Manual for SOA Exam MLC.

Chapter 4. Life Insurance.

Section 4.1. Introduction to life insurance.

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Level benefit insurance in the continuous case

In this chapter, we will consider a cashflow of contingent payments, i.e. the payments depend on uncertain events modeled as a random variable.

Definition 1

The **(APV) actuarial present value** *of a cashflow of payments is the expectation of its present value at the time of purchase of this cashflow.*

The expected present value is also called the **expected present value** and the **net single premium**.
The present value of a cashflow of payments can be random because many reasons. A possibility that payments are made only with a certain probability. A **contingent cashflow** is a cashflow whose payments are uncertain. Usually, we are able to estimate the probability that a contingent payment is made.

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Recall that i is the annual effective rate of interest, $v = (1 + i)^{-1}$ is the annual discount factor, $\delta = \ln(1 + i)$ is the force of interest (or continuously compounded annual rate of interest).

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Example 1

Consider the contingent cashflow

Payment	C_1	C_2	$\cdots$	C_m
Probability that payment is made	p_1	p_2	$\cdots$	p_m
Time (in years)	t_1	t_2	$\cdots$	t_m

Here, p_j, $1 \leq j \leq m$, *is the probability that* j*–th payment* C_j *is made. Compute the actuarial present value of this contingent cashflow.*

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Here, p_j, $1 \le j \le m$, is the probability that j–th payment C_j is made. Compute the actuarial present value of this contingent cashflow.

Solution: Let $\delta_j = \begin{cases} 1 & \text{if the } j-\text{th payment is made,} \\ 0 & \text{if the } j-\text{th payment is not made.} \end{cases}$

The present value random variable of this cashflow is $\sum_{j=1}^m C_j(1+i)^{-t_j}\delta_j$. The actuarial present value of this cashflow is

$$E\left[\sum_{j=1}^m C_j(1+i)^{-t_j}\delta_j\right] = \sum_{j=1}^m C_j(1+i)^{-t_j}p_j = \sum_{j=1}^m C_j v^{t_j} p_j.$$

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We consider an insurance policy on a certain entity. Let T be the age–at–death of this entity. Under this insurance policy, the policyholder receives a payment at a certain time in the future. Both the amount of the payment and the payment date depend on T. Let b_t be the benefit payment made when failure happens at time t. Let v_t be the discount factor when failure happens at time t. The present value of the benefit payment is denoted by

$$\overline{Z} = b_T v_T.$$

The actuarial present value of this benefit is

$$E[\overline{Z}] = \int_0^\infty b_t v_t f_T(t)\, dt.$$

The bar over X is to denote that the continuous r.v. T is used. When the entity in the insurance contract is (x), T is T_x and

$$E[\overline{Z}] = \int_0^\infty b_t v_t f_{T_x}(t)\, dt = \int_0^\infty b_t v_t \cdot {}_t p_x \mu_{x+t}\, dt.$$

If the benefit payment is made at the time of death, then $v_t = v^t$.

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Example 2

For a whole life insurance on (60)*, you are given:*
(i) Death benefits are paid at the moment of death.
(ii) Mortality follows the de Moivre model with terminal age 100.
(iii) $i = 7\%$.
(iv) $b_t = (20000)(1.04)^t$, $t \geq 0$.
Calculate the mean and the standard deviation of the present value random variable for this insurance.

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Calculate the mean and the standard deviation of the present value random variable for this insurance.

Solution: The present value random variable is

$$Z = b_{T_{60}} v^{T_{60}} = (20000)(1.04)^{T_{60}}(1.07)^{-T_{60}} = (20000)\left(\frac{1.04}{1.07}\right)^{T_{60}}.$$

The density of T_{60} is

$$f_{T_{60}}(t) = \frac{1}{40},\ 0 \leq t \leq 40.$$

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Hence,

$$E[Z] = \int_0^{40} (20000)\left(\frac{1.04}{1.07}\right)^t \frac{1}{40}\,dt = \frac{(20000)\left(\frac{1.04}{1.07}\right)^t}{40\ln(1.04/1.07)}\Bigg|_0^{40}$$

$$= \frac{(20000)(\left(\frac{1.04}{1.07}\right)^{40} - 1)}{40\ln(1.04/1.07)} = 11945.06573,$$

$$E[Z^2] = \int_0^{40} (20000)^2\left(\frac{1.04}{1.07}\right)^{2t} \frac{1}{40}\,dt = \frac{(20000)^2(\left(\frac{1.04}{1.07}\right)^{80} - 1)}{80\ln(1.04/1.07)}$$

$$= 157748208.7,$$

$$\mathrm{Var}(Z) = 157748208.7 - (11945.06573)^2 = 15063613.41,$$

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In some cases, these insurance products depend on the time interval of failure K. If b_t and v_t are constant functions in each interval $(k-1, k]$, then T and K are in the same interval $(k-1, k]$, $b_T = b_K$ and $v_T = v_K$. In this case the present value of the benefit payment is

$$Z = b_K v_K.$$

The actuarial present value of the benefit payment is

$$E[Z] = \sum_{k=1}^{\infty} b_k v_k \mathbb{P}\{K = k\} = \sum_{k=1}^{\infty} b_k v_k \mathbb{P}\{k-1 \le T < k\}.$$

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When the entity in the insurance contract is (x), K is K_x and

$$E[Z] = \sum_{k=1}^{\infty} b_k v_k \mathbb{P}\{K_x = k\} = \sum_{k=1}^{\infty} b_k v_k \cdot {}_{k-1|}q_x.$$

Recall that

$$\mathbb{P}\{K_x = k\} = \mathbb{P}\{k-1 \le T_x < k\} = \mathbb{P}\{k-1 \le X - x < k | X > x\}$$
$$= \frac{s(x+k-1) - s(x+k)}{s(x)} = {}_{k-1|}q_x = {}_{k-1}p_x \cdot q_{x+k-1} = {}_{k-1}p_x - {}_k p_x.$$

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Example 3

A four–year warranty in a digital television will pay \$$400(5-k)$ *if the television breaks during the* k*–th year,* $k = 1, \dots, 4$*. The payment will be paid at the end of the year. The effective annual discount rate is 4%. The survival function is* $s(x) = \frac{1000}{(x+10)^3}$*,* $x \geq 0$*. Find the actuarial present value of this warranty benefit.*

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Solution: The actuarial present value of the warranty benefit is

$$\sum_{k=1}^{4} 400(5-k)v^k(s(k-1)-s(k)) = 712.1391022.$$

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$$
\begin{aligned}
&\sum_{k=1}^{4} 400(5-k)v^k(s(k-1)-s(k))\\
=&(1600)(0.96)\left(\frac{1000}{(10)^3}-\frac{1000}{(1+10)^3}\right)\\
&+(1200)(0.96)^2\left(\frac{1000}{(1+10)^3}-\frac{1000}{(2+10)^3}\right)\\
&+(800)(0.96)^3\left(\frac{1000}{(2+10)^3}-\frac{1000}{(3+10)^3}\right)\\
&+(400)(0.96)^4\left(\frac{1000}{(3+10)^3}-\frac{1000}{(4+10)^3}\right)\\
=&381.98046582+190.89406461+87.43850706\\
&+30.82606472+21.00000000=712.1391022.
\end{aligned}
$$

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