Manual for SOA Exam MLC.

Chapter 4. Life Insurance.

Section 4.3. Further properties of the APV for discrete insurance.

Extract from:
"Arcones' Manual for the SOA Exam MLC. Fall 2009 Edition".
available at http://www.actexmadriver.com/

Further properties of the APV for insurance

The following table shows the definition of all the variables in the previous section:

type of insurance	payment
whole life insurance	$Z_x = \nu^{K_x}$
n–year term life insurance	$Z^1_{x:\overline{n}\vert} = \nu^{K_x} I(K_x \le n)$
n–year deferred life insurance	$_{n\vert}Z_x = \nu^{K_x} I(n < K_x)$
n–year pure endowment life insurance	$Z_{x:\overset{1}{\overline{n}\vert}} = \nu^{n} I(n < K_x)$
n–year endowment life insurance	$Z_{x:\overline{n}\vert} = \nu^{\min(K_x, n)}$

Theorem 1

We have that

$$Z_x = Z^1_{x:\overline{n}|} + {}_n|Z_x,$$
$$A_x = A^1_{x:\overline{n}|} + {}_n|A_x,$$
$${}^2A_x = {}^2A^1_{x:\overline{n}|} + {}^2{}_n|A_x,$$
$$\mathrm{Var}(Z_x) = \mathrm{Var}(Z^1_{x:\overline{n}|}) + \mathrm{Var}({}_n|Z_x) - 2A^1_{x:\overline{n}|} \cdot {}_n|A_x$$
$$\mathrm{Cov}(Z^1_{x:\overline{n}|}, {}_n|Z_x) = -A^1_{x:\overline{n}|} \cdot {}_n|A_x = \frac{1}{2}\left({A^1_{x:\overline{n}|}}^2 + {}_n|{A_x}^2 - {A_x}^2\right).$$

Proof: We have that

$$Z^1_{x:\overline{n}|} + {}_{n|}Z_x = \nu^{K_x} I(K_x \le n) + \nu^{K_x} I(n < K_x) = \nu^{K_x} = Z_x.$$

and

$${Z^1_{x:\overline{n}|}}^2 + {{}_{n|}Z_x}^2 = \nu^{2K_x} I(K_x \le n) + \nu^{2K_x} I(n < K_x) = \nu^{2K_x} = Z_x^2.$$

Hence,

$$A_x = A^1_{x:\overline{n}|} + {}_{n|}A_x$$

and

$${}^2A_x = {}^2A^1_{x:\overline{n}|} + {}^2{}_{n|}A_x.$$

Thus,

$$\begin{aligned}\mathrm{Var}(Z_x) &= {}^2A_x - (A_x)^2 = {}^2A^1_{x:\overline{n}|} + {}^2{}_{n|}A_x - \left(A^1_{x:\overline{n}|} + {}_{n|}A_x\right)^2 \\ &= {}^2A^1_{x:\overline{n}|} - \left(A^1_{x:\overline{n}|}\right)^2 + {}^2{}_{n|}A_x - ({}_{n|}A_x)^2 - 2A^1_{x:\overline{n}|} \cdot {}_{n|}A_x \\ &= \mathrm{Var}(Z^1_{x:\overline{n}|}) + \mathrm{Var}({}_{n|}Z_x) - 2A^1_{x:\overline{n}|} \cdot {}_{n|}A_x\end{aligned}$$

Since

$$Z^1_{x:\overline{n}|} \cdot {}_n|Z_x = \nu^{K_x} I(K_x \le n) \cdot \nu^{K_x} I(n < K_x) = 0.$$

we get that

$$\begin{aligned} &\mathrm{Cov}(Z^1_{x:\overline{n}|}, {}_n|Z_x) = E[Z^1_{x:\overline{n}|} \cdot {}_n|Z_x] - E[Z^1_{x:\overline{n}|}]E[{}_n|Z_x] \\ =& - E[Z^1_{x:\overline{n}|}]E[{}_n|Z_x] = -A^1_{x:\overline{n}|} \cdot {}_n|A_x. \end{aligned}$$

Finally, we have

$$\begin{aligned} &\frac{1}{2}\left({A^1_{x:\overline{n}|}}^2 + {}_n|{A_x}^2 - {A_x}^2 \right) \\ =&\frac{1}{2}\left({A^1_{x:\overline{n}|}}^2 + {}_n|{A_x}^2 - \left(A^1_{x:\overline{n}|} + {}_n|A_x \right)^2 \right) = -A^1_{x:\overline{n}|} \cdot {}_n|A_x. \end{aligned}$$

Example 1

Suppose that

$$E[Z_x] = 0.75, \mathrm{Var}(Z_x) = 0.45, E[Z^1_{x:\overline{n}|}] = 0.5, \mathrm{Var}(Z^1_{x:\overline{n}|}) = 0.2.$$

Find $E[{}_n|Z_x]$ *and* $\mathrm{Var}({}_n|Z_x)$.

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Find $E[{}_n|Z_x]$ *and* $\mathrm{Var}({}_n|Z_x)$.

Solution: Since $Z_x = Z^1_{x:\overline{n}|} + {}_n|Z_x$,

$$0.75 = A_x = A^1_{x:\overline{n}|} + {}_n|A_x = (0.5) + {}_n|A_x.$$

So, ${}_n|A_x = 0.25$. Since $Z_x^2 = (Z^1_{x:\overline{n}|})^2 + ({}_n|Z_x)^2$,

$$0.45 + (0.75)^2 = E[Z_x^2] = E[(Z^1_{x:\overline{n}|})^2] + E[({}_n|Z_x)^2]$$
$$= 0.2 + (0.5)^2 + E[({}_n|Z_x)^2]$$

Hence,

$$E[({}_n|Z_x)^2] = 0.45 + (0.75)^2 - 0.2 - (0.5)^2 = 0.5625.$$

and

$$\mathrm{Var}({}_n|Z_x) = 0.5625 - (0.25)^2 = 0.5.$$

Theorem 2

We have that

$$
\begin{aligned}
Z_{x:\overline{n}|} &= Z^{1}_{x:\overline{n}|} + Z_{x:\overset{\;1}{\overline{n}|}},\\
A_{x:\overline{n}|} &= A^{1}_{x:\overline{n}|} + A_{x:\overset{\;1}{\overline{n}|}},\\
{}^{2}A_{x:\overline{n}|} &= {}^{2}A^{1}_{x:\overline{n}|} + {}^{2}A_{x:\overset{\;1}{\overline{n}|}},\\
\mathrm{Var}(Z_{x:\overline{n}|}) &= \mathrm{Var}(Z^{1}_{x:\overline{n}|}) + \mathrm{Var}(Z_{x:\overset{\;1}{\overline{n}|}}) - 2A^{1}_{x:\overline{n}|} \cdot A_{x:\overset{\;1}{\overline{n}|}}\\
\mathrm{Cov}(Z^{1}_{x:\overline{n}|}, Z_{x:\overset{\;1}{\overline{n}|}}) &= -A^{1}_{x:\overline{n}|} \cdot A_{x:\overset{\;1}{\overline{n}|}} = \frac{1}{2}\left({A^{1}_{x:\overline{n}|}}^{2} + {A_{x:\overset{\;1}{\overline{n}|}}}^{2} - {A_{x:\overline{n}|}}^{2} \right).
\end{aligned}
$$

Proof.

The proof is similar to the proof of the previous theorem. So, we only sketch the proof. We have

$$Z^{1}_{x:\overline{n}|} + Z_{x:\overline{n}|}^{\;\;1} = \nu^{K_x} I(K_x \leq n) + \nu^n I(n < K_x) = \nu^{\min(K_x,n)} = Z_{x:\overline{n}|}.$$

and

$${Z^{1}_{x:\overline{n}|}}^2 + {Z_{x:\overline{n}|}^{\;\;1}}^2 = \nu^{2K_x} I(K_x \leq n) + \nu^{2n} I(n < K_x) = \nu^{2\min(K_x,n)} = Z^2_{x:\overline{n}|}.$$

Proceeding as the proof of the previous theorem, the proof follows. □

Example 2

Suppose that

$$E[Z^1_{x:\overline{n}|}] = 0.5, \mathrm{Var}(Z^1_{x:\overline{n}|}) = 0.35, \nu^n = 0.4, {}_np_x = 0.6.$$

Find $E[Z_{x:\overline{n}|}]$ *and* $\mathrm{Var}(Z_{x:\overline{n}|})$.

Example 2

Suppose that

$$E[Z_{x:\overline{n}|}^{1}] = 0.5, \mathrm{Var}(Z_{x:\overline{n}|}^{1}) = 0.35, \nu^n = 0.4, {}_np_x = 0.6.$$

Find $E[Z_{x:\overline{n}|}]$ *and* $\mathrm{Var}(Z_{x:\overline{n}|})$*.*

Solution: We have that

$$Z_{x:\overline{n}|} = Z_{x:\overline{n}|}^{1} + Z_{x:\overset{1}{\overline{n}|}} = Z_{x:\overline{n}|}^{1} + \nu^n I(n < K_x),$$

$$E[Z_{x:\overline{n}|}] = E[Z_{x:\overline{n}|}^{1}] + \nu^n {}_np_x = (0.5) + (0.4)(0.6) = 0.74.$$

We also have that

$$E[Z_{x:\overline{n}|}{}^2] = E[Z_{x:\overline{n}|}^{1}{}^2] + E[Z_{x:\overset{1}{\overline{n}|}}{}^2]$$

$$= \mathrm{Var}(Z_{x:\overline{n}|}^{1}) + \left(E[Z_{x:\overline{n}|}^{1}]\right)^2 + \nu^{2n} {}_np_x$$

$$= 0.35 + (0.5)^2 + (0.4)^2(0.6) = 0.696.$$

Hence,

$$\mathrm{Var}(Z_{x:\overline{n}|}^{1}) = 0.696 - (0.74)^2 = 0.1484.$$

Theorem 3

We have that

$$ {}_{n|}A_x = A_{x:\overline{n}|}^{\;\;1} A_{x+n} = {}_nE_x A_{x+n}. $$

Proof: By the change of variables $k = j - n$,

$$ A_{x:\overline{n}|}^{\;\;1} A_{x+n} = v^n {}_np_x \sum_{k=1}^{\infty} v^k \cdot {}_{k-1|}q_{x+n} = \sum_{k=1}^{\infty} v^{n+k} {}_{n+k-1|}q_x $$

$$ = \sum_{k=n+1}^{\infty} v^k \cdot {}_{k-1|}q_x = {}_{n|}A_x. $$

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$$ = \sum_{k=n+1}^{\infty} v^k \cdot {}_{k-1|}q_x = {}_{n|}A_x. $$

Corollary 1

$$ A_x = A_{x:\overline{n}|}^{1} + {}_nE_x A_{x+n}. $$

Example 3

An insurance company offers a 10*–year deferred life insurance for an individual aged 25, which will pay $250000 at the end of the year of his death. Suppose that* $p_x = 0.95$*, for each* $x \geq 0$*, and* $\delta = 0.065$*. Find the actuarial present value of this life insurance.*

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Solution: We have that

$$A_{x+n} = \frac{q_{x+n}}{q_{x+n} + i} = \frac{1 - 0.95}{1 - 0.95 + e^{0.065} - 1} = \frac{0.05}{e^{0.065} - 0.95}.$$

and

$$(250000)_{10|}A_x = (250000)_{10}E_x A_{x+10}$$
$$= (250000)(0.95)^{10} e^{-(10)(0.065)} \frac{0.05}{e^{0.065} - 0.95} = 33348.69979.$$

Example 4

Using $i = 6\%$ *and the life table in the textbook, find:*
(i) $A_{40:\overset{1}{\overline{10}|}}$. *(ii)* ${}_{10|}A_{40}$. *(iii)* $A^1_{40:\overline{10}|}$. *(iv)* $A_{40:\overline{10}|}$.

Example 4

Using $i = 6\%$ and the life table in the textbook, find:
(i) $A_{40:\overset{1}{\overline{10}|}}$. *(ii)* ${}_{10|}A_{40}$. *(iii)* $A^1_{40:\overline{10}|}$. *(iv)* $A_{40:\overline{10}|}$.

Solution: (i) We have that

$$A_{40:\overset{1}{\overline{10}|}} = \nu^{10}{}_{10}p_{40} = (1.06)^{-10}\frac{\ell_{50}}{\ell_{40}} = (1.06)^{-10}\frac{93735}{96517} = 0.5422996406.$$

(ii)

$$_{10|}A_{40} = {}_{10}E_{40}A_{50} = (0.5422996406)(0.20696) = 0.1122343.$$

(iii)

$$A^1_{40:\overline{10}|} = A_{40} - {}_{10|}A_{40} = 0.13264 - 0.1122343 = 0.0204057.$$

(iv)

$$A_{40:\overline{10}|} = A^1_{40:\overline{10}|} + {}_{10}E_{40} = 0.0204057 + 0.5422996406 = 0.5627053406.$$

Example 5

Suppose that $\delta = 0.04$ *and* (x) *has force of mortality* $\mu = 0.03$*. Calculate:*

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Calculate:
(i) A_x and $\mathrm{Var}(Z_x)$.

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Calculate:
(i) A_x and $\mathrm{Var}(Z_x)$.
Solution: (i) We have that

$$A_x = \frac{e^{\mu} - 1}{e^{\delta+\mu} - 1} = \frac{e^{0.03} - 1}{e^{0.04+0.03} - 1} = 0.4200151407,$$

$$^2A_x = \frac{e^{\mu} - 1}{e^{\delta+\mu} - 1} = \frac{e^{0.03} - 1}{e^{(2)(0.04)+0.03} - 1} = 0.2619112429,$$

$$\mathrm{Var}(Z_x) = {}^2A_x - (A_x)^2 = 0.2619112429 - (0.4200151407)^2$$

$$=0.08549852448.$$

Example 5

Suppose that $\delta = 0.04$ *and* (x) *has force of mortality* $\mu = 0.03$.
Calculate:
(ii) $A_{\overset{1}{x}:\overline{10|}}$ and $\mathrm{Var}(Z_{\overset{1}{x}:\overline{10|}})$.

Example 5

Suppose that $\delta = 0.04$ *and* (x) *has force of mortality* $\mu = 0.03$. *Calculate:*

(ii) $A_{\overset{1}{x}:\overline{10}|}$ and $\mathrm{Var}(Z_{\overset{1}{x}:\overline{10}|})$.

Solution: (ii) We have that

$$A_{\overset{1}{x}:\overline{10}|} = e^{-10\delta}\,{}_{10}p_x = e^{-(10)(0.04)}e^{-(10)(0.03)} = e^{-0.7} = 0.4965853038,$$

$$^2A_{\overset{1}{x}:\overline{10}|} = e^{-10(2)\delta}\,{}_{10}p_x = e^{-(10)(2)(0.04)}e^{-(10)(0.03)} = e^{-0.11}$$

$$=0.3328710837,$$

$$\mathrm{Var}(Z_{\overset{1}{x}:\overline{10}|}) = {}^2A_{\overset{1}{x}:\overline{10}|} - A_{\overset{1}{x}:\overline{10}|}{}^2 = 0.3328710837 - (0.4965853038)^2$$

$$=0.08627411975.$$

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Suppose that $\delta = 0.04$ *and* (x) *has force of mortality* $\mu = 0.03$.
Calculate:
(iii) ${}_{10|}A_x$ and $\mathrm{Var}({}_{10|}Z_x)$.

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Suppose that $\delta = 0.04$ *and* (x) *has force of mortality* $\mu = 0.03$.
Calculate:
(iii) ${}_{10|}A_x$ and $\mathrm{Var}({}_{10|}Z_x)$.

Solution: (iii) We have that

$${}_{10|}A_x = {}_nE_x A_{x+n} = (0.4965853038)(0.4200151407) = 0.2085733462,$$

$${}^2{}_{10|}A_x = {}^2{}_nE_x \cdot {}^2A_{x+n} = (0.2619112429)(0.3328710837)$$

$$=0.08718267926,$$

$$\mathrm{Var}({}_{10|}A_x) = 0.08718267926 - (0.2085733462)^2 = 0.04367983851.$$

Example 5

Suppose that $\delta = 0.04$ *and* (x) *has force of mortality* $\mu = 0.03$.
Calculate:
(iv) $A^{1}_{x:\overline{10}|}$ and $\mathrm{Var}(Z^{1}_{x:\overline{n}|})$.

Example 5

Suppose that $\delta = 0.04$ *and* (x) *has force of mortality* $\mu = 0.03$.
Calculate:
(iv) $A^{1}_{x:\overline{10}|}$ and $\mathrm{Var}(Z^{1}_{x:\overline{n}|})$.

Solution: (iv) We have that

$$A^{1}_{x:\overline{10}|} = A_x - {}_{10|}A_x = 0.4200151407 - 0.2085733462 = 0.2114417945,$$

$${}^{2}A^{1}_{x:\overline{10}|} = {}^{2}A_x - {}^{2}{}_{10|}A_x = 0.2619112429 - 0.08718267926$$

$$=0.1747285636,$$

$$\mathrm{Var}(Z^{1}_{x:\overline{10}|}) = {}^{2}A^{1}_{x:\overline{10}|} - {A^{1}_{x:\overline{10}|}}^{2} = 0.1747285636 - (0.2114417945)^2$$

$$=0.1300209311.$$

Example 5

Suppose that $\delta = 0.04$ *and* (x) *has force of mortality* $\mu = 0.03$. *Calculate:*

(v) $A_{x:\overline{10}|}$ and $\mathrm{Var}(_{x:\overline{10}|})$.

Example 5

Suppose that $\delta = 0.04$ *and* (x) *has force of mortality* $\mu = 0.03$.
Calculate:
(v) $A_{x:\overline{10}|}$ and $\mathrm{Var}(_{x:\overline{10}|})$.

Solution: (v) We have that

$$A_{x:\overline{10}|} = A^{1}_{x:\overline{10}|} + A_{x:\overset{1}{\overline{10}|}} = 0.2114417945 + 0.4965853038$$

$$=0.7080270983,$$

$$^{2}A_{x:\overline{10}|} = {}^{2}A^{1}_{x:\overline{10}|} + {}^{2}A_{x:\overset{1}{\overline{10}|}} = 0.1747285636 + 0.3328710837$$

$$=0.5075996473,$$

$$\mathrm{Var}(Z^{1}_{x:\overline{10}|}) = {}^{2}A^{1}_{x:\overline{10}|} - A^{1}_{x:\overline{10}|}{}^{2} = 0.5075996473 - (0.7080270983)^2$$

$$=0.006297275373.$$

