Manual for SOA Exam MLC.

Chapter 4. Life Insurance.

Section 4.2. Payments paid at the end of the year of death.

Extract from:
"Arcones' Manual for the SOA Exam MLC. Fall 2009 Edition".
available at http://www.actexmadriver.com/

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Definition 1

The present value payment of a whole life insurance of a unit payment made at the end of the year of the death is denoted by Z_x.

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Notice that $Z_x = \nu^{K_x}$, where K_x is the time interval of death of (x), i.e. $K_x = k$ if the failure takes place in the interval $(k-1, k]$, where $k = 1, 2, \dots$. In other words, $K_x = k$, if $T_x \in (k-1, k]$.

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Definition 2

The actuarial present value payment of a whole life insurance of a unit payment made at the end of the year of the death is denoted by A_x, *i.e.*

$$A_x = E[Z_x] = E[\nu^{K_x}].$$

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The actuarial present value payment of a whole life insurance of a unit payment made at the end of the year of the death is denoted by A_x, i.e.

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The actuarial present value payment of a whole life insurance with payment C is CA_x.

In general this type of insurance applies when a payment is made on the first policy anniversary following failure of a certain entity. Let K be the interval of failure of this entity. The present value of the benefit payment is $Z = \nu^K$.

We have that

$$A_x = E[Z_x] = \sum_{k=1}^{\infty} \nu^k \mathbb{P}\{K_x = k\},$$

and

$$E[Z_x^2] = \sum_{k=1}^{\infty} \nu^{2k} \mathbb{P}\{K_x = k\}.$$

Letting $\nu' = \nu^2$, we have that

$$E[Z_x^2] = \sum_{k=1}^{\infty} (\nu')^k \mathbb{P}\{K_x = k\}.$$

We denote by ${}^2A_x$ the value of A_x when twice the force of interest is used. This means that ${}^2A_x$ is A_x when the discount factor is changed from ν to ν^2 (or the interest rate i is changed from i to $(1+i)^2 - 1$). It is easy to see that ${}^2A_x = E[Z_x^2]$. Hence,

$$\mathrm{Var}(Z_x) = {}^2A_x - A_x^2.$$

Since

$$\mathbb{P}\{K_x = k\} = \mathbb{P}\{k-1 < T_x \le k\} = \mathbb{P}\{k-1 < X - x \le k | X > x\}$$
$$= \frac{s(x+k-1) - s(x+k)}{s(x)} = {}_{k-1}|q_x = {}_{k-1}p_x \cdot q_{x+k-1} = {}_{k-1}p_x - {}_{k}p_x.$$

we have that:

Theorem 1

$$A_x = \sum_{k=1}^{\infty} \nu^k \mathbb{P}\{K_x = k\} = \sum_{k=1}^{\infty} \nu^k \cdot {}_{k-1}|q_x.$$

$${}^2A_x = \sum_{k=1}^{\infty} \nu^{2k} \mathbb{P}\{K_x = k\} = \sum_{k=1}^{\infty} \nu^{2k} \cdot {}_{k-1}|q_x.$$

and

$$\mathrm{Var}(Z_x) = {}^2A_x - A_x^2.$$

Example 1

Jane is 30 years old. She buys a whole life policy insurance which will pay \$20000 at the end of the year of her death. Suppose that $p_x = 0.9$*, for each* $x \geq 0$*, and* $i = 5\%$*. Find the actuarial present value of this life insurance.*

Example 1

Jane is 30 years old. She buys a whole life policy insurance which will pay \$20000 at the end of the year of her death. Suppose that $p_x = 0.9$, *for each* $x \geq 0$, *and* $i = 5\%$. *Find the actuarial present value of this life insurance.*

Solution: The actuarial present value is

$$(20000)A_{30} = (20000)\sum_{k=1}^{\infty} \nu^k {}_{k-1}p_{30} \cdot q_{30+k-1}$$

$$=(20000)\sum_{k=1}^{\infty}(1.05)^{-k}(0.9)^{k-1}(1-0.9)$$

$$=(20000)(1.05)^{-1}(1-0.9)\sum_{k=1}^{\infty}(1.05)^{-(k-1)}(0.9)^{k-1}$$

$$=\frac{(20000)(1.05)^{-1}(1-0.9)}{1-(1.05)^{-1}(0.9)} = 13333.33333.$$

Example 2

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a whole life policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$.

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Consider the life table

x	80	81	82	83	84	85	86
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An 80–year old buys a whole life policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$*.*

(i) Find the actuarial present value of this life insurance.

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An 80–year old buys a whole life policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$.

(i) Find the actuarial present value of this life insurance.

Solution: (i) The actuarial present value of this life insurance is

$$(50000)A_{80} = (50000)\sum_{k=1}^{\infty} \nu^k \frac{\ell_{80+k-1} - \ell_{80+k}}{\ell_{80}}$$

$$=(50000)(1.065)^{-1}\frac{250-217}{250} + (50000)(1.065)^{-2}\frac{217-161}{250}$$

$$+ (50000)(1.065)^{-3}\frac{161-107}{250} + (50000)(1.065)^{-4}\frac{107-62}{250}$$

$$+ (50000)(1.065)^{-5}\frac{62-28}{250} + (50000)(1.065)^{-6}\frac{28-0}{250}$$

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An 80–year old buys a whole life policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$.

(i) Find the actuarial present value of this life insurance.

$$
\begin{aligned}
=&(50000)(1.065)^{-1}\frac{250-217}{250} + (50000)(1.065)^{-2}\frac{217-161}{250}\\
&+ (50000)(1.065)^{-3}\frac{161-107}{250} + (50000)(1.065)^{-4}\frac{107-62}{250}\\
&+ (50000)(1.065)^{-5}\frac{62-28}{250} + (50000)(1.065)^{-6}\frac{28-0}{250}\\
=&6197.183099 + 9874.583967 + 8940.770191 + 6995.907818\\
&+ 4963.189688 + 3837.871065 = 40809.50583.
\end{aligned}
$$

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x	80	81	82	83	84	85	86
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An 80–year old buys a whole life policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$*.*
(ii) Find the probability that the APV of this life insurance is adequate to cover this insurance.

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Consider the life table

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
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An 80–year old buys a whole life policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$*.*
(ii) Find the probability that the APV of this life insurance is adequate to cover this insurance.

Solution: (ii) We have that 40809.50583 is adequate if $40809.50583 > (50000)(1.065)^{-K_x}$, which is equivalent to $K_x > \frac{-\ln(40809.50583/50000)}{\ln(1.065)} = 3.225226088$. The probability that the APV is adequate is

$$\mathbb{P}\{K_x \geq 4\} = \mathbb{P}\{T_x > 3\} = \frac{107}{250} = 0.428.$$

Theorem 2

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$$A_x = \frac{a_{\overline{\omega - x}|i}}{\omega - x}.$$

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Proof: The density of T_x is $f_{T_x}(t) = \frac{1}{\omega-x}$, if $0 \le t \le \omega - x$. Hence,

$$\mathbb{P}\{K_x = k\} = \mathbb{P}\{k-1 < T_x \le k\} = \int_{k-1}^{k} \frac{1}{\omega - x}\, dt = \frac{1}{\omega - x}$$

and

$$A_x = \sum_{k=1}^{\infty} \nu^k \mathbb{P}\{K_x = k\} = \sum_{k=1}^{\omega-x} \nu^k \frac{1}{\omega - x} = \frac{a_{\overline{\omega-x}|i}}{\omega - x}.$$

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and

$$A_x = \sum_{k=1}^{\infty} \nu^k \mathbb{P}\{K_x = k\} = \sum_{k=1}^{\omega - x} \nu^k \frac{1}{\omega - x} = \frac{a_{\overline{\omega-x}|i}}{\omega - x}.$$

It follows from the previous theorem that

$${}^{m}A_x = \frac{a_{\overline{\omega-x}|(1+i)^m - 1}}{\omega - x}.$$

Example 3

Rose is 40 years old. She buys a whole life policy insurance which will pay \$200,000 at the end of the year of her death. Suppose that the de Moivre model holds with terminal age 120*. Find the mean and the standard deviation of the present value of this life insurance under the annual effective rate of interest of* $i = 10\%$.

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Rose is 40 years old. She buys a whole life policy insurance which will pay $200,000 at the end of the year of her death. Suppose that the de Moivre model holds with terminal age 120*. Find the mean and the standard deviation of the present value of this life insurance under the annual effective rate of interest of* $i = 10\%$.

Solution: The actuarial present value of this life insurance is

$$(200000)A_{40} = (200000)\frac{a_{\overline{80}|0.1}}{80} = 24987.79535.$$

The second moment of the present value of this life insurance is

$$(200000)^2 \cdot {}^2A_{40} = (200000)^2 \frac{a_{\overline{80}|(1.1)^2-1}}{80} = 2380951814.$$

The variance of the present value of this life insurance is

$$\mathrm{Var}(Z_x) = 2380951814 - (24987.79535)^2 = 1756561898.$$

The standard deviation of the present value of this life insurance is

$$\sqrt{1756561898} = 41911.35763.$$

Theorem 3

If the probability function of time interval of failure is is

$$\mathbb{P}\{K = k\} = p^{k-1}(1-p), k = 1, 2, \dots$$

then

$$A = E[Z] = \frac{q}{q+i},$$

where $q = 1 - p$.

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then

$$A = E[Z] = \frac{q}{q+i},$$

where $q = 1 - p$.

Proof: We have that

$$A = \sum_{k=1}^{\infty}(1+i)^{-k}\mathbb{P}\{K = k\} = \sum_{k=1}^{\infty}(1+i)^{-k}p^{k-1}(1-p)$$

$$= \frac{(1-p)(1+i)^{-1}}{1-(1+i)^{-1}p} = \frac{1-p}{1+i-p} = \frac{1-p}{1+i-p} = \frac{q}{q+i}.$$

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By the previous theorem,

$${}^{m}A = \frac{q}{q+(1+i)^m - 1}.$$

Example 4

A benefit of \$500 is paid at the end of the year of failure, of a home electronic product. Let K be the end of the year of failure. Suppose that $\mathbb{P}\{K = k\} = \frac{(0.95)^k}{19}$, $k = 1, 2 \dots$. The annual effective interest rate is $i = 6\%$. Calculate:

(i) The actuarial present value of this benefit.

(ii) The standard deviation of the present value of this benefit.

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A benefit of \$500 is paid at the end of the year of failure, of a home electronic product. Let K be the end of the year of failure. Suppose that $\mathbb{P}\{K = k\} = \frac{(0.95)^k}{19}$, $k = 1, 2 \dots$. The annual effective interest rate is $i = 6\%$. Calculate:

(i) The actuarial present value of this benefit.

(ii) The standard deviation of the present value of this benefit.

Solution: (i) For this distribution, $\mathbb{P}\{K = k\} = p^{k-1}(1 - p)$ holds with $p = 0.95$. Let $Z = (500)\nu^K$ be the present value of this benefit. We have that

$$E[Z] = (500)\frac{q}{q + i} = (500)\frac{0.05}{0.05 + 0.06} = 227.2727273.$$

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A benefit of \$500 is paid at the end of the year of failure, of a home electronic product. Let K be the end of the year of failure. Suppose that $\mathbb{P}\{K=k\} = \frac{(0.95)^k}{19}$, $k = 1, 2 \ldots$. The annual effective interest rate is $i = 6\%$. Calculate:

(i) The actuarial present value of this benefit.

(ii) The standard deviation of the present value of this benefit.

Solution: (ii) We have that

$$E[Z^2] = (500)^2 \frac{q}{q + (1+i)^2 - 1} = (500)^2 \frac{0.05}{0.05 + (1.06)^2 - 1} = 72004.60$$

$$\mathrm{Var}(Z) = 72004.60829 - (227.2727273)^2 = 20351.71572.$$

The standard deviation is $\sqrt{20351.71572} = 142.6594396$.

Corollary 1

Under a constant force of mortality μ,

$$A_x = \frac{q_x}{i + q_x} = \frac{e^{\mu} - 1}{e^{\delta+\mu} - 1}.$$

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Proof: Under a constant force of mortality μ,

$$\mathbb{P}\{K_x = k\} = p_x^{k-1}(1 - p_x), k = 1, 2, \dots$$

Hence, Theorem 3 applies and

$$A_x = \frac{1 - p_x}{1 + i - p_x} = \frac{q_x}{i + q_x}.$$

Using that $q_x = 1 - e^{-\mu}$, we get that

$$\frac{q_x}{i + q_x} = \frac{1 - e^{-\mu}}{e^{\delta} - 1 + 1 - e^{-\mu}} = \frac{e^{\mu} - 1}{e^{\delta+\mu} - 1}.$$

Example 5

Mariah is 40 years old. She buys a whole life policy insurance which will pay $150,000 at the end of the year of her death. Suppose that the force of mortality is $\mu = 0.01$ *and* $\delta = 0.07$*. Find the mean and the standard deviation of the present value random variable of this life insurance.*

Example 5

Mariah is 40 years old. She buys a whole life policy insurance which will pay \$150,000 at the end of the year of her death. Suppose that the force of mortality is $\mu = 0.01$ and $\delta = 0.07$. Find the mean and the standard deviation of the present value random variable of this life insurance.

Solution: The actuarial present value is

$$(150000)A_{40} = (150000)\frac{q_x}{q_x + i} = \frac{(150000)(1 - e^{-0.01})}{1 - e^{-0.01} + e^{0.07} - 1} = 18100.34985.$$

We have that

$$(150000)^2 \cdot {}^2A_{40} = (150000)^2 \frac{q_x}{q_x + (1+i)^2 - 1}$$

$$= \frac{(150000)(1 - e^{-0.01})}{1 - e^{-0.01} + e^{(2)(0.07)} - 1} = 1397286233,$$

$$\mathrm{Var}((150000)Z_{40}) = 1397286233 - (18100.35)^2 = 1069663568,$$

$$\sqrt{\mathrm{Var}((150000)Z_{40})} = \sqrt{1069663568} = 32705.71155.$$

Theorem 4

For each $x > 0$,

$$A_x = \nu q_x + \nu p_x A_{x+1}$$

and

$$^2A_x = \nu^2 q_x + \nu^2 p_x \cdot {}^2A_{x+1}.$$

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$${}^2A_x = \nu^2 q_x + \nu^2 p_x \cdot {}^2A_{x+1}.$$

Proof: By the change of variables $k = j + 1$, and using ${}_jp_x = p_x \cdot {}_{j-1}p_{x+1}$,

$$\begin{aligned}A_x &= \sum_{k=1}^{\infty} \nu^k {}_{k-1}p_x \cdot q_{x+k-1} = \nu q_x + \sum_{k=2}^{\infty} \nu^k {}_{k-1}p_x \cdot q_{x+k-1} \\ &= \nu q_x + \sum_{j=1}^{\infty} \nu^{j+1} {}_jp_x \cdot q_{x+j} = \nu q_x + \nu p_x \sum_{j=1}^{\infty} \nu^j \cdot {}_{j-1}p_{x+1} \cdot q_{x+j} \\ &= \nu q_x + \nu p_x A_{x+1}.\end{aligned}$$

Example 6

Jess and Jane buy a whole life policy insurance on the day of their birthdays. Both policies will pay $50000 at the end of the year of death. Jess is 45 years old and the net single premium of her insurance is $25000. Jane is 44 years old and the net single premium of her insurance is $23702. Suppose that $i = 0.06$. *Find the probability that a 44–year old will die within one year.*

Example 6

Jess and Jane buy a whole life policy insurance on the day of their birthdays. Both policies will pay \$50000 at the end of the year of death. Jess is 45 years old and the net single premium of her insurance is \$25000. Jane is 44 years old and the net single premium of her insurance is \$23702. Suppose that $i = 0.06$. *Find the probability that a 44–year old will die within one year.*

Solution: We know that $A_{44} = 23702$, $A_{45} = 25000$ and $i = 0.06$. We need to find q_{44}. Using that

$$23702 = (50000)A_{44} = (50000)\nu q_{44} + (50000)\nu p_{44} A_{45}$$
$$=(50000)(1.06)^{-1}(1 - p_{44}) + (1.06)^{-1} p_{44}(25000),$$

we get that

$$p_{44} = \frac{(50000)(1.06)^{-1} - 23702}{(50000)(1.06)^{-1} - (1.06)^{-1}(25000)} = \frac{50000 - 23702(1.06)}{50000 - 25000} = 0.9950352$$

and $q_{44} = 0.0049648$.

n–year term life insurance.

Definition 3

A life insurance policy is called an n–**th term life insurance policy** *if it pays a face value if the insured dies within n years of the issue of the policy.*

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Definition 4

The present value of an n–th term life insurance policy which pays a unit face value at the end of the year of the death is denoted by $Z^{1}_{x:\overline{n}|}$.

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The present value of an n–th term life insurance policy which pays a unit face value at the end of the year of the death is denoted by $Z^1_{x:\overline{n}|}$.

We have that

$$Z^1_{x:\overline{n}|} = \nu^{K_x} I(K_x \le n) = \begin{cases} \nu^{K_x} & \text{if } K_x \le n, \\ 0 & \text{if } n < K_x. \end{cases}$$

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The present value of an n–th term life insurance policy which pays a unit face value at the end of the year of the death is denoted by $Z^1_{x:\overline{n}|}$.

We have that

$$Z^1_{x:\overline{n}|} = \nu^{K_x} I(K_x \leq n) = \begin{cases} \nu^{K_x} & \text{if } K_x \leq n, \\ 0 & \text{if } n < K_x. \end{cases}$$

In $Z^1_{x:\overline{n}|}$, the upper index ^{1} means that the future lifetime of x must fail before n years.

Definition 5

The actuarial present value of an n–th term life insurance policy which pays a unit face value at the end of the year of the death is $A^1_{x:\overline{n}|}$.

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Theorem 5

$$A^1_{x:\overline{n}|} = E[Z^1_{x:\overline{n}|}] = \sum_{k=1}^{n} \nu^k \mathbb{P}\{K_x = k\} = \sum_{k=1}^{n} \nu^k \cdot {}_{k-1}|q_x,$$

$${}^2A^1_{x:\overline{n}|} = E[{Z^1_{x:\overline{n}|}}^2] = \sum_{k=1}^{n} \nu^{2k} \mathbb{P}\{K_x = k\} = \sum_{k=1}^{n} \nu^{2k} \cdot {}_{k-1}|q_x.$$

Hence, $\mathrm{Var}(Z^1_{x:\overline{n}|}) = {}^2A^1_{x:\overline{n}|} - {A^1_{x:\overline{n}|}}^2$.

Example 7

Suppose that $i = 0.05$, $q_x = 0.05$ *and* $q_{x+1} = 0.02$. *Find* $A^{1}_{x:\overline{2}|}$ *and* $\mathrm{Var}(Z^{1}_{x:\overline{2}|})$.

Example 7

Suppose that $i = 0.05$, $q_x = 0.05$ *and* $q_{x+1} = 0.02$. *Find* $A^{1}_{x:\overline{2}|}$ *and* $\mathrm{Var}(Z^{1}_{x:\overline{2}|})$.

Solution: We have that

$$A^{1}_{x:\overline{2}|} = \nu q_x + \nu^2 p_x q_{x+1}$$

$$=(1.05)^{-1}(0.05) + (1.05)^{-2}(1 - 0.05)(0.02) = 0.06485260771,$$

$${}^2A^{1}_{x:\overline{2}|} = \nu^2 q_x + \nu^4 p_x q_{x+1}$$

$$=(1.05)^{-2}(0.05) + (1.05)^{-4}(1 - 0.05)(0.02) = 0.06098282094,$$

$$\mathrm{Var}(Z^{1}_{x:\overline{2}|}) = {}^2A^{1}_{x:\overline{2}|} - \left(A^{1}_{x:\overline{2}|}\right)^2$$

$$=0.06098282094 - (0.06485260771)^2 = 0.05677696021.$$

Example 8

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a three–year life policy insurance which will pay \$50000 at the end of the year of his death. $i = 6.5\%$.

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(i) Find the actuarial present value of this life insurance.

Solution: (i) The actuarial present value of this life insurance is

$$
\begin{aligned}
(50000)A^{1}_{80:\overline{3}|} &= (50000)\sum_{k=1}^{3} \nu^k \frac{\ell_{80+k-1} - \ell_{80+k}}{\ell_{80}} \\
=&(50000)(1.065)^{-1}\frac{250-217}{250} + (50000)(1.065)^{-2}\frac{217-161}{250} \\
&+ (50000)(1.065)^{-3}\frac{161-107}{250} \\
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An 80–year old buys a three–year life policy insurance which will pay \$50000 at the end of the year of his death. $i = 6.5\%$.

(ii) Find the probability that APV of this life insurance is adequate to cover this insurance.

Solution: (ii) 25012.54 is adequate if

$$25012.53726 > (50000)(1.065)^{-K_x} I(K_x \leq 3).$$

$25012.54 > (50000)(1.065)^{-K_x}$ is equivalent to $K_x > \frac{-\ln(25012.54/50000)}{\ln(1.065)} = 10.999$. The probability that the APV is adequate is $\mathbb{P}\{K_x \geq 4\} = \mathbb{P}\{T_x > 3\} = \frac{107}{250} = 0.428$.

Theorem 6

Under the de Moivre model, if $n \le \omega - x$,

$$A^{1}_{x:\overline{n}|} = \frac{a_{\overline{n}|}}{\omega - x}.$$

Proof: The density of T_x is $f_{T_x}(t) = \frac{1}{\omega - x}$, if $0 \le t \le \omega - x$, and

$$\mathbb{P}\{K_x = k\} = \mathbb{P}\{k - 1 < T_x \le k\} = \int_{k-1}^{k} \frac{1}{\omega - x}\, dt = \frac{1}{\omega - x}.$$

Hence,

$$A^{1}_{x:\overline{n}|} = \sum_{k=1}^{n} \nu^k \mathbb{P}\{K_x = k\} = \sum_{k=1}^{n} \nu^k \frac{1}{\omega - x} = \frac{a_{\overline{n}|}}{\omega - x}.$$

Example 9

Abigail is 45 years old. She buys a whole life policy insurance which will pay $250,000 at the end of the year of her death if death happens before 20 years. Suppose that the de Moivre model holds with terminal age 110*. Find the mean and the standard deviation of the present value of this life insurance under* $i = 7.5\%$.

Example 9

Abigail is 45 years old. She buys a whole life policy insurance which will pay \$250,000 at the end of the year of her death if death happens before 20 years. Suppose that the de Moivre model holds with terminal age 110. *Find the mean and the standard deviation of the present value of this life insurance under* $i = 7.5\%$.

Solution: The actuarial present value of this life insurance is

$$(250000)A^{1}_{45:\overline{20}|} = (250000)\frac{a_{\overline{20}|0.075}}{65} = 39209.58215.$$

The second moment of the present value of this life insurance is

$$(250000)^2 \cdot {}^2A^{1}_{45:\overline{20}|} = (250000)^2\frac{a_{\overline{20}|(1.075)^2-1}}{65} = 5836148593.$$

The variance of the present value of this life insurance is

$$\mathrm{Var}(Z_x) = 5836148593 - (39209.58215)^2 = 4298757261.$$

The standard deviation of the present value of this life insurance is

$$\sqrt{4298757261} = 65564.90876.$$

Theorem 7

If the probability function of time interval of failure is is

$$\mathbb{P}\{K_x = k\} = p_x^{k-1}(1 - p_x), k = 1, 2, \dots$$

then

$$A^1_{x:\overline{n}|} = (1 - p_x^n v^n)\frac{q_x}{q_x + i}.$$

Theorem 7

If the probability function of time interval of failure is is

$$\mathbb{P}\{K_x = k\} = p_x^{k-1}(1 - p_x), k = 1, 2, \dots$$

then

$$A^1_{x:\overline{n}|} = (1 - p_x^n v^n)\frac{q_x}{q_x + i}.$$

Proof: We have that

$$A^1_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \mathbb{P}\{K_x = k\} = \sum_{k=1}^{n} v^k p_x^{k-1}(1 - p_x)$$

$$=(1 - p_x)v \sum_{k=1}^{n}(vp_x)^{k-1} = (1 - p_x)v \frac{1 - (vp_x)^n}{1 - vp_x}$$

$$=(1 - (vp_x)^n)\frac{1 - p_x}{1 + i - p_x} = (1 - p_x^n v^n)\frac{q_x}{q_x + i}.$$

Corollary 2

Under a constant force of mortality μ and force of interest δ,

$$A^{1}_{x:\overline{n}|} = (1 - p_x^n v^n)\frac{q_x}{q_x + i} = (1 - e^{-n(\mu+\delta)})\frac{1 - e^{-\mu}}{e^{\delta} - e^{-\mu}}.$$

Corollary 2

Under a constant force of mortality μ and force of interest δ,

$$A^{1}_{x:\overline{n}|} = (1 - p_x^n v^n)\frac{q_x}{q_x + i} = (1 - e^{-n(\mu+\delta)})\frac{1 - e^{-\mu}}{e^{\delta} - e^{-\mu}}.$$

Proof: In this case, $p_x = e^{-\mu}$ and $1 + i = e^{\delta}$. So,

$$A^{1}_{x:\overline{n}|} = (1 - p_x^n v^n)\frac{q_x}{q_x + i} = (1 - e^{-n(\mu+\delta)})\frac{1 - e^{-\mu}}{1 - e^{-\mu} + e^{\delta} - 1}$$

$$=(1 - e^{-n(\mu+\delta)})\frac{1 - e^{-\mu}}{e^{\delta} - e^{-\mu}}.$$

Example 10

Suppose that $\delta = 0.04$ *and* (x) *has force of mortality* $\mu = 0.03$. *Find* $A^{1}_{x:\overline{10}|}$ *and* $\mathrm{Var}(Z^{1}_{x:\overline{10}|})$.

Solution:

Example 10

Suppose that $\delta = 0.04$ *and* (x) *has force of mortality* $\mu = 0.03$. *Find* $A^{1}_{x:\overline{10}|}$ *and* $\mathrm{Var}(Z^{1}_{x:\overline{10}|})$.

Solution:

We have that

$$A^{1}_{x:\overline{10}|} = (1 - e^{-n(\mu+\delta)})\frac{q_x}{q_x + i}$$

$$=(1 - e^{-(10)(0.03+0.04)})\frac{1 - e^{-0.03}}{1 - e^{-0.03} + e^{0.04} - 1}$$

$$=(1 - e^{-0.7})\frac{1 - e^{-0.03}}{e^{0.04} - e^{-0.03}} = 0.2114417945,$$

$${}^{2}A^{1}_{x:\overline{10}|} = (1 - e^{-n(\mu+2\delta)})\frac{q_x}{q_x + (1+i)^2 - 1}$$

$$=(1 - e^{-(10)(0.03+(2)(0.04))})\frac{1 - e^{-0.03}}{1 - e^{-0.03} + e^{(2)(0.04)} - 1}$$

$$=(1 - e^{-1.1})\frac{1 - e^{-0.03}}{e^{0.08} - e^{-0.03}} = 0.1747285636,$$

Example 10

Suppose that $\delta = 0.04$ and (x) has force of mortality $\mu = 0.03$. Find $A^{1}_{x:\overline{10}|}$ and $\mathrm{Var}(Z^{1}_{x:\overline{10}|})$.

Solution:

$$\mathrm{Var}(Z^{1}_{x:\overline{10}|}) = {}^{2}A^{1}_{x:\overline{10}|} - {A^{1}_{x:\overline{10}|}}^{2}$$
$$= 0.1747285636 - (0.2114417945)^{2} = 0.1300209311.$$

Theorem 8

For $n \geq 1$,

$$A^{1}_{x:\overline{n}|} = \nu q_x + \nu p_x A^{1}_{x+1:\overline{n-1}|}.$$

Proof: By the change of variables $k = j + 1$,

$$\begin{aligned}
A^{1}_{x:\overline{n}|} &= \sum_{k=1}^{n} \nu^k {}_{k-1}p_x \cdot q_{x+k-1} \\
&= \nu q_x + \sum_{k=2}^{n} \nu^k p_x \cdot {}_{k-2}p_{x+1} \cdot q_{x+k-1} \\
&= \nu q_x + \sum_{j=1}^{n-1} \nu^{j+1} p_x \cdot {}_{j-1}p_{x+1} \cdot q_{x+j} \\
&= \nu q_x + \nu p_x \sum_{j=1}^{n-1} \nu^j \cdot {}_{j-1}p_{x+1} \cdot q_{x+1+j-1} \\
&= \nu q_x + \nu p_x A^{1}_{x+1:\overline{n-1}|}.
\end{aligned}$$

Example 11

Using $i = 0.05$ *and a certain life table* $A^{1}_{37:\overline{10}|} = 0.52$. *Suppose that an actuary revises this life table and changes* p_{37} *from* 0.95 *to* 0.96. *Other values in the life table are unchanged. Find* $A^{1}_{37:\overline{10}|}$ *using the revised life table.*

Example 11

Using $i = 0.05$ *and a certain life table* $A^1_{37:\overline{10}|} = 0.52$*. Suppose that an actuary revises this life table and changes* p_{37} *from* 0.95 *to* 0.96*. Other values in the life table are unchanged. Find* $A^1_{37:\overline{10}|}$ *using the revised life table.*

Solution: We have that

$$A^1_{37:\overline{10}|} = \nu(1 - p_{37}) + \nu p_{37} A^1_{38:\overline{9}|}.$$

Hence, under the old table

$$A^1_{38:\overline{9}|} = \frac{0.52 - (1.05)^{-1}(1 - 0.95)}{(1.05)^{-1}(0.95)} = 0.5221052632.$$

Since $A^1_{38:\overline{9}|}$ does not depend on p_{37}, using the revised life table $A^1_{38:\overline{9}|} = 0.5221052632$. Hence, using the revised life table

$$A^1_{37:\overline{10}|} = (1.05)^{-1}(1 - 0.96) + (1.05)^{-1}(0.96)(0.5221052632)$$
$$=0.5154486216.$$

n–year deferred life insurance.

Definition 6

*A life insurance policy is called an n–**year deferred life insurance** if it pays a face value if the insured dies at least n years after the issue of the policy.*

Definition 7

The present value of an n–year deferred life insurance with unit payment paid at the end of the end of the year of death is denoted by ${}_{n|}Z_x$.

We have that

$$ {}_{n|}Z_x = \nu^{K_x} I(n < K_x) = \begin{cases} 0 & \text{if } K_x \le n \\ \nu^{K_x} & \text{if } n < K_x. \end{cases} $$

Definition 8

The actuarial present value of a n–year deferred life insurance with unit payment paid at the end of the end of the year of death is denoted by ${}_{n}|A_x$.

Theorem 9

$$ {}_{n}|A_x = E[{}_{n}|Z_x] = \sum_{k=n+1}^{\infty} \nu^k \mathbb{P}\{K_x = k\} = \sum_{k=n+1}^{\infty} \nu^k \cdot {}_{k-1}|q_x, $$

$$ {}^{2}{}_{n}|A_x = E[{}_{n}|Z_x{}^2] = \sum_{k=n+1}^{\infty} \nu^{2k} \mathbb{P}\{K_x = k\} = \sum_{k=n+1}^{\infty} \nu^{2k} \cdot {}_{k-1}|q_x. $$

$$ \mathrm{Var}({}_{n}|Z_x) = {}^{2}{}_{n}|A_x - {}_{n}|A_x{}^2. $$

Example 12

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a three–year deferred policy insurance which will pay $50000 at the end of the year of his death. Suppose that $i = 6.5\%$.

Example 12

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a three–year deferred policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$*.*

(i) Find the actuarial present value of this life insurance.

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x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a three–year deferred policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$.

(i) Find the actuarial present value of this life insurance.

$$(50000)_{3|}A_{80} = (50000)\sum_{k=4}^{\infty} \nu^k \frac{\ell_{80+k-1} - \ell_{80+k}}{\ell_{80}}$$

$$=(50000)(1.065)^{-4}\frac{107-62}{250} + (50000)(1.065)^{-5}\frac{62-28}{250}$$

$$+ (50000)(1.065)^{-6}\frac{28-0}{250}$$

$$=6995.907818 + 4963.189688 + 3837.871065 = 15796.96857.$$

Example 12

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a three–year deferred policy insurance which will pay $50000 at the end of the year of his death. Suppose that $i = 6.5\%$.

(ii) Find the probability that APV of this life insurance is adequate to cover this insurance.

Example 12

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a three–year deferred policy insurance which will pay \$50000 at the end of the year of his death. Suppose that $i = 6.5\%$.

(ii) Find the probability that APV of this life insurance is adequate to cover this insurance.

Solution: (ii) We have that 15796.96857 is adequate if $15796.96857 > (50000)(1.065)^{-K_x} I(3 < K_x)$. If $K_x \geq 4$, $15796.96857 > (50000)(1.065)^{-K_x} I(3 < K_x)$ is equivalent to $K_x > \frac{-\ln(15796.96857/50000)}{\ln(1.065)} = 18.29628616$. So, 15796.96857 is adequate when $K_x \leq 3$. The probability that APV is adequate is

$$\mathbb{P}\{K_x \leq 3\} = \mathbb{P}\{T_x \leq 3\} = \frac{250 - 107}{250} = 0.572.$$

Theorem 10

Under the De Moivre model, if $n \le \omega - x$,

$$ {}_{n|}A_x = \frac{v^n a_{\overline{\omega-x-n}|i}}{\omega - x}. $$

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$$ {}_{n|}A_x = \frac{v^n a_{\overline{\omega-x-n}|i}}{\omega - x}. $$

Proof:

$$ {}_{n|}A_x = \sum_{k=n+1}^{\infty} v^k \cdot {}_{k-1|}q_x = \sum_{k=n+1}^{\omega-x} v^k \frac{1}{\omega - x} = v^n \sum_{k=1}^{\omega-x-n} v^k \frac{1}{\omega - x} $$

$$ = v^n \frac{a_{\overline{\omega-x-n}|i}}{\omega - x}. $$

Example 13

Rose is 40 years old. She buys a 25-year deferred life policy insurance which will pay $200,000 at the end of the year of her death. Suppose that the De Moivre model holds with terminal age 120*. Find the mean and the standard deviation of the present value of this life insurance under the annual effective rate of interest of* $i = 10\%$.

Example 13

Rose is 40 years old. She buys a 25-year deferred life policy insurance which will pay $200,000 at the end of the year of her death. Suppose that the De Moivre model holds with terminal age 120*. Find the mean and the standard deviation of the present value of this life insurance under the annual effective rate of interest of* $i = 10\%$.

Solution:
The actuarial present value of this life insurance is

$$(200000)_{25|}A_{40} = (200000)v^{25}\frac{a_{\overline{55}|0.1}}{80} = 2295.195308.$$

The second moment of the present value of this life insurance is

$$(200000)^2 \cdot {}_{25|}{}^2A_{40} = (200000)^2 v^{50}\frac{a_{\overline{55}|(0.1)(2+0.1)}}{80} = 20281697.51.$$

Example 13

Rose is 40 years old. She buys a 25-year deferred life policy insurance which will pay \$200,000 at the end of the year of her death. Suppose that the De Moivre model holds with terminal age 120. Find the mean and the standard deviation of the present value of this life insurance under the annual effective rate of interest of $i = 10\%$.

Solution:
The variance of the present value of this life insurance is

$$\mathrm{Var}({}_{n}|Z_x) = 20281697.51 - (2295.195308)^2 = 15013776.01.$$

The standard deviation of the present value of this life insurance is

$$\sqrt{15013776.01} = 3874.761413.$$

Theorem 11

Under constant force of mortality μ,

$$ {}_{n|}A_x = e^{-n(\mu+\delta)} \frac{q_x}{q_x + i}. $$

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$$ {}_{n|}A_x = e^{-n(\mu+\delta)} \frac{q_x}{q_x + i}. $$

Proof:

$$ {}_{n|}A_x = \sum_{k=n+1}^{\infty} v^k \cdot {}_{k-1|}q_x = \sum_{k=n+1}^{\infty} e^{-k\delta} p_x^{k-1} q_x $$

$$ = \sum_{k=1}^{\infty} e^{-(k+n)\delta} p_x^{k+n-1} q_x = e^{-n(\mu+\delta)} \sum_{k=1}^{\infty} e^{-k\delta} p_x^{k-1} q_x $$

$$ = e^{-n(\mu+\delta)} q_x e^{-\delta} \frac{1}{1 - q_x e^{-\delta}} = e^{-n(\mu+\delta)} \frac{q_x}{q_x + i}. $$

Example 14

An insurance company offers a 10*–year deferred life insurance for individuals aged 25, which will pay \$250000 at the end of the year of his death. Suppose that* $p_x = 0.95$*, for each* $x \geq 0$*, and* $\delta = 0.065$*.*

(i) Find the expected value and the variance of the present value of this life insurance.

Example 14

An insurance company offers a 10*–year deferred life insurance for individuals aged 25, which will pay \$250000 at the end of the year of his death. Suppose that* $p_x = 0.95$*, for each* $x \geq 0$*, and* $\delta = 0.065$*.*

(i) Find the expected value and the variance of the present value of this life insurance.

Solution: (i) The expected value is

$$(250000) \cdot {}_{10|}A_{25} = \frac{(250000)e^{-(0.065)(10)}(0.95)^{10}(0.05)}{e^{0.065} - 1 + 0.05} = 33348.70,$$

$$(250000)^2 \cdot {}^2{}_{10|}A_{25} = \frac{(250000)^2 e^{-2(0.065)(10)}(0.95)^{10}(0.05)}{e^{(2)(0.065)} - 1 + 0.05}$$

$=2700448959,$

$$\mathrm{Var}\left(\sum_{j=1}^{50} {}_{10|}Z_x\right) = 2700448959 - (33348.70)^2 = 1588313181.$$

Example 14

An insurance company offers a 10*–year deferred life insurance for individuals aged 25, which will pay $250000 at the end of the year of his death. Suppose that* $p_x = 0.95$, *for each* $x \geq 0$, *and* $\delta = 0.065$.

(ii) Find the mean and the variance of the present value of the aggregate present value for 50 insurance contracts.

Example 14

An insurance company offers a 10–year deferred life insurance for individuals aged 25, which will pay \$250000 at the end of the year of his death. Suppose that $p_x = 0.95$, for each $x \geq 0$, and $\delta = 0.065$.

(ii) Find the mean and the variance of the present value of the aggregate present value for 50 insurance contracts.

Solution: (ii) The aggregate APV of these 50 lives is $\sum_{j=1}^{50}(250000)_{10|}Z_{x_j}$, where $_{10|}Z_{x_j}$ are i.i.d.r.v.'s.

$$E\left[\sum_{j=1}^{50} {}_{10|}Z_{x_j}\right] = (50)(250000) \cdot {}_{10|}A_{25} = (50)(33348.70)$$

$$=1667434.99,$$

$$\mathrm{Var}\left(\sum_{j=1}^{50} {}_{10|}Z_{x_j}\right) = (50)(1588313181) = 79415659050.$$

Example 14

An insurance company offers a 10*–year deferred life insurance for individuals aged 25, which will pay $250000 at the end of the year of his death. Suppose that* $p_x = 0.95$*, for each* $x \geq 0$*, and* $\delta = 0.065$*.*

(iii) Using the normal approximation, calculate the amount such that the probability that the aggregate APV of these 50 lives is less than this amount is 0.95.

Example 14

An insurance company offers a 10–*year deferred life insurance for individuals aged 25, which will pay \$250000 at the end of the year of his death. Suppose that* $p_x = 0.95$*, for each* $x \geq 0$*, and* $\delta = 0.065$*.*

(iii) Using the normal approximation, calculate the amount such that the probability that the aggregate APV of these 50 lives is less than this amount is 0.95.

Solution: (iii) The amount is the 95 percentile of a normal distribution with mean $E[\sum_{j=1}^{50} {}_{10|}Z_{x_j}]$ and variance $\mathrm{Var}(\sum_{j=1}^{50} {}_{10|}Z_{x_j})$, which is

$$(1667434.9895) + (1.6448536)\sqrt{79415659050} = 2130967.63.$$

Theorem 12

For each $x > 0$,

$$_{n|}A_x = \nu^{n+1}{}_n p_x \cdot q_{x+n} + {}_{n+1|}A_x$$

Theorem 12

For each $x > 0$,

$$ {}_{n|}A_x = \nu^{n+1} {}_n p_x \cdot q_{x+n} + {}_{n+1|}A_x $$

Proof: We have that

$$ \begin{aligned} {}_{n|}A_x &= \sum_{k=n+1}^{\infty} \nu^k {}_{k-1}p_x \cdot q_{x+k-1} \\ &= \nu^{n+1} {}_n p_x \cdot q_{x+n} + \sum_{k=n+2}^{\infty} \nu^k {}_{k-1}p_x \cdot q_{x+k-1} \\ &= \nu^{n+1} {}_n p_x \cdot q_{x+n} + {}_{n+1|}A_x. \end{aligned} $$

Theorem 12

For each $x > 0$,

$$ {}_{n|}A_x = \nu^{n+1}{}_{n}p_x \cdot q_{x+n} + {}_{n+1|}A_x $$

Proof: We have that

$$ \begin{aligned} {}_{n|}A_x = & \sum_{k=n+1}^{\infty} \nu^k{}_{k-1}p_x \cdot q_{x+k-1} \\ = & \nu^{n+1}{}_{n}p_x \cdot q_{x+n} + \sum_{k=n+2}^{\infty} \nu^k{}_{k-1}p_x \cdot q_{x+k-1} \\ = & \nu^{n+1}{}_{n}p_x \cdot q_{x+n} + {}_{n+1|}A_x. \end{aligned} $$

It follows from the previous theorem that

$$ {}^2{}_{n|}A_x = v^{2(n+1)}{}_{n}p_x \cdot q_{x+n} + {}^2{}_{n+1|}A_x. $$

Example 15

Suppose that ${}_{14|}A_{35} = 0.24$, $i = 8\%$, ${}_{14}p_{35} = 0.7$, $q_{49} = 0.03$.
Find ${}_{15|}A_{35}$.

Example 15

Suppose that ${}_{14|}A_{35} = 0.24$, $i = 8\%$, ${}_{14}p_{35} = 0.7$, $q_{49} = 0.03$. *Find* ${}_{15|}A_{35}$.

Solution:

$$ {}_{n|}A_x = v^{n+1}\, {}_{n}p_x \cdot q_{x+n-1} + {}_{n+1|}A_x, $$

we get that

$$ 0.24 = (1.08)^{-15}(0.7)(0.03) + {}_{15|}A_{35} $$

and ${}_{15|}A_{35} = 0.24 - (1.08)^{-15}(0.7)(0.03) = 0.233379924195716$.

n–year pure endowment life insurance

Definition 9

*A life insurance policy is called an n–***year pure endowment life insurance** *if it pays a face value in n years when the insured dies at least n years from the issue of the policy.*

Definition 10

The present value of an n–year pure endowment life insurance with unit payment is denoted by $Z_{x:\overset{1}{\overline{n|}}}$.

We have that

$$Z_{x:\overset{1}{\overline{n|}}} = \nu^n I(n < K_x) = \begin{cases} 0 & \text{if } K_x \le n \\ \nu^n & \text{if } n < K_x \end{cases}$$

We use the upper index ^{1} to mean that the n–th term must fail before x.

Definition 11

The actuarial present value of a n–year pure endowment life insurance with unit payment is denoted by $A_{x:\overline{n}|}^{\;\;1}$ *and by* ${}_nE_x$*.*

Theorem 13

We have that

$$A_{x:\overline{n}|}^{\;\;1} = {}_nE_x = E[Z_{x:\overline{n}|}^{\;\;1}] = \nu^n \mathbb{P}\{K_x > n\} = \nu^n \frac{s(x+n)}{s(x)} = \nu^n \cdot {}_np_x,$$

$${}^2A_{x:\overline{n}|}^{\;\;1} = E[{Z_{x:\overline{n}|}^{\;\;1}}^2] = \nu^{2n} \mathbb{P}\{K_x > n\} = \nu^{2n} \frac{s(x+n)}{s(x)} = \nu^{2n} \cdot {}_np_x,$$

$$\mathrm{Var}(Z_{x:\overline{n}|}^{\;\;1}) = {}^2A_{x:\overline{n}|}^{\;\;1} - {A_{x:\overline{n}|}^{\;\;1}}^2 = \nu^{2n} \cdot {}_np_x \cdot {}_nq_x.$$

Example 16

An insurance company has 100 clients age 30 which will receive a payment of 50,000 at the end of 10 years if they are alive. Suppose that the probability that a life age 30 will die within 10 years is 0.02. The current annual effective rate of interest is 9%.
Find the mean and the variance of the present value of the total payments which the insurance company will made.

Example 16

An insurance company has 100 clients age 30 which will receive a payment of 50,000 at the end of 10 years if they are alive. Suppose that the probability that a life age 30 will die within 10 years is 0.02. The current annual effective rate of interest is 9%.
Find the mean and the variance of the present value of the total payments which the insurance company will made.

Solution: We have that

$$A_{x:\overset{1}{\overline{n}|}} = \nu^n \mathbb{P}\{K_x > n\} = (1.09)^{-10}(0.98) = 0.4139626,$$
$$\mathrm{Var}(Z_{x:\overset{1}{\overline{n}|}}) = (1.09)^{-20}(0.98)(1-0.98) = 0.003497266.$$

Let Y be the total of amount of payments made at the 100 clients. We have that

$$E[Y] = (100)(50000)(0.4139626) = 2069813,$$
$$\mathrm{Var}(Y) = (100)(50000)^2(0.003497266) = 874316500.$$

n–year endowment life insurance.

Definition 12

*A life insurance policy is called an n–***year endowment life insurance** *if it makes a payment when either death happens before n years, or at the end of the n years if death happens after n years.*

Definition 13

The present value of an n–year endowment life insurance with unit payment paid at end of year of death is denoted by $Z_{x:\overline{n}|}$.

Notice that the payment is paid at the end of year of death or in n years whatever comes first. We have that

$$Z_{x:\overline{n}|} = \nu^{\min(K_x,n)} = \begin{cases} \nu^{K_x} & \text{if } K_x \leq n \\ \nu^n & \text{if } n < K_x \end{cases}$$

Definition 14

The actuarial present value of an n–year endowment life insurance with unit payment paid at end of year of death is denoted by $A_{x:\overline{n}|}$.

Theorem 14

$$A_{x:\overline{n}|} = E[Z_{x:\overline{n}|}] = \sum_{k=1}^{n} \nu^k \mathbb{P}\{K_x = k\} + \nu^n \mathbb{P}\{K_x > n\}$$

$$= \sum_{k=1}^{n-1} \nu^k \mathbb{P}\{K_x = k\} + \nu^n \mathbb{P}\{K_x \geq n\}$$

$$= \sum_{k=1}^{n} \nu^k {}_{k-1|}q_x + \nu^n {}_n p_x$$

$${}^2A_{x:\overline{n}|} = E[Z^2_{x:\overline{n}|}] = \sum_{k=1}^{n} \nu^{2k} \mathbb{P}\{K_x = k\} + \nu^{2n} \mathbb{P}\{K_x > n\}$$

$$= \cdots$$

$$\mathrm{Var}(Z_{x:\overline{n}|}) = {}^2A_{x:\overline{n}|} - A_{x:\overline{n}|}^2.$$

- $Z_{x:\overline{1}|} = \nu$. So, $A_{x:\overline{1}|} = \nu$, ${}^2A_{x:\overline{1}|} = \nu^2$ and $\mathrm{Var}(Z_{x:\overline{1}|}) = 0$.

- $Z_{x:\overline{1}|} = \nu$. So, $A_{x:\overline{1}|} = \nu$, ${}^2A_{x:\overline{1}|} = \nu^2$ and $\mathrm{Var}(Z_{x:\overline{1}|}) = 0$.
- $Z_{x:\overline{2}|} = \nu I(T_x \le 1) + \nu^2 I(T_x > 1)$. So, $A_{x:\overline{1}|} = \nu q_x + \nu^2 p_x$, and $\mathrm{Var}(Z_{x:\overline{1}|}) = \nu^2(1-\nu)^2 p_x q_x$.

Example 17

Suppose that $i = 0.05$, $q_x = 0.05$ *and* $q_{x+1} = 0.02$. *Find* $A_{x:\overline{2}|}$ *and* $\mathrm{Var}(Z_{x:\overline{2}|})$.

Example 17

Suppose that $i = 0.05$, $q_x = 0.05$ *and* $q_{x+1} = 0.02$. *Find* $A_{x:\overline{2}|}$ *and* $\mathrm{Var}(Z_{x:\overline{2}|})$.

Solution: We have that

$$A_{x:\overline{2}|} = \nu q_x + \nu^2 p_x = (1.05)^{-1}(0.05) + (1.05)^{-2}(1 - 0.05)$$
$$= 0.9092970522,$$
$$\mathrm{Var}(Z_{x:\overline{2}|}) = (1.05)^{-2}(1 - (1.05)^{-1})^2(0.05)(1 - 0.05)$$
$$= 0.000097695860391.$$

Example 18

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a three-year endowment policy insurance which will pay $50000. Suppose that $i = 6.5\%$.

Example 18

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a three-year endowment policy insurance which will pay \$50000. Suppose that $i = 6.5\%$.

(i) Find the actuarial present value of this life insurance.

Example 18

Consider the life table

x	*80*	*81*	*82*	*83*	*84*	*85*	*86*
ℓ_x	*250*	*217*	*161*	*107*	*62*	*28*	*0*

An 80–year old buys a three-year endowment policy insurance which will pay \$50000. Suppose that $i = 6.5\%$.

(i) Find the actuarial present value of this life insurance.

Solution: (i) The actuarial present value of this life insurance is

$$(50000)A_{80:\overline{3}|} = (50000)\sum_{k=1}^{3}\nu^k\frac{\ell_{80+k-1}-\ell_{80+k}}{\ell_{80}} + (50000)\nu^{2n}\frac{\ell_{80+3}}{\ell_{80}}$$

$$=(50000)(1.065)^{-1}\frac{250-217}{250} + (50000)(1.065)^{-2}\frac{217-161}{250}$$

$$+(50000)(1.065)^{-3}\frac{161-107}{250} + (50000)(1.065)^{-3}\frac{107}{250}$$

$$=6197.183099 + 9874.583967 + 8940.770191 + 17715.97056 = 42728.50$$

Example 18

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ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a three-year endowment policy insurance which will pay $50000. Suppose that $i = 6.5\%$.

(ii) Find the probability that APV of this life insurance is adequate to cover this insurance.

Example 18

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a three-year endowment policy insurance which will pay \$50000. Suppose that $i = 6.5\%$.

(ii) Find the probability that APV of this life insurance is adequate to cover this insurance.

Solution: (ii) We have that 42728.50782 is adequate if $42728.50782 > (50000)(1.065)^{-\min(K_x,3)}$, which is equivalent to $\min(K_x, 3) > \frac{-\ln(42728.50782/50000)}{\ln(1.065)} = 2.495548689$. The probability that the APV is adequate is

$$\mathbb{P}\{K_x \geq 3\} = \mathbb{P}\{T_x \geq 2\} = \frac{161}{250} = 0.644.$$

Theorem 15

Under the De Moivre model, if $n \le \omega - x$,

$$A_{x:\overline{n}|} = \frac{a_{\overline{n}|i}}{\omega - x} + v^n \frac{\omega - x - n}{\omega - x}.$$

Theorem 15

Under the De Moivre model, if $n \le \omega - x$,

$$A_{x:\overline{n}|} = \frac{a_{\overline{n}|i}}{\omega - x} + v^n \frac{\omega - x - n}{\omega - x}.$$

Proof:

$$A_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \, {}_{k-1|}q_x + v^n \cdot {}_n p_x = \sum_{k=1}^{n} v^k \frac{1}{\omega - x} + v^n \frac{\omega - x - n}{\omega - x}$$

$$= \frac{a_{\overline{n}|i}}{\omega - x} + v^n \frac{\omega - x - n}{\omega - x}.$$

Example 19

A 10–year endowment insurance pays \$20000 at the end of the year of failure, or \$20000 for survival to time 10, whichever occurs first. Find the actuarial present value of this endowment insurance for a 40–year old if $s(x) = \frac{100-x}{100}$, $0 \le x \le 100$, *and* $i = 7.5\%$.

Example 19

A 10–year endowment insurance pays \$20000 at the end of the year of failure, or \$20000 for survival to time 10, whichever occurs first. Find the actuarial present value of this endowment insurance for a 40–year old if $s(x) = \frac{100-x}{100}$, $0 \le x \le 100$, *and* $i = 7.5\%$.

Solution: We have De Moivre's law with terminal age 100. The actuarial present value of this insurance is

$$(20000)A_{40:\overline{10}|} = (20000)\frac{a_{\overline{10}|i}}{60} + (20000)v^{10}\frac{60-10}{60}$$
$$=2288.026985 + 8086.565472 = 10374.59246.$$

m–year deferred n–year term life insurance.

Definition 15

*A life insurance policy is called an m–**year deferred** n–**year term life insurance** if it makes a payment if death happens during the period of n years that starts m years from now.*

Definition 16

The present value of an n–year deferred n–year term life insurance with unit payment paid at end of year of death is denoted by ${}_{m|n}Z_x$.

$$ {}_{m|n}Z_x = v^{K_x} I(m+1 \le K_x \le n) = \begin{cases} v^{K_x} & \text{if } m+1 \le K_x \le n, \\ 0 & \text{else.} \end{cases} $$

Definition 17

The actuarial present value of an m–year deferred n–year term life insurance with unit payment paid at end of year of death is ${}_{m|n}A_x$.

Theorem 16

$$ {}_{m|n}A_x = E[{}_{m|n}Z_x] = \sum_{k=m+1}^{m+n} v^k \cdot {}_{k-1|}q_x, $$

$$ {}^2{}_{m|n}A_{x:\overline{n}|} = E[({}_{m|n}Z_x)^2] = \sum_{k=m+1}^{m+n} v^k \cdot {}_{k-1|}q_x. $$

Theorem 16

$$ {}_{m|n}A_x = E[{}_{m|n}Z_x] = \sum_{k=m+1}^{m+n} v^k \cdot {}_{k-1|}q_x, $$

$$ {}^2{}_{m|n}A_{x:\overline{n}|} = E[({}_{m|n}Z_x)^2] = \sum_{k=m+1}^{m+n} v^k \cdot {}_{k-1|}q_x. $$

Proof: We have that

$$ {}_{m|n}A_x = E[{}_{m|n}Z_x] = \sum_{k=m+1}^{m+n} v^k \mathbb{P}\{K_x = k\} == \sum_{k=m+1}^{m+n} v^k \cdot {}_{k-1|}q_x, $$

$$ {}^2{}_{m|n}A_{x:\overline{n}|} = E[({}_{m|n}Z_x)^2] = \sum_{k=m+1}^{m+n} v^{2k} \mathbb{P}\{K_x = k\} = \sum_{k=m+1}^{m+n} v^k \cdot {}_{k-1|}q_x. $$

Theorem 16

$$ {}_{m|n}A_x = E[{}_{m|n}Z_x] = \sum_{k=m+1}^{m+n} v^k \cdot {}_{k-1|}q_x, $$

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$$ \mathrm{Var}({}_{m|n}Z_x) = {}_{m|n}{}^2A_{x:\overline{n}|} - {}_{m|n}A_x{}^2. $$

Theorem 17

Under De Moivre's law, if $n + m \le \omega - x$,

$$ {}_{m|n}A_x = \frac{v^m a_{\overline{n}|}}{\omega - x}. $$

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$$ {}_{m|n}A_x = \frac{v^m a_{\overline{n}|}}{\omega - x}. $$

Proof:

$$ {}_{m|n}A_x = \sum_{k=m+1}^{m+n} v^k \cdot {}_{k-1|}q_x = \sum_{k=m+1}^{m+n} v^k \frac{1}{\omega - x} = \sum_{k=1}^{n} v^{m+k} \frac{1}{\omega - x} $$

$$ = v^m \sum_{k=1}^{n} v^k \frac{1}{\omega - x} = \frac{v^m a_{\overline{n}|}}{\omega - x}. $$

Theorem 18

$$ {}_{m|n}A_x = {}_mE_x A^{1}_{x+m:\overline{n}|}. $$

Theorem 18

$$ {}_{m|n}A_x = {}_mE_x A^{1}_{x+m:\overline{n}|}. $$

Proof: By a previous theorem, ${}_mp_x \cdot {}_{k-1|}q_{x+m} = {}_{m+k-1|}q_x$.
Hence,

$$ {}_mE_x A^{1}_{x+m:\overline{n}|} = v^m \cdot {}_mp_x \sum_{k=1}^{n} v^k \cdot {}_{k-1|}q_{x+m} = \sum_{k=1}^{n} v^{m+k} \cdot {}_{m+k-1|}q_{x+m} $$

$$ = \sum_{k=m+1}^{m+n} v^k \cdot {}_{k-1|}q_x = {}_{m|n}A_x. $$

Theorem 19

Under the constant force of mortality model,

$$ {}_{m|n}A_x = e^{-m(\delta+\mu)} \left(1 - e^{-n(\delta+\mu)}\right) \frac{q_x}{q_x + i}. $$

Theorem 19

Under the constant force of mortality model,

$$_{m|n}A_x = e^{-m(\delta+\mu)}\left(1 - e^{-n(\delta+\mu)}\right)\frac{q_x}{q_x + i}.$$

Proof:

$$_{m|n}A_x = {}_mE_x A^{1}_{x+m:\overline{n}|} = e^{-m(\delta+\mu)}\left(1 - e^{-n(\delta+\mu)}\right)\frac{q_x}{q_x + i}.$$

Theorem 20

$${}_{m|n}A_x = {}_{m|}A_x - {}_{m+n|}A_x.$$

Theorem 20

$${}_{m|n}A_x = {}_{m|}A_x - {}_{m+n|}A_x.$$

Proof: We have that

$${}_{m|}A_x - {}_{m+n|}A_x = \sum_{k=m+1}^{\infty} v^k \cdot {}_{k-1|}q_x - \sum_{k=m+n+1}^{\infty} v^k \cdot {}_{k-1|}q_x$$

$$= \sum_{k=m+1}^{m+n} v^k \cdot {}_{k-1|}q_x = {}_{m|n}A_x.$$

Theorem 21

$$A_x = A^{1}_{x:\overline{m}|} + {}_{m|n}A_x + {}_{m+n|}A_x.$$

Theorem 21

$A_x = A^1_{x:\overline{m}|} + {}_{m|n}A_x + {}_{m+n|}A_x.$

Proof: We have that

$$A^1_{x:\overline{m}|} + {}_{m|n}A_x + {}_{m+n|}A_x = A^1_{x:\overline{m}|} + {}_{m|}A_x - {}_{m+n|}A_x + {}_{m+n|}A_x = A^1_{x:\overline{m}|} + $$

Theorem 22

$$ {}_{m|n}A_x = A^{1}_{x:\overline{m+n}|} - A^{1}_{x:\overline{m}|}. $$

Theorem 22

$$ {}_{m|n}A_x = A^1_{x:\overline{m+n}|} - A^1_{x:\overline{m}|}. $$

Proof: We have that

$$ A^1_{x:\overline{m+n}|} - A^1_{x:\overline{m}|} = \sum_{k=1}^{m+n} v^k \cdot {}_{k-1|}q_x - \sum_{k=1}^{m} v^k \cdot {}_{k-1|}q_x $$

$$ = \sum_{k=m+1}^{m+n} v^k \cdot {}_{k-1|}q_x = {}_{m|n}A_x. $$

