{250} & \text{if } 1 \le t \le 2, \\ \frac{54}{250} & \text{if } 2 \le t \le 3, \\ \frac{45}{250} & \text{if } 3 \le t \le 4, \\ \frac{34}{250} & \text{if } 4 \le t \le 5, \\ \frac{28}{250} & \text{if } 5 \le t \le 6. \end{cases}$$

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(i) calculate the complete expected life at 80 using that $\overset{\circ}{e}_x = \int_0^\infty t f_{T_{80}}(t)dt$.

Solution: (i) So,

$$\overset{\circ}{e}_x = \int_0^\infty t f_{T_{80}}(t) = \int_0^1 t\frac{33}{250} + \int_1^2 t\frac{56}{250} + \int_2^3 t\frac{54}{250} + \int_3^4 t\frac{45}{250} + \int_4^5 t\frac{34}{250} + \int_4^6 t\frac{28}{250}$$

$$= \frac{1}{2}\frac{33}{250} + \frac{3}{2}\frac{56}{250} + \frac{5}{2}\frac{54}{250} + \frac{7}{2}\frac{45}{250} + \frac{9}{2}\frac{34}{250} + \frac{11}{2}\frac{28}{250}.$$

Example 3

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming linear interpolation,

(ii) calculate the complete expected life at 80 using that $\overset{\circ}{e}_x = \int_0^\infty {}_tp_x dt$.

Example 3

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming linear interpolation,

(ii) calculate the complete expected life at 80 using that $\overset{\circ}{e}_x = \int_0^\infty {}_tp_x\,dt$.

Solution: (ii) By Example 2,

$$
{}_tp_{80} = \begin{cases} \frac{250-33t}{250} & \text{if } 0 \le t \le 1, \\ \frac{217-56(t-1)}{250} & \text{if } 1 \le t \le 2, \\ \frac{161-54(t-2)}{250} & \text{if } 2 \le t \le 3, \\ \frac{107-45(t-3)}{250} & \text{if } 3 \le t \le 4, \\ \frac{62-34(t-4)}{250} & \text{if } 4 \le t \le 5, \\ \frac{28-28(t-5)}{250} & \text{if } 5 \le t \le 6. \end{cases}
$$

Example 3

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming linear interpolation,

(ii) calculate the complete expected life at 80 using that $\overset{\circ}{e}_x = \int_0^\infty {}_t p_x dt$.

Solution: (ii) So,

$$\overset{\circ}{e}_x = \int_0^\infty {}_t p_x dt = \int_0^1 \frac{250 - 33t}{250} dt + \int_1^2 \frac{217 - 56(t-1)}{250} dt + \int_2^3 \frac{161 - 54(t-2)}{250} dt + \int_3^4 \frac{107 - 45(t-3)}{250} dt + \int_4^5 \frac{62 - 34(t-4)}{250} dt + \int_5^6 \frac{28 - 28(t-5)}{250} dt$$

$=2.8$

Example 3

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming linear interpolation,

(iii) calculate the complete expected life at 80 using that $\overset{\circ}{e}_x = e_x + \frac{1}{2}$.

Example 3

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming linear interpolation,

(iii) calculate the complete expected life at 80 using that $\overset{\circ}{e}_x = e_x + \frac{1}{2}$.

Solution: (iii) We have that

$$e_{80} = \sum_{k=1}^{\infty} \frac{\ell_{80+k}}{\ell_{80}} = \frac{217 + 161 + 107 + 62 + 28}{250} = 2.3$$

So, $\overset{\circ}{e}_{80} = 2.3 + 0.5 = 2.8$.

Example 3

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming linear interpolation,

(iv) calculate $\overset{\circ}{e}_{80:\overline{3}|}$.

Example 3

Consider the life table

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*

Assuming linear interpolation,

(iv) calculate $\overset{\circ}{e}_{80:\overline{3}|}$.

Solution: (iv)

$$\overset{\circ}{e}_{80:\overline{3}|} = \frac{L_{80} + L_{81} + L_{82}}{\ell_{80}} = \frac{\frac{250+217}{2} + \frac{217+161}{2} + \frac{161+107}{2}}{250} = 2.226.$$

Theorem 4

For each x,
(i) $\mathbb{P}\{K(x) = k\} = \frac{d_{x+k}}{\ell_x}$, $k = 0, 1, 2 \dots$
(ii) S_x *has a uniform distribution on the interval* $(0, 1)$.
(iii) $K(x)$ *and* S_x *are independent r.v.'s.*

Proof: For each x, each $k \geq 0$ and each $0 \leq t < 1$,

$$\mathbb{P}\{K(x) = k, S_x \leq t\} = \mathbb{P}\{k < T(x) \leq k+t\} = \frac{\ell_{x+k} - \ell_{x+k+t}}{\ell_x} = \frac{td_{x+k}}{\ell_x}.$$

Letting $t \to 1$, we get that

$$\mathbb{P}\{K(x) = k\} = \frac{d_{x+k}}{l_x}.$$

We also have that

$$\mathbb{P}\{S_x \leq t\} = \sum_{k=0}^{\infty} \mathbb{P}\{K(x) = k, S_x \leq t\} = \sum_{k=0}^{\infty} \frac{td_{x+k}}{\ell_x} = t.$$

Hence, for each $k \geq 0$ and each $0 \leq t \leq 1$,

$$\mathbb{P}\{K(x) = k, S_x \leq t\} = \mathbb{P}\{K(x) = k\}\mathbb{P}\{S_x \leq t\}$$

which implies that $K(x)$ and S_x are independent r.v.'s.

Corollary 1

Under the assumption of uniform distribution of deaths:
(i) $\overset{\circ}{e}_x = e_x + \frac{1}{2}$.
(ii) $\mathrm{Var}(T(x)) = \mathrm{Var}(K(x)) + \frac{1}{12}$.

Corollary 1

Under the assumption of uniform distribution of deaths:
(i) $\overset{\circ}{e}_x = e_x + \frac{1}{2}$.
(ii) $\mathrm{Var}(T(x)) = \mathrm{Var}(K(x)) + \frac{1}{12}$.

Proof: (i) Since $T(x) = K(x) + S_x$,
$\overset{\circ}{e}_x = E[T(x)] = E[K(x)] + E[S_x] = e_x + \frac{1}{2}$.

Corollary 1

Under the assumption of uniform distribution of deaths:
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Proof: (i) Since $T(x) = K(x) + S_x$,
$\overset{\circ}{e}_x = E[T(x)] = E[K(x)] + E[S_x] = e_x + \frac{1}{2}$.
(ii) Since $T(x) = K(x) + S_x$ and $K(x)$ and S_x are independent,

$$\mathrm{Var}(T(x)) = \mathrm{Var}(K(x)) + \mathrm{Var}(S(x)) = \mathrm{Var}(K(x)) + \frac{1}{12}.$$

Exponential interpolation.

Under exponential interpolation, we assume that $\ell_{x+t} = ab^t$, for $0 \leq t \leq 1$, where a and b depend on x. Since $\ell_x = ab^0$ and $\ell_{x+1} = ab^1$, we get that $a = \ell_x$, $b = \frac{\ell_{x+1}}{\ell_x} = p_x$, and

$$\ell_{x+t} = ab^t = \ell_x p_x^t = \ell_x \left(\frac{\ell_{x+1}}{\ell_x}\right)^t = (\ell_x)^{1-t}(\ell_{x+1})^t. \qquad (2)$$

We will see that force of mortality is constant between x and $x+1$. The form obtained using exponential interpolation is also called the **constant force of mortality form of the number of living**.

Theorem 5

Under a exponential form for the number of living, for each nonnegative integer x and each $0 \leq t < 1$:

(i) ${}_tp_x = p_x^t$.

(ii) ${}_tq_x = 1 - (1 - q_x)^t$.

(iii) $f_{T_x}(t) = -p_x^t \log p_x$.

(iv) $\mu_{x+t} = -\log p_x$.

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Proof: (i) ${}_tp_x = \frac{\ell_{x+t}}{\ell_x} = p_x^t$.

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Under a exponential form for the number of living, for each nonnegative integer x and each $0 \leq t < 1$:

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Proof: (i) ${}_t p_x = \frac{\ell_{x+t}}{\ell_x} = p_x^t$.
(ii) ${}_t q_x = 1 - {}_t p_x = 1 - p_x^t = 1 - (1 - q_x)^t$.
(iii) $f_{T(x)}(t)(t) = -\frac{d}{dt} \log({}_t p_x) = -p_x^t \log p_x$.

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Example 4

Using the life table in Section 4.1 and exponential interpolation, find:

(i) ${}_{0.75}p_{80}$

(ii) ${}_{2.25}p_{80}$.

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Solution: (i) We have that

$$ {}_{0.75}p_{80} = p_{80}^{0.75} = \left(\frac{\ell_{81}}{\ell_{80}}\right)^{0.75} = \left(\frac{50987}{53925}\right)^{0.75} = 0.958852885. $$

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(ii) We have that

$${}_{2.25}p_{80} = p_{80}p_{81} \cdot {}_{0.25}p_{82} = \frac{\ell_{81}}{\ell_{80}}\frac{\ell_{82}}{\ell_{81}}\left(\frac{\ell_{83}}{l_{82}}\right)^{0.25}$$
$$=\frac{50987}{53925}\frac{47940}{50987}\left(\frac{44803}{47940}\right)^{0.25} = 0.8450154997.$$

Theorem 6

Given $t \geq 0$, let k be the nonnegative integer such that $k \leq t < k+1$. Under exponential interpolation:

(i) $s(t) = \frac{\ell_k}{\ell_0} p_k^{t-k}$.

(ii) $f_X(t) = \frac{\ell_k}{\ell_0} p_k^t(-\log p_k)$.

(iii) $f_{T(x)}(t)(t) = {}_k p_x \cdot p_{x+k}^t(-\log p_{x+k})$, $0 \leq t \leq 1$.

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(iii) $f_{T(x)}(t)(t) = {}_k p_x \cdot p_{x+k}^t(-\log p_{x+k})$, $0 \leq t \leq 1$.

Proof: (i) By (2), for each integer x and each $0 \leq t \leq 1$,

$$s(x+t) = \frac{\ell_{x+t}}{\ell_0} = \frac{\ell_x}{\ell_0}\left(\frac{\ell_{x+1}}{\ell_x}\right)^t = \frac{\ell_x}{\ell_0} p_x^t.$$

Hence, for $t \geq 0$ and $k \leq t < k+1$,

$$s(t) = s(k+t-k) = \frac{\ell_k}{\ell_0} p_k^{t-k}.$$

Theorem 6

Given $t \geq 0$, let k be the nonnegative integer such that $k \leq t < k+1$. Under exponential interpolation:
(i) $s(t) = \frac{\ell_k}{\ell_0} p_k^{t-k}$.
(ii) $f_X(t) = \frac{\ell_k}{\ell_0} p_k^t (-\log p_k)$.
(iii) $f_{T(x)}(t)(t) = {}_k p_x \cdot p_{x+k}^t (-\log p_{x+k})$, $0 \leq t \leq 1$.

Proof: (ii)

$$f_X(t) = -\frac{d}{dt} s(t) = \frac{\ell_k}{\ell_0} p_k^t (-\log p_k).$$

Theorem 6

Given $t \geq 0$, let k be the nonnegative integer such that $k \leq t < k+1$. Under exponential interpolation:
(i) $s(t) = \frac{\ell_k}{\ell_0} p_k^{t-k}$.
(ii) $f_X(t) = \frac{\ell_k}{\ell_0} p_k^t (-\log p_k)$.
(iii) $f_{T(x)}(t)(t) = {}_k p_x \cdot p_{x+k}^t (-\log p_{x+k})$, $0 \leq t \leq 1$.

Proof: (iii)

$$f_{T(x)}(t) = \frac{f_X(x+t)}{s(x)} = \frac{\frac{\ell_{x+k}}{\ell_0} p_{x+k}^t (-\log p_{x+k})}{\frac{\ell_x}{\ell_0}} = {}_k p_x \cdot p_{x+k}^t (-\log p_{x+k})$$

Example 5

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

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Consider the life table

x	80	81	82	83	84	85	86
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(i) Using exponential interpolation, calculate ℓ_{80+t}, $0 \leq t \leq 6$.

Example 5

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Using exponential interpolation, calculate ℓ_{80+t}, $0 \le t \le 6$.

Solution: (i) Using that $\ell_{x+t} = \ell_x \left(\frac{\ell_{x+1}}{\ell_x}\right)^t$, $0 \le t \le 1$,

$$\ell_{80+t} = \begin{cases} 250\left(\frac{217}{250}\right)^t & \text{if } 0 \le t \le 1, \\ 217\left(\frac{161}{217}\right)^{t-1} & \text{if } 1 \le t \le 2, \\ 161\left(\frac{107}{161}\right)^{t-2} & \text{if } 2 \le t \le 3, \\ 107\left(\frac{62}{107}\right)^{t-3} & \text{if } 3 \le t \le 4, \\ 62\left(\frac{28}{62}\right)^{t-4} & \text{if } 4 \le t \le 5, \\ 0 & \text{if } 5 < t \le 6. \end{cases}$$

Example 5

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(ii) Using exponential interpolation, calculate ${}_tp_{80}$, $0 \le t \le 6$.

Example 5

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(ii) Using exponential interpolation, calculate ${}_tp_{80}$, $0 \le t \le 6$.

Solution: (ii) Using that ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$,

$$
{}_tp_{80} = \begin{cases} \frac{250}{250}\left(\frac{217}{250}\right)^{t} & \text{if } 0 \le t \le 1, \\ \frac{217}{250}\left(\frac{161}{217}\right)^{t-1} & \text{if } 1 \le t \le 2, \\ \frac{161}{250}\left(\frac{107}{161}\right)^{t-2} & \text{if } 2 \le t \le 3, \\ \frac{107}{250}\left(\frac{62}{107}\right)^{t-3} & \text{if } 3 \le t \le 4, \\ \frac{62}{250}\left(\frac{28}{62}\right)^{t-4} & \text{if } 4 \le t \le 5, \\ 0 & \text{if } 5 < t \le 6. \end{cases}
$$

Example 5

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(iii) Using exponential interpolation, calculate $\mu(80+t)$, $0 \leq t \leq 6$.

Example 5

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(iii) Using exponential interpolation, calculate $\mu(80+t)$, $0 \le t \le 6$.

Solution: (iii) Using that $\mu(80+t) = -\frac{d(\log({}_tp_{80}))}{dt}$,

$$\mu(80+t) = \begin{cases} -\log\left(\frac{217}{250}\right) & \text{if } 0 \le t \le 1, \\ -\log\left(\frac{161}{217}\right) & \text{if } 1 \le t \le 2, \\ -\log\left(\frac{107}{161}\right) & \text{if } 2 \le t \le 3, \\ -\log\left(\frac{62}{107}\right) & \text{if } 3 \le t \le 4, \\ -\log\left(\frac{28}{62}\right) & \text{if } 4 \le t \le 5, \\ \infty & \text{if } 5 < t \le 6. \end{cases}$$

Example 5

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(iv) Using exponential interpolation, calculate the density function of the future life T_{80}.

Example 5

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(iv) Using exponential interpolation, calculate the density function of the future life T_{80}.

Solution: (iv) Using that $f_{T_{80}}(t) = -\frac{d({}_tp_{80})}{dt}$,

$$f_{T_{80}}(t) = \begin{cases} \frac{250}{250}\left(\frac{217}{250}\right)^{t}\left(-\log\left(\frac{217}{250}\right)\right) & \text{if } 0 \le t \le 1, \\ \frac{217}{250}\left(\frac{161}{217}\right)^{t-1}\left(-\log\left(\frac{161}{217}\right)\right) & \text{if } 1 \le t \le 2, \\ \frac{161}{250}\left(\frac{107}{161}\right)^{t-2}\left(-\log\left(\frac{107}{161}\right)\right) & \text{if } 2 \le t \le 3, \\ \frac{107}{250}\left(\frac{62}{107}\right)^{t-3}\left(-\log\left(\frac{62}{107}\right)\right) & \text{if } 3 \le t \le 4, \\ \frac{62}{250}\left(\frac{28}{62}\right)^{t-4}\left(-\log\left(\frac{28}{62}\right)\right) & \text{if } 4 \le t \le 5, \\ 0 & \text{if } 5 < t \le 6. \end{cases}$$

Theorem 7

Under an exponential form for ℓ_{x+t},

(i) $L_x = \frac{d_x}{-\log p_x}$.

(ii) $T_x = \sum_{k=x}^{\infty} \frac{d_k}{-\log p_k}$.

(iii) $m_x = -\log p_x$.

(iv) $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{d_k}{-\ell_x \log p_k}$.

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(iv) $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{d_k}{-\ell_x \log p_k}$.

Proof: (i)

$$L_x = \int_0^1 \ell_{x+t}\, dt = \int_0^1 \ell_x p_x^t\, dt = \frac{\ell_x p_x^t}{\log p_x}\Big|_0^1 = \frac{\ell_x(p_x - 1)}{\log p_x}$$

$$= \frac{\ell_x q_x}{-\log p_x} = \frac{d_x}{-\log p_x}.$$

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(iv) $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{d_k}{-\ell_x \log p_k}$.

Proof: (i)

$$L_x = \int_0^1 \ell_{x+t}\, dt = \int_0^1 \ell_x p_x^t\, dt = \frac{\ell_x p_x^t}{\log p_x}\Big|_0^1 = \frac{\ell_x (p_x - 1)}{\log p_x}$$
$$= \frac{\ell_x q_x}{-\log p_x} = \frac{d_x}{-\log p_x}.$$

(ii) $T_x = \sum_{k=x}^{\infty} L_x = \sum_{k=x}^{\infty} \frac{d_k}{-\log p_k}$.

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(i) $L_x = \frac{d_x}{-\log p_x}$.
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(iv) $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{d_k}{-\ell_x \log p_k}$.

Proof: (i)

$$L_x = \int_0^1 \ell_{x+t}\, dt = \int_0^1 \ell_x p_x^t\, dt = \frac{\ell_x p_x^t}{\log p_x} \Big|_0^1 = \frac{\ell_x(p_x - 1)}{\log p_x}$$

$$= \frac{\ell_x q_x}{-\log p_x} = \frac{d_x}{-\log p_x}.$$

(ii) $T_x = \sum_{k=x}^{\infty} L_x = \sum_{k=x}^{\infty} \frac{d_k}{-\log p_k}$.
(iii) $m_x = \frac{d_x}{L_x} = -\log p_x$.

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$$L_x = \int_0^1 \ell_{x+t}\,dt = \int_0^1 \ell_x p_x^t\,dt = \frac{\ell_x p_x^t}{\log p_x}\Big|_0^1 = \frac{\ell_x(p_x - 1)}{\log p_x}$$

$$= \frac{\ell_x q_x}{-\log p_x} = \frac{d_x}{-\log p_x}.$$

(ii) $T_x = \sum_{k=x}^{\infty} L_x = \sum_{k=x}^{\infty} \frac{d_k}{-\log p_k}$.

(iii) $m_x = \frac{d_x}{L_x} = -\log p_x$.

(iv) $\overset{\circ}{e}_x = \frac{T_x}{\ell_x} = \sum_{k=x}^{\infty} \frac{d_k}{-\ell_x \log p_k}$.

Example 6

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming exponential interpolation,

Example 6

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x	80	81	82	83	84	85	86
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Assuming exponential interpolation,

(i) calculate the complete expected life at 80 using that $\overset{\circ}{e}_x = \int_0^\infty {}_t p_x \, dt$.

Example 6

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming exponential interpolation,

(i) calculate the complete expected life at 80 using that $\overset{\circ}{e}_x = \int_0^\infty {}_tp_x\,dt$.

Solution: (i) By Example 5,

$$
{}_tp_{80} = \begin{cases}
\frac{250}{250}\left(\frac{217}{250}\right)^{t} & \text{if } 0 \le t \le 1, \\
\frac{217}{250}\left(\frac{161}{217}\right)^{t-1} & \text{if } 1 \le t \le 2, \\
\frac{161}{250}\left(\frac{107}{161}\right)^{t-2} & \text{if } 2 \le t \le 3, \\
\frac{107}{250}\left(\frac{62}{107}\right)^{t-3} & \text{if } 3 \le t \le 4, \\
\frac{62}{250}\left(\frac{28}{62}\right)^{t-4} & \text{if } 4 \le t \le 5, \\
0 & \text{if } 5 < t \le 6.
\end{cases}
$$

Example 6

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming exponential interpolation,

(i) calculate the complete expected life at 80 using that $\mathring{e}_x = \int_0^\infty {}_tp_x dt$.

Solution: (i) So,

$$\mathring{e}_x = \int_0^\infty {}_tp_x dt = \int_0^1 \frac{250}{250}\left(\frac{217}{250}\right)^t dt + \int_1^2 \frac{217}{250}\left(\frac{161}{217}\right)^{t-1} dt$$

$$+ \int_2^3 \frac{161}{250}\left(\frac{107}{161}\right)^{t-2} dt + \int_3^4 \frac{107}{250}\left(\frac{62}{107}\right)^{t-3} dt$$

$$+ \int_4^5 \frac{62}{250}\left(\frac{28}{62}\right)^{t-4} dt = 2.712484924.$$

Example 6

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming exponential interpolation,

(ii) calculate the complete expected life at 80 using that $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{d_k}{-\ell_x \log p_k}$.

Example 6

Consider the life table

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*

Assuming exponential interpolation,

(ii) calculate the complete expected life at 80 using that $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{d_k}{-\ell_x \log p_k}$.

Solution: (ii) Using that $d_x = \ell_x - \ell_{x+1}$ and $p_x = \frac{\ell_{x+1}}{\ell_x}$, we get that

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
d_x	33	56	54	45	34	28	0
p_x	$\frac{217}{250}$	$\frac{161}{217}$	$\frac{107}{161}$	$\frac{62}{107}$	$\frac{28}{62}$	0	0

Example 6

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming exponential interpolation,

(ii) calculate the complete expected life at 80 using that $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{d_k}{-\ell_x \log p_k}$.

Solution: (ii) Hence,

$$\overset{\circ}{e}_{80} = \sum_{k=80}^{\infty} \frac{d_k}{-\ell_x \log p_k}$$

$$= \frac{33}{250\left(-\log\left(\frac{217}{250}\right)\right)} + \frac{56}{250\left(-\log\left(\frac{161}{217}\right)\right)} + \frac{54}{250\left(-\log\left(\frac{107}{161}\right)\right)} + \frac{45}{250\left(-\log\left(\frac{62}{107}\right)\right)} + \frac{34}{250\left(-\log\left(\frac{28}{62}\right)\right)} = 2.712484924.$$

Example 6

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming exponential interpolation,

(iii) calculate $\overset{\circ}{e}_{80:\overline{3}|}$.

Example 6

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Assuming exponential interpolation,

(iii) calculate $\overset{\circ}{e}_{80:\overline{3}|}$.

Solution: (iii)

$$\overset{\circ}{e}_{80:\overline{3}|} = \sum_{k=80}^{82} \frac{d_k}{-\ell_{80} \log p_k}$$

$$= \frac{33}{250\left(-\log\left(\frac{217}{250}\right)\right)} + \frac{56}{250\left(-\log\left(\frac{161}{217}\right)\right)} + \frac{54}{250\left(-\log\left(\frac{107}{161}\right)\right)}$$

$$= 2.211545729.$$

Harmonic interpolation.

Under **harmonic interpolation**, we assume that $\ell_{x+t} = \frac{1}{a+tb}$, for $0 \leq t \leq 1$, where a and b depend on x. This implies that $\frac{1}{\ell_{x+t}}$ is a linear function. Hence,

$$\frac{1}{\ell_{x+t}} = (1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}$$

and

$$\ell_{x+t} = \frac{1}{(1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}}. \quad (3)$$

A function of the form $\frac{1}{a+bx}$ is called a hyperbolic function. Harmonic interpolation of the number of living is also called the **hyperbolic form of the number of living**. If the number of living follows harmonic interpolation, we say that it satisfies the **Balducci assumption**.

Theorem 8

Under the Balducci assumption for ℓ_{x+t},

(i) ${}_tp_x = \frac{p_x}{t+(1-t)p_x} = \frac{1-q_x}{1-(1-t)q_x}$, $0 \leq t \leq 1$.

(ii) ${}_tq_x = \frac{tq_x}{1-(1-t)q_x}$, $0 \leq t \leq 1$.

(iii) $\mu_{x+t} = \frac{1-p_x}{t+(1-t)p_x} = \frac{q_x}{1-(1-t)q_x}$.

(iv) $f_{T(x)}(t) = \frac{p_x(1-p_x)}{(t+(1-t)p_x)^2} = \frac{q_x(1-q_x)}{(1-(1-t)q_x)^2}$, $0 \leq t \leq 1$.

Theorem 8

Under the Balducci assumption for ℓ_{x+t}*,*

(i) ${}_tp_x = \frac{p_x}{t+(1-t)p_x} = \frac{1-q_x}{1-(1-t)q_x}$, $0 \le t \le 1$.

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(iii) $\mu_{x+t} = \frac{1-p_x}{t+(1-t)p_x} = \frac{q_x}{1-(1-t)q_x}$.

(iv) $f_{T(x)}(t) = \frac{p_x(1-p_x)}{(t+(1-t)p_x)^2} = \frac{q_x(1-q_x)}{(1-(1-t)q_x)^2}$, $0 \le t \le 1$.

Proof: (i) We have that

$$ {}_tp_x = \frac{\ell_{x+t}}{\ell_x} = \frac{1}{(1-t) + t\frac{\ell_x}{\ell_{x+1}}} = \frac{1}{(1-t) + t\frac{1}{p_x}} = \frac{p_x}{t+(1-t)p_x}. $$

and

$$ \frac{p_x}{t+(1-t)p_x} = \frac{1-q_x}{t+(1-t)(1-q_x)} = \frac{1-q_x}{1-(1-t)q_x}. $$

Theorem 8

Under the Balducci assumption for ℓ_{x+t},

(i) ${}_tp_x = \frac{p_x}{t+(1-t)p_x} = \frac{1-q_x}{1-(1-t)q_x}$, $0 \le t \le 1$.

(ii) ${}_tq_x = \frac{tq_x}{1-(1-t)q_x}$, $0 \le t \le 1$.

(iii) $\mu_{x+t} = \frac{1-p_x}{t+(1-t)p_x} = \frac{q_x}{1-(1-t)q_x}$.

(iv) $f_{T(x)}(t) = \frac{p_x(1-p_x)}{(t+(1-t)p_x)^2} = \frac{q_x(1-q_x)}{(1-(1-t)q_x)^2}$, $0 \le t \le 1$.

Proof: (ii)

$$ {}_tq_x = 1 - {}_tp_x = 1 - \frac{1-q_x}{1-(1-t)q_x} = \frac{tq_x}{1-(1-t)q_x}. $$

Theorem 8

Under the Balducci assumption for ℓ_{x+t}*,*

(i) ${}_tp_x = \frac{p_x}{t+(1-t)p_x} = \frac{1-q_x}{1-(1-t)q_x}$, $0 \le t \le 1$.

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(iii) $\mu_{x+t} = \frac{1-p_x}{t+(1-t)p_x} = \frac{q_x}{1-(1-t)q_x}$.

(iv) $f_{T(x)}(t) = \frac{p_x(1-p_x)}{(t+(1-t)p_x)^2} = \frac{q_x(1-q_x)}{(1-(1-t)q_x)^2}$, $0 \le t \le 1$.

Proof: (iii) We have that

$$\mu_{x+t} = -\frac{d}{dt}\log {}_tp_x = -\frac{d}{dt}\log \frac{p_x}{t+(1-t)p_x}$$

$$=\frac{d}{dt}\log(t+(1-t)p_x) = \frac{1-p_x}{t+(1-t)p_x},$$

$$\mu_{x+t} = -\frac{d}{dt}\log {}_tp_x = -\frac{d}{dt}\log \frac{1-q_x}{1-(1-t)q_x}$$

$$=\frac{d}{dt}\log(1-(1-t)q_x) = \frac{q_x}{1-(1-t)q_x}.$$

Theorem 8

Under the Balducci assumption for ℓ_{x+t},

(i) ${}_tp_x = \frac{p_x}{t+(1-t)p_x} = \frac{1-q_x}{1-(1-t)q_x}$, $0 \leq t \leq 1$.

(ii) ${}_tq_x = \frac{tq_x}{1-(1-t)q_x}$, $0 \leq t \leq 1$.

(iii) $\mu_{x+t} = \frac{1-p_x}{t+(1-t)p_x} = \frac{q_x}{1-(1-t)q_x}$.

(iv) $f_{T(x)}(t) = \frac{p_x(1-p_x)}{(t+(1-t)p_x)^2} = \frac{q_x(1-q_x)}{(1-(1-t)q_x)^2}$, $0 \leq t \leq 1$.

Proof: (iv) $f_{T(x)}(t) = {}_tp_x\mu_{x+t} = \frac{p_x(1-p_x)}{(t+(1-t)p_x)^2} = \frac{q_x(1-q_x)}{(1-(1-t)q_x)^2}$.

Theorem 9

Under the Balducci assumption, ${}_{1-t}q_{x+t} = (1-t)q_x$.

Theorem 9

Under the Balducci assumption, ${}_{1-t}q_{x+t} = (1-t)q_x$.

Proof: We have that

$$
\begin{aligned}
{}_{1-t}q_{x+t} &= \frac{s(x+t) - s(x+1)}{s(x+t)} = 1 - \frac{p_x}{{}_tp_x} \\
&= 1 - p_x \frac{t + (1-t)p_x}{p_x} = (1-t)q_x.
\end{aligned}
$$

Example 7

Using the life table in Section 4.1 and harmonic interpolation, find:
(i) ${}_{0.75}p_{80}$
(ii) ${}_{2.25}p_{80}$.

Example 7

Using the life table in Section 4.1 and harmonic interpolation, find:
(i) ${}_{0.75}p_{80}$
(ii) ${}_{2.25}p_{80}$.

Solution: (i) We have that

$$ {}_{0.75}p_{80} = \frac{\frac{\ell_{81}}{\ell_{80}}}{0.75 + (1-0.75)\frac{\ell_{81}}{\ell_{80}}} = \frac{\frac{50987}{53925}}{0.75 + (0.25)\frac{50987}{53925}} = 0.9585734295. $$

(ii) We have that

$$ {}_{2.25}p_{80} = p_{80}p_{81} \cdot {}_{0.25}p_{82} = \frac{\ell_{81}}{\ell_{80}}\frac{\ell_{82}}{\ell_{81}}\frac{\frac{\ell_{83}}{\ell_{82}}}{0.25 + (1-0.25)\frac{\ell_{83}}{\ell_{82}}} $$

$$ = \frac{50987}{53925}\frac{47940}{50987}\frac{\frac{44803}{47940}}{0.25 + (0.75)\frac{44803}{47940}} = 0.8737185905. $$

Theorem 10

Given $t \geq 0$, let k be the nonnegative integer such that $k \leq t < k+1$. Under the Balducci assumption

(i) $s(t) = \frac{\ell_k}{\ell_0} \frac{p_k}{1-(1-t+k)(1-p_k)}$.

(ii) $f_X(t) = \frac{\ell_k}{\ell_0} \frac{p_k(1-p_k)}{(1-(1-t+k)(1-p_k))^2}$.

(iii) $f_{T_x}(t) = {}_k p_x \cdot \frac{p_{x+k}(1-p_{x+k})}{(1-(1-t+k)(1-p_{x+k}))^2}$, $0 \leq t \leq 1$.

Theorem 10

Given $t \geq 0$, let k be the nonnegative integer such that $k \leq t < k+1$. Under the Balducci assumption

(i) $s(t) = \frac{\ell_k}{\ell_0}\frac{p_k}{1-(1-t+k)(1-p_k)}$.

(ii) $f_X(t) = \frac{\ell_k}{\ell_0}\frac{p_k(1-p_k)}{(1-(1-t+k)(1-p_k))^2}$.

(iii) $f_{T_x}(t) = {}_kp_x \cdot \frac{p_{x+k}(1-p_{x+k})}{(1-(1-t+k)(1-p_{x+k}))^2}$, $0 \leq t \leq 1$.

Proof: (i) For each integer x and each $0 \leq t \leq 1$,

$$s(x+t) = \frac{\ell_{x+t}}{\ell_0} = \frac{\ell_x}{\ell_0}\frac{1}{(1-t)+t\frac{\ell_x}{\ell_{x+1}}} = \frac{\ell_x}{\ell_0}\frac{p_x}{(1-t)p_x+t}$$

$$=\frac{\ell_x}{\ell_0}\frac{p_x}{1-(1-t)(1-p_x)}.$$

Hence, for $t \geq 0$ and $k \leq t < k+1$,

$$s(t) = s(k+t-k) = \frac{\ell_k}{\ell_0}\frac{p_k}{1-(1-t+k)(1-p_k)}.$$

Theorem 10

Given $t \geq 0$, let k be the nonnegative integer such that $k \leq t < k+1$. Under the Balducci assumption
(i) $s(t) = \frac{\ell_k}{\ell_0} \frac{p_k}{1-(1-t+k)(1-p_k)}$.
(ii) $f_X(t) = \frac{\ell_k}{\ell_0} \frac{p_k(1-p_k)}{(1-(1-t+k)(1-p_k))^2}$.
(iii) $f_{T_x}(t) = {}_kp_x \cdot \frac{p_{x+k}(1-p_{x+k})}{(1-(1-t+k)(1-p_{x+k}))^2}$, $0 \leq t \leq 1$.

Proof: (ii)

$$f_X(t) = -\frac{d}{dt}s(t) = \frac{\ell_k}{\ell_0} \frac{p_k(1-p_k)}{(1-(1-t+k)(1-p_k))^2}.$$

Theorem 10

Given $t \geq 0$, let k be the nonnegative integer such that $k \leq t < k+1$. Under the Balducci assumption

(i) $s(t) = \frac{\ell_k}{\ell_0} \frac{p_k}{1-(1-t+k)(1-p_k)}$.

(ii) $f_X(t) = \frac{\ell_k}{\ell_0} \frac{p_k(1-p_k)}{(1-(1-t+k)(1-p_k))^2}$.

(iii) $f_{T_x}(t) = {}_kp_x \cdot \frac{p_{x+k}(1-p_{x+k})}{(1-(1-t+k)(1-p_{x+k}))^2}$, $0 \leq t \leq 1$.

Proof: (iii)

$$f_{T_x}(t) = \frac{f_X(x+t)}{s(x)} = \frac{f_X(x+k+t-k)}{s(x)} = \frac{\frac{\ell_{x+k}}{\ell_0} \frac{p_{x+k}(1-p_{x+k})}{(1-(1-t+k)(1-p_{x+k}))^2}}{\frac{\ell_x}{\ell_0}}$$

$$= {}_kp_x \cdot \frac{p_{x+k}(1-p_{x+k})}{(1-(1-t+k)(1-p_{x+k}))^2}.$$

Example 8

Using harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Example 8

Using harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Calculate ℓ_{80+t}, $0 \leq t \leq 6$.

Example 8

Using harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Calculate ℓ_{80+t}, $0 \le t \le 6$.

Solution: (i) Using that $\ell_{x+t} = \ell_x \left(\frac{\ell_{x+1}}{\ell_x}\right)^t$, $0 \le t \le 1$,

$$\ell_{80+t} = \begin{cases} \frac{1}{(1-t)\frac{1}{250}+t\frac{1}{217}} & \text{if } 0 \le t \le 1, \\ \frac{1}{(1-(t-1))\frac{1}{217}+(t-1)\frac{1}{161}} & \text{if } 1 \le t \le 2, \\ \frac{1}{(1-(t-2))\frac{1}{161}+(t-2)\frac{1}{107}} & \text{if } 2 \le t \le 3, \\ \frac{1}{(1-(t-3))\frac{1}{107}+(t-3)\frac{1}{62}} & \text{if } 3 \le t \le 4, \\ \frac{1}{(1-(t-4))\frac{1}{62}+(t-4)\frac{1}{28}} & \text{if } 4 \le t \le 5, \\ 0 & \text{if } 5 < t \le 6. \end{cases}$$

Example 8

Using harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(ii) Calculate ${}_tp_{80}$, $0 \le t \le 6$.

Example 8

Using harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(ii) Calculate ${}_tp_{80}$, $0 \le t \le 6$.

Solution: (ii) Using that ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$,

$$
{}_tp_{80} = \begin{cases}
\frac{1}{(1-t)\frac{250}{250}+t\frac{250}{217}} & \text{if } 0 \le t \le 1, \\
\frac{1}{(1-(t-1))\frac{250}{217}+(t-1)\frac{250}{161}} & \text{if } 1 \le t \le 2, \\
\frac{1}{(1-(t-2))\frac{250}{161}+(t-2)\frac{250}{107}} & \text{if } 2 \le t \le 3, \\
\frac{1}{(1-(t-3))\frac{250}{107}+(t-3)\frac{250}{62}} & \text{if } 3 \le t \le 4, \\
\frac{1}{(1-(t-4))\frac{250}{62}+(t-4)\frac{250}{28}} & \text{if } 4 \le t \le 5, \\
0 & \text{if } 5 < t \le 6.
\end{cases}
$$

Example 8

Using harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(iii) Calculate the density function of T_{80}.

Example 8

Using harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(iii) Calculate the density function of T_{80}.

Solution: (iii) Using that $f_{T_{80}}(t) = -\frac{d({}_t p_x)}{dt}$,

$$f_{T_{80}}(t) = \begin{cases} \dfrac{\frac{250}{217} - \frac{250}{250}}{\left((1-t)\frac{250}{250} + t\frac{250}{217}\right)^2} & \text{if } 0 \le t \le 1, \\ \dfrac{\frac{250}{161} - \frac{250}{217}}{\left((1-(t-1))\frac{250}{217} + (t-1)\frac{250}{161}\right)^2} & \text{if } 1 \le t \le 2, \\ \dfrac{\frac{250}{107} - \frac{250}{161}}{\left((1-(t-2))\frac{250}{161} + (t-2)\frac{250}{107}\right)^2} & \text{if } 2 \le t \le 3, \\ \dfrac{\frac{250}{62} - \frac{250}{107}}{\left((1-(t-3))\frac{250}{107} + (t-3)\frac{250}{62}\right)^2} & \text{if } 3 \le t \le 4, \\ \dfrac{\frac{250}{28} - \frac{250}{62}}{\left((1-(t-4))\frac{250}{62} + (t-4)\frac{250}{28}\right)^2} & \text{if } 4 \le t \le 5, \\ 0 & \text{if } 5 \le t \le 6 \end{cases}$$

Theorem 11

Under the Balducci assumption for ℓ_{x+t},

(i) $L_x = \frac{-\ell_{x+1} \log p_x}{q_x}$.

(ii) $T_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \log p_k}{q_k}$.

(iii) $m_x = \frac{q_x^2}{-p_x \log p_x}$.

(iv) $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \log p_k}{\ell_x q_k}$.

Theorem 11

Under the Balducci assumption for ℓ_{x+t},

(i) $L_x = \frac{-\ell_{x+1}\log p_x}{q_x}$.

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(iii) $m_x = \frac{q_x^2}{-p_x \log p_x}$.

(iv) $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1}\log p_k}{\ell_x q_k}$.

Proof: (i)

$$L_x = \int_0^1 \ell_{x+t}\,dt = \int_0^1 \frac{1}{(1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}}\,dt$$

$$= \frac{\log\left((1-t)\frac{1}{\ell_x} + t\frac{1}{\ell_{x+1}}\right)}{\frac{1}{\ell_{x+1}} - \frac{1}{\ell_x}}\Bigg|_0^1 = \frac{\log\frac{1}{\ell_{x+1}} - \log\frac{1}{\ell_x}}{\frac{1}{\ell_{x+1}} - \frac{1}{\ell_x}}$$

$$= \frac{\ell_x\ell_{x+1}\log\frac{\ell_x}{\ell_{x+1}}}{\ell_x - \ell_{x+1}} = \frac{-\ell_{x+1}\log p_x}{q_x}.$$

Theorem 11

Under the Balducci assumption for ℓ_{x+t},

(i) $L_x = \frac{-\ell_{x+1} \log p_x}{q_x}$.

(ii) $T_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \log p_k}{q_k}$.

(iii) $m_x = \frac{q_x^2}{-p_x \log p_x}$.

(iv) $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \log p_k}{\ell_x q_k}$.

Proof: (ii) $T_x = \sum_{k=x}^{\infty} L_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \log p_k}{q_k}$.

Theorem 11

Under the Balducci assumption for ℓ_{x+t},

(i) $L_x = \frac{-\ell_{x+1} \log p_x}{q_x}$.

(ii) $T_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \log p_k}{q_k}$.

(iii) $m_x = \frac{q_x^2}{-p_x \log p_x}$.

(iv) $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \log p_k}{\ell_x q_k}$.

Proof: (iii) $m_x = \frac{d_x}{L_x} = \frac{d_x q_x}{-\ell_{x+1} \log p_x} = \frac{\ell_x d_x q_x}{-\ell_{x+1}\ell_x \log p_x} = \frac{q_x^2}{-p_x \log p_x}$.

Theorem 11

Under the Balducci assumption for ℓ_{x+t},

(i) $L_x = \frac{-\ell_{x+1} \log p_x}{q_x}$.

(ii) $T_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \log p_k}{q_k}$.

(iii) $m_x = \frac{q_x^2}{-p_x \log p_x}$.

(iv) $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \log p_k}{\ell_x q_k}$.

Proof: (iv) $\overset{\circ}{e}_x = \frac{T_x}{\ell_x} = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \log p_k}{\ell_x q_k}$.

Example 9

Use harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Example 9

Use harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Calculate $\mathring{e}_{80}$ using that $\mathring{e}_x = \int_0^\infty {}_tp_x dt$.

Example 9

Use harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Calculate $\overset{\circ}{e}_{80}$ using that $\overset{\circ}{e}_x = \int_0^\infty {}_tp_x dt$.

Solution: (i) By Example 5,

$$
{}_tp_{80} = \begin{cases}
\frac{1}{(1-t)\frac{250}{250}+t\frac{250}{217}} & \text{if } 0 \le t \le 1, \\
\frac{1}{(1-(t-1))\frac{250}{217}+(t-1)\frac{250}{161}} & \text{if } 1 \le t \le 2, \\
\frac{1}{(1-(t-2))\frac{250}{161}+(t-2)\frac{250}{107}} & \text{if } 2 \le t \le 3, \\
\frac{1}{(1-(t-3))\frac{250}{107}+(t-3)\frac{250}{62}} & \text{if } 3 \le t \le 4, \\
\frac{1}{(1-(t-4))\frac{250}{62}+(t-4)\frac{250}{28}} & \text{if } 4 \le t \le 5, \\
0 & \text{if } 5 < t \le 6.
\end{cases}
$$

Example 9

Use harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(i) Calculate $\overset{\circ}{e}_{80}$ using that $\overset{\circ}{e}_x = \int_0^\infty {}_tp_x dt$.

$$\overset{\circ}{e}_{80} = \int_0^1 \frac{1}{(1-t)\frac{250}{250} + t\frac{250}{217}}\, dt + \int_1^2 \frac{1}{(1-(t-1))\frac{250}{217} + (t-1)\frac{250}{161}}\, dt$$
$$+ \int_2^3 \frac{1}{(1-(t-2))\frac{250}{161} + (t-2)\frac{250}{107}}\, dt$$
$$+ \int_3^4 \frac{1}{(1-(t-3))\frac{250}{107} + (t-3)\frac{250}{62}}\, dt$$
$$+ \int_4^5 \frac{1}{(1-(t-4))\frac{250}{62} + (t-4)\frac{250}{28}}\, dt = 2.681292266.$$

Example 9

Use harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(ii) Calculate $\overset{\circ}{e}_{80}$ using that $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \log p_k}{\ell_x q_k}$.

Example 9

Use harmonic interpolation for the life table

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*

(ii) Calculate $\overset{\circ}{e}_{80}$ using that $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \log p_k}{\ell_x q_k}$.

Solution: (ii) Using that $d_x = \ell_x - \ell_{x+1}$, $p_x = \frac{\ell_{x+1}}{\ell_x}$ and $q_x = \frac{\ell_x - \ell_{x+1}}{\ell_x}$, we get that

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0
d_x	33	56	54	45	34	28	0
p_x	$\frac{217}{250}$	$\frac{161}{217}$	$\frac{107}{161}$	$\frac{62}{107}$	$\frac{28}{62}$	0	0
q_x	$\frac{33}{250}$	$\frac{56}{217}$	$\frac{54}{161}$	$\frac{45}{107}$	$\frac{34}{62}$	0	0

Example 9

Use harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(ii) Calculate $\overset{\circ}{e}_{80}$ using that $\overset{\circ}{e}_x = \sum_{k=x}^{\infty} \frac{-\ell_{k+1} \log p_k}{\ell_x q_k}$.

Solution: (ii)Hence,

$$\overset{\circ}{e}_{80} = \sum_{k=80}^{\infty} \frac{-\ell_{k+1} \log p_k}{\ell_{80} q_k}$$

$$= \frac{-(217) \log\left(\frac{217}{250}\right)}{(250)\left(\frac{33}{250}\right)} + \frac{-(161) \log\left(\frac{161}{217}\right)}{(250)\left(\frac{56}{217}\right)} + \frac{-(107) \log\left(\frac{107}{161}\right)}{(250)\left(\frac{54}{161}\right)}$$

$$+ \frac{-(62) \log\left(\frac{62}{107}\right)}{(250)\left(\frac{45}{107}\right)} + \frac{-(28) \log\left(\frac{28}{62}\right)}{(250)\left(\frac{34}{62}\right)} = 2.681292266.$$

Example 9

Use harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(iii) Calculate $\overset{\circ}{e}_{80:\overline{3}|}$.

Example 9

Use harmonic interpolation for the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

(iii) Calculate $\overset{\circ}{e}_{80:\overline{3}|}$.

Solution: (iii)

$$\overset{\circ}{e}_{80:\overline{3}|} = \sum_{k=80}^{82} \frac{-\ell_{k+1} \log p_k}{\ell_{80} q_k} = \frac{-(217)\log\left(\frac{217}{250}\right)}{(250)\left(\frac{33}{250}\right)} + \frac{-(161)\log\left(\frac{161}{217}\right)}{(250)\left(\frac{56}{217}\right)}$$

$$+ \frac{-(107)\log\left(\frac{107}{161}\right)}{(250)\left(\frac{54}{161}\right)}$$

$= 2.197149575.$

