Manual for SOA Exam MLC.

Chapter 4. Life Insurance.

Section 4.6. Level benefit insurance in the continuous case.

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Whole life insurance.

Definition 1

The present value of a unit payment whole life insurance paid at the moment of of death is denoted by $\overline{Z}_x$*.*

We have that $\overline{Z}_x = \nu^{T_x} = e^{-\delta T_x}$.

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Definition 2

The actuarial present value of a unit payment whole life insurance paid at the moment of of death is denoted by $\overline{A}_x$.

We have that

$$\overline{A}_x = E[\overline{Z}_x] = \int_0^\infty \nu^t f_{T_x}(t)\,dt = \int_0^\infty \nu^t\,{}_tp_x \mu_{x+t}\,dt.$$

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$$\overline{A}_x = E[\overline{Z}_x] = \int_0^\infty \nu^t f_{T_x}(t)\, dt = \int_0^\infty \nu^t\, {}_tp_x \mu_{x+t}\, dt.$$

We define

$${}^m\overline{A}_x = E[\overline{Z}_x^m] = \int_0^\infty \nu^{mt} f_{T_x}(t)\, dt = \int_0^\infty \nu^{mt}\, {}_tp_x \mu_{x+t}\, dt.$$

We have that $\mathrm{Var}(\overline{Z}_x) = {}^2\overline{A}_x - \overline{A}_x^2$.

Theorem 1

The cumulative distribution function of $\overline{Z}_x = e^{-\delta T_x}$ *is*

$$F_{\overline{Z}_x}(z) = \begin{cases} 0 & \textit{if } z < 0, \\ S_{T_x}\left(\frac{-\ln z}{\delta}\right) & \textit{if } 0 < z \leq 1, \\ 1 & \textit{if } 1 \leq z. \end{cases}$$

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Proof: If $0 < z \leq 1$,

$$F_{\overline{Z}_x}(z) = \mathbb{P}\{\overline{Z}_x \leq z\} = \mathbb{P}\{e^{-\delta T_x} \leq z\} = \mathbb{P}\{-\ln(z)/\delta \leq T_x\}$$
$$= S_{T_x}\left(\frac{-\ln z}{\delta}\right).$$

Theorem 2

$\overline{Z}_x$ is a continuous r.v. with density

$$f_{\overline{Z}_x}(z) = \frac{f_{T_x}\left(-\frac{\ln z}{\delta}\right)}{\delta z}, e^{-\delta(\omega - x)} \le z \le 1.$$

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Proof: For $0 < z < 1$,

$$f_{\overline{Z}_x}(z) = \frac{d}{dz} S_{T_x}\left(\frac{-\ln z}{\delta}\right) = f_{T_x}(-(\ln z)/\delta)\frac{1}{z\delta}$$

Notice that if $0 < z < e^{-\delta(\omega - x)}$, $-(\ln z)/\delta > \omega - x$ and

$$f_{T_x}(-(\ln z)/\delta)\frac{1}{z\delta} = 0.$$

Corollary 1

Under the de Moivre model with terminal age ω, $\overline{Z}_x$ is a continuous r.v. with cumulative distribution function

$$F_{\overline{Z}_x}(z) = \frac{\delta(\omega - x) + \ln(z)}{\delta(\omega - x)}, \quad e^{-\delta(\omega - x)} \leq z \leq 1.$$

and density function

$$f_{\overline{Z}_x}(z) = \frac{1}{\delta z(\omega - x)}, \quad e^{-\delta(\omega - x)} \leq z \leq 1.$$

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Proof: We have that $\int_a^\infty f_{T_x}(t)\,dt = \frac{\omega - x - a}{\omega - x}$, if $0 \leq a \leq \omega - x$. Hence, for $e^{-\delta(\omega - x)} \leq z \leq 1$,

$$F_{\overline{Z}_x}(z) = \int_{-(\ln z)/\delta}^\infty f_{T_x}(t)\,dt = \frac{\omega - x + (\ln z)/\delta}{\omega - x} = \frac{\delta(\omega - x) + \ln(z)}{\delta(\omega - x)}.$$

$$f_{\overline{Z}_x}(z) = F'_{\overline{Z}_x}(z) = \frac{1}{\delta z(\omega - x)}, \quad e^{-\delta(\omega - x)} \leq z \leq 1.$$

Corollary 2

Under constant force of mortality μ, $\overline{Z}_x$ is a continuous r.v. with cumulative distribution function

$$F_{\overline{Z}_x}(z) = z^{\frac{\mu}{\delta}}, \quad 0 \le z \le 1.$$

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$$f_{\overline{Z}_x}(z) = \frac{\mu}{\delta} z^{\frac{\mu}{\delta}-1}, \quad 0 \le z \le 1.$$

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Proof: We have that $\int_a^\infty f_{T_x}(t)\,dt = e^{-\mu a}$, if $0 \le a < \infty$. Hence, for $0 \le z \le 1$,

$$F_{\overline{Z}_x}(z) = \int_{-(\ln z)/\delta}^{\infty} f_{T_x}(t)\,dt = e^{\mu(\ln z)/\delta} = z^{\frac{\mu}{\delta}}.$$

$\overline{Z}_x$ has density

$$f_{\overline{Z}_x}(z) = F'_{\overline{Z}_x}(z) = \frac{\mu}{\delta} z^{\frac{\mu}{\delta}-1}, \quad 0 \le z \le 1.$$

Suppose that T_x has a $(1-p)$–th quantile ξ_{1-p} such that

$$\mathbb{P}\{T_x < \xi_{1-p}\} = 1 - p = \mathbb{P}\{T_x \leq \xi_{1-p}\}.$$

Given $b, \delta > 0$, $h(t) = be^{-\delta t}$, $t \geq 0$, is a decreasing function. By a previous theorem, the p–th quantile of $Z = be^{-\delta T_x}$ is $be^{-\delta \xi_{1-p}}$.

Example 1

A benefit of \$500 is paid at the failure time T of a home electronic product. The pdf of the time of failure of the product is

$$f_T(t) = \begin{cases} \frac{t}{50} & \text{if } 0 \le t \le 10, \\ 0 & \text{else.} \end{cases}$$

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Solution: (i) By the change of variables $t \ln(1.075) = y$,

$$(500)\overline{A}_x = (500)\int_0^{10} (1.075)^{-t} \frac{t}{50}\, dt$$

$$=(500)\int_0^{10\ln(1.075)} e^{-y} \frac{y}{50(\ln(1.075))^2}\, dy$$

$$=\frac{10}{(\ln(1.075))^2}(-e^{-y})(y+1)\Big|_0^{10\ln(1.075)}$$

$$=\frac{10}{(\ln(1.075))^2}\left(1-(1.075)^{-10}(10\ln(1.075)+1)\right) = 313.3879498.$$

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(ii) Find the density of present value of this benefit.

Solution: (ii) Let $Z = (500)(1.075)^{-T}$ be the present value of the insurance benefit. We have that $T = -\frac{\ln(Z/500)}{\ln(1.075)}$. The density of Z is

$$f_Z(z) = f_T\left(-\frac{\ln(z/500)}{\ln(1.075)}\right)\left|\frac{d}{dz}\left(-\frac{\ln(z/500)}{\ln(1.075)}\right)\right|$$
$$= \frac{-\ln(z/500)}{50\ln(1.075)}\frac{1}{z\ln(1.075)} = \frac{-\ln(z/500)}{50z(\ln(1.075))^2},$$
$$\text{if } 500(1.075)^{-10} \le z \le 500.$$

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Solution: (iii) Let ζ_p be the p–th quantile of Z. Since $y = h(t) = (500)(1.075)^{-t}$ is decreasing, we have that $\zeta_p = h(\xi_{1-p})$, where ζ_{1-p} is a $(1-p)$–th quantile of T. We have that

$$1 - p = \mathbb{P}\{T \le \zeta_{1-p}\} = \frac{\zeta_{1-p}^2}{100}$$

and $\xi_{1-p} = 10\sqrt{1-p}$. Hence,

$$\zeta_p = h(\xi_{1-p}) = h(10\sqrt{1-p}) = 500(1.075)^{-10\sqrt{1-p}}.$$

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(iii) Find the 25, 50 and 75 percentiles of the present value random variable of this benefit.

Solution: (iii) Hence,

$$\zeta_p = h(\xi_{1-p}) = h(10\sqrt{1-p}) = 500(1.075)^{-10\sqrt{1-p}}.$$

The 25 percentile of the benefit is $\zeta_{0.25} = 500(1.075)^{-10\sqrt{0.75}} = 290.6742245$.

The 50 percentile of the benefit is $\zeta_{0.5} = 500(1.075)^{-10\sqrt{0.5}} = 348.2793162$.

The 75 percentile of the benefit is $\zeta_{0.75} = 500(1.075)^{-10\sqrt{0.25}} = 417.300441$.

Recall that the present value of a continuous annuity with unit rate is

$$\bar{a}_{\overline{n}|i} = \int_0^n \nu^t \, dt = \frac{1 - \nu^{-n}}{\delta}.$$

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For the de Moivre model with terminal age ω,

$$\overline{A}_x = \frac{\overline{a}_{\overline{\omega - x}|i}}{\omega - x} = \frac{1 - \nu^{\omega - x}}{\delta(\omega - x)} = \frac{1 - e^{-\delta(\omega - x)}}{\delta(\omega - x)}.$$

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Proof: The density T_x of is $f_{T_x}(t) = \frac{1}{\omega - x}$, $0 \leq x \leq \omega$. Hence,

$$\overline{A}_x = \int_0^\infty e^{-t\delta} \frac{f(x+t)}{s(x)} \, dt = \int_0^{\omega - x} \nu^t \frac{1}{\omega - x} \, dt$$

$$= \frac{\overline{a}_{\overline{\omega - x}|i}}{\omega - x} = \frac{1 - \nu^{\omega - x}}{\delta(\omega - x)} = \frac{1 - e^{-\delta(\omega - x)}}{\delta(\omega - x)}.$$

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$$= \frac{\overline{a}_{\overline{\omega-x}|i}}{\omega - x} = \frac{1 - \nu^{\omega-x}}{\delta(\omega - x)} = \frac{1 - e^{-\delta(\omega-x)}}{\delta(\omega - x)}.$$

By the previous theorem, ${}^m\overline{A}_x = \frac{1 - e^{-m\delta(\omega-x)}}{m\delta(\omega-x)}$.

Example 2

Julia is 40 year old. She buys a whole life policy insurance which will pay \$50,000 at the time of her death. Suppose that the survival function is $s(x) = 1 - \frac{x}{100}$, $0 \leq x \leq 100$*. Suppose that the continuously compounded rate of interest is* $\delta = 0.05$.

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(i) Find the actuarial present value of the benefit of this life insurance.

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(i) Find the actuarial present value of the benefit of this life insurance.

Solution: (i) The density of T_{40} is $f_{T_{40}}(t) = \frac{1}{60}$, $0 \leq t \leq 60$. Hence,

$$\overline{A}_{40} = \frac{\overline{a}_{\overline{60}|i}}{60} = \frac{1 - e^{-(0.05)(60)}}{(0.05)(60)} = \frac{1 - e^{-3}}{3} = 0.3167376439.$$

The actuarial present value is $(50000)(0.3167376439) = 15836.88219$.

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(ii) Find the standard deviation of the present value of the benefit of this life insurance.

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(ii) Find the standard deviation of the present value of the benefit of this life insurance.

Solution: (ii) We have that

$$^2\overline{A}_{40} = \frac{\overline{a}_{\overline{60}|(1+i)^2-1}}{60} = \frac{1 - e^{-(2)(0.05)(60)}}{(2)(0.05)(60)} = \frac{1 - e^{-6}}{6} = 0.1662535413,$$

$$\mathrm{Var}(50000\overline{Z}_{40}) = (50000)^2(0.1662535413 - (0.3167376439)^2)$$

$$=164827015.6.$$

The standard deviation of $50000\overline{Z}_{40}$ is $\sqrt{164827015.6} = 12838.4974$.

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(iii) Find the density of the present value of the benefit of this life insurance.

Solution: (iii) The present value is $Y = 50000\overline{Z}_{40} = (50000)e^{-(0.05)T_{40}}$. Let $h : (0, 60) \to ((50000)e^{-3}, 50000)$ be defined by $h(t) = (50000)e^{-(0.05)t}$. Then, $h^{-1}(y) = -\frac{\ln(y/50000)}{0.05}$ and the density of Y is

$$f_Y(y) = f_{T_{40}}(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right| = \frac{1}{60} \frac{1}{y(0.05)} = \frac{1}{3y},$$

if $(50000)e^{-3} \le y \le 50000$.

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Julia is 40 year old. She buys a whole life policy insurance which will pay $50,000 at the time of her death. Suppose that the survival function is $s(x) = 1 - \frac{x}{100}$, $0 \le x \le 100$. *Suppose that the continuously compounded rate of interest is* $\delta = 0.05$.

(iv) Find the median of the present value of the benefit of this life insurance.

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(iv) Find the median of the present value of the benefit of this life insurance.

Solution: (iv) Let m be the median of the present value of the benefit of this life insurance. Let m' be the median of T_{40}. We have that $m = h(m')$, where $h(x) = (50000)e^{-(0.05)x}$, $x \ge 0$. We have

$$0.5 = \mathbb{P}\{T_{40} \le m'\} = \frac{m'}{60}$$

and $m' = 30$. Hence, $m = h(m') = (50000)e^{-(30)(0.05)} = 11156.50801$.

Theorem 4

For the constant force of mortality model with mortality rate μ, $\overline{A}_x = \frac{\mu}{\mu+\delta}$.

Proof.

Since $f(x) = \mu e^{-\mu x}$

$$\overline{A}_x = \int_0^\infty e^{-t\delta} \frac{f(x+t)}{s(x)}\, dt = \int_0^\infty e^{-t\delta} \mu e^{-\mu t}\, dt$$

$$= \frac{-\mu e^{-t(\mu+\delta)}}{\mu+\delta} \Big|_0^\infty = \frac{\mu}{\mu+\delta}.$$

□

By the previous theorem,

$$^m\overline{A}_x = \frac{\mu}{\mu+m\delta}.$$

Example 3

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(i) Find the APV of the benefit of the a life insurance to (x) with unity payment paid the time of the death.

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Solution: (i) We have that

$$\overline{A}_x = \frac{\mu}{\mu + \delta} = \frac{0.05}{0.05 + 0.06} = 0.4545454545.$$

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(ii) Find the variance of the present value of the benefit of a life insurance to (x) with unity payment paid the time of the death.

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(ii) Find the variance of the present value of the benefit of a life insurance to (x) with unity payment paid the time of the death.

Solution: (ii) We have that

$$ {}^2\overline{A}_x = \frac{\mu}{\mu + 2\delta} = \frac{0.05}{0.05 + (2)(0.06)} = 0.2941176471. $$

and

$$ \mathrm{Var}(\overline{Z}_x) = 0.2941176471 - (0.4545454545)^2 = 0.08750607689. $$

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(iii) Find the density of the present value of the benefit of a life insurance to (x) with unity payment paid the time of the death.

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(iii) Find the density of the present value of the benefit of a life insurance to (x) with unity payment paid the time of the death.

Solution: (iii) $\overline{Z}_x = e^{-(0.06)T_x}$. Let $h : (0,\infty) \to (0,1)$ be defined by $h(t) = e^{-(0.06)t}$. Then, $h^{-1}(z) = \frac{-\ln z}{0.06}$. Hence, the density of $\overline{Z}_x$ is

$$f_{\overline{Z}_x}(z) = f_{T_{40}}(h^{-1}(z)) \left| \frac{d}{dz} h^{-1}(z) \right| = (0.05) e^{-(0.05)\frac{-\ln z}{0.06}} \frac{1}{z(0.06)}$$

$$= \frac{5z^{-1/6}}{6}, 0 \le z \le 1.$$

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The force of interest is 0.06. (x) *has a constant force of mortality of* 0.05.

(iv) Find the first and third quartile of the present value of the benefit of a life insurance to (x) with unity payment paid the time of the death.

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The force of interest is 0.06*.* (x) *has a constant force of mortality of* 0.05*.*

(iv) Find the first and third quartile of the present value of the benefit of a life insurance to (x) with unity payment paid the time of the death.

Solution: (iv) Let ξ_p be the p–th percentile of T_x. We have that

$$p = \mathbb{P}\{T_x \le \xi_p\} = 1 - e^{-(0.05)\xi_p},$$

So, $\xi_p = \frac{-\ln(1-p)}{0.05}$. Let ζ_p be the p–th percentile of $Z_x = e^{-(0.06)T_x}$. Then, $\zeta_p = e^{-(0.06)\xi_{1-p}} = e^{-(0.06)\frac{-\ln p}{0.05}} = p^{6/5}$. The first quartile of $\overline{Z}_x$ is $(0.25)^{6/5} = 0.1894645708138$. The third quartile of $\overline{Z}_x$ is $(0.75)^{6/5} = 0.708065633471176$.

n–year term life insurance.

For an n–year term life insurance, a payment is made if the failure happens within the n–year term of an insurance commencing at issue. So, a payment is made only if the failure happens before n years.

Definition 3

The present value of a unit payment n–year term life insurance paid at the moment of of death is denoted by $\overline{Z}^{1}_{x:\overline{n}|}$.

We have that

$$\overline{Z}^{1}_{x:\overline{n}|} = \nu^{T_x} I(T_x \le n) = \begin{cases} \nu^{T_x} & \text{if } T_x \le n \\ 0 & \text{if } n < T_x \end{cases}$$

Definition 4

The actuarial present value of a unit payment n–year term life insurance paid at the moment of of death is denoted by $\overline{A}^{1}_{x:\overline{n}|}$.

We have that

$$\overline{A}^{1}_{x:\overline{n}|} = E[\overline{Z}^{1}_{x:\overline{n}|}] = \int_0^n \nu^t f_{T_x}(t)\,dt = \int_0^n \nu^t\, {}_tp_x \mu_{x+t}\,dt.$$

We denote by ${}^m\overline{A}_x$ to the APV of a unit payment n–year term life insurance paid at the moment of of death at m times the original force of interest. We have that

$${}^m\overline{A}^{1}_{x:\overline{n}|} = E[\overline{Z}^{1\ m}_{x:\overline{n}|}] = \int_0^n \nu^{mt} f_{T_x}(t)\,dt = \int_0^n \nu^{mt}\, {}_tp_x \mu_{x+t}\,dt$$

It is easy to see that

$$\mathrm{Var}(\overline{Z}^{1}_{x:\overline{n}|}) = {}^2A^{1}_{x:\overline{n}|} - {A^{1}_{x:\overline{n}|}}^2.$$

Theorem 5

The cumulative distribution function of $\overline{Z}^{1}_{x:\overline{n}|} = e^{-\delta T_x} I(T_x \le n)$ *is*

$$F_{\overline{Z}^{1}_{x:\overline{n}|}}(z) = \begin{cases} 0 & \text{if } z < 0, \\ {}_n p_x & \text{if } 0 \le z \le e^{-n\delta}, \\ S_{T_x}\left(\frac{-\ln z}{\delta}\right) & \text{if } e^{-n\delta} \le z \le 1, \\ 1 & \text{if } 1 \le z. \end{cases}$$

Proof: If $0 \le z < 1$, then

$$
\begin{aligned}
&F_{\overline{Z}^1_{x:\overline{n}|}}(z) = \mathbb{P}\{\overline{Z}^1_{x:\overline{n}|} \le z\} \\
=&\mathbb{P}\{e^{-\delta T_x} I(T_x \le n) \le z\} = \mathbb{P}\{T_x > n\} + \mathbb{P}\{T_x \le n, e^{-\delta T_x} \le z\} \\
=&\mathbb{P}\{T_x > n\} + \mathbb{P}\{T_x \le n, e^{-\delta T_x} \le z\} \\
=&\mathbb{P}\{T_x > n\} + \mathbb{P}\{T_x \le n, -\ln(z)/\delta \le T_x\} \\
=&\mathbb{P}\{-\ln(z)/\delta \le T_x \le n\} + \mathbb{P}\{n < T_x\}.
\end{aligned}
$$

If $0 \le z < e^{-n\delta}$, then

$$
F_{\overline{Z}^1_{x:\overline{n}|}}(z) = \mathbb{P}\{-\ln(z)/\delta \le T_x \le n\} + \mathbb{P}\{n < T_x\} = \mathbb{P}\{n < T_x\} = {}_np_x.
$$

If $e^{-n\delta} \le z \le 1$,

$$
\begin{aligned}
&F_{\overline{Z}^1_{x:\overline{n}|}}(z) = \mathbb{P}\{-\ln(z)/\delta \le T_x \le n\} + \mathbb{P}\{n < T_x\} \\
=&\mathbb{P}\{-\ln(z)/\delta \le T_x\} = S_{T_x}\left(\frac{-\ln z}{\delta}\right).
\end{aligned}
$$

Theorem 6

Suppose $n \le \omega - x$. $\overline{Z}^{1}_{x:\overline{n}|}$ has a mixed distribution. The probability density function of the continuous part of $\overline{Z}^{1}_{x:\overline{n}|}$ is

$$f_{\overline{Z}^{1}_{x:\overline{n}|}}(z) = \begin{cases} \frac{f_{T_x}(-(\ln z)/\delta)}{\delta z} & \text{if } e^{-n\delta} \le z \le 1, \\ 0 & \text{else.} \end{cases}$$

The probability mass function of the discrete part of $\overline{Z}^{1}_{x:\overline{n}|}$ is

$$p_{\overline{Z}^{1}_{x:\overline{n}|}}(z) = \begin{cases} {}_n p_x & \text{if } z = 0, \\ 0 & \text{else.} \end{cases}$$

Theorem 7

Under the de Moivre model, if $n \le \omega - x$,

$$\overline{A}^{1}_{x:\overline{n}|} = \frac{\overline{a}_{\overline{n}|}}{\omega - x} = \frac{1-\nu^n}{\delta(\omega - x)} = \frac{1-e^{-\delta n}}{\delta(\omega - x)}.$$

Proof: The density T_x of is $f_{T_x}(t) = \frac{1}{\omega - x}$, $0 \le x \le \omega$. Hence,

$$\overline{A}^{1}_{x:\overline{n}|} = \int_0^n e^{-t\delta} \frac{f(x+t)}{s(x)}\,dt = \int_0^n \nu^t \frac{1}{\omega - x}\,dt$$
$$= \frac{\overline{a}_{\overline{n}|i}}{\omega - x} = \frac{1-\nu^n}{\delta(\omega - x)} = \frac{1-e^{-\delta n}}{\delta(\omega - x)}.$$

Example 4

Julia is 40 year old. She buys a 15–year term life policy insurance which will pay \$50,000 at the time of her death. Suppose that the survival function is $s(x) = 1 - \frac{x}{100}$, $0 \le x \le 100$. *Suppose that the continuously compounded rate of interest is* $\delta = 0.05$.

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(i) Find the actuarial present value of the benefit of this life insurance.

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(i) Find the actuarial present value of the benefit of this life insurance.

Solution: (i) The density of T_{40} is $f_{T_{40}}(t) = \frac{1}{60}$, $0 \le t \le 60$. Hence,

$$\overline{A}^{1}_{40:\overline{15}|} = \frac{\overline{a}_{\overline{15}|i}}{60} = \frac{1 - e^{-(15)(0.05)}}{(60)(0.05)} = \frac{1 - e^{-0.75}}{3} = 0.1758778158.$$

The actuarial present value is $(50000)(0.1758778158) = 8793.89079$.

Example 4

Julia is 40 year old. She buys a 15–year term life policy insurance which will pay $50,000 at the time of her death. Suppose that the survival function is $s(x) = 1 - \frac{x}{100}$, $0 \le x \le 100$. *Suppose that the continuously compounded rate of interest is* $\delta = 0.05$.

(ii) Find the standard deviation of the present value of the benefit of this life insurance.

Example 4

Julia is 40 year old. She buys a 15–year term life policy insurance which will pay \$50,000 at the time of her death. Suppose that the survival function is $s(x) = 1 - \frac{x}{100}$, $0 \leq x \leq 100$. *Suppose that the continuously compounded rate of interest is* $\delta = 0.05$.

(ii) Find the standard deviation of the present value of the benefit of this life insurance.

Solution: (ii) We have that

$$^2\overline{A}^1_{40:\overline{15}|} = \frac{\overline{a}_{\overline{15}|(1+i)^2-1}}{60} = \frac{1 - e^{-(15)(2)(0.05)}}{(60)(2)(0.05)} = \frac{1 - e^{-1.5}}{6} = 0.1294783066$$

$$\mathrm{Var}(50000\overline{Z}^1_{40:\overline{15}|}) = (50000)^2(0.1294783066 - (0.1758778158)^2)$$

$$= 246363251.273553.$$

The standard deviation of $50000\overline{Z}^1_{40:\overline{15}|}$ is $\sqrt{246363251.273553} = 15695.9628973043$.

Theorem 8

Under constant force of mortality μ,

$$\overline{A}^{1}_{x:\overline{n}|} = \frac{(1 - e^{-n(\mu+\delta)})\mu}{\mu + \delta}.$$

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Proof:

$$\overline{A}^{1}_{x:\overline{n}|} = \int_0^n e^{-t\delta} {}_tp_x \mu_x(t)\, dt = \int_0^n e^{-t\delta} \mu e^{-t\mu}\, dt$$

$$= \int_0^n \mu e^{-t(\delta+\mu)}\, dt = \frac{\mu(1 - e^{-n(\mu+\delta)})}{\mu + \delta}.$$

Example 5

Find the APV of a 15*-year term life insurance to* (x) *with unity payment if* $\delta = 0.06$ *and* (x) *has a constant force of mortality* $\mu = 0.05$*.*

Example 5

Find the APV of a 15-year term life insurance to (x) with unity payment if $\delta = 0.06$ and (x) has a constant force of mortality $\mu = 0.05$.

Solution: We have that

$$\overline{A}^{1}_{x:\overline{15}|} = \frac{(0.05)(1 - e^{-(15)(0.05+0.06)})}{0.05 + 0.06} = \frac{(0.05)(1 - e^{-(15)(0.11)})}{0.11}$$
$$= 0.3672500415.$$

n–year deferred life insurance.

In the case of n–year deferred life insurance, a payment is made only if the failure happens at least n years following policy issue.

Definition 5

The present value of a unit payment n–year deferred life insurance paid at the moment of of death is denoted by ${}_{n|}\overline{Z}_x$.

We have that

$$ {}_{n|}\overline{Z}_x = e^{-\delta T_x} I(n < T_x) = \begin{cases} 0 & \text{if } T_x \le n, \\ e^{-\delta T_x} & \text{if } n < T_x. \end{cases} $$

Definition 6

The actuarial present value of a unit payment n–year deferred life insurance paid at the moment of of death is denoted by ${}_{n|}\overline{A}_x$.

We have that

$$ {}_{n|}\overline{A}_x = E[{}_{n|}\overline{Z}_x] = \int_n^\infty e^{-\delta t} f_{T_x}(t)\, dt = \int_n^\infty e^{-\delta t}\, {}_t p_x \mu_{x+t}\, dt $$

We denote by ${}^m\overline{A}_x$ to the APV of a unit payment n–year deferred life insurance paid at the moment of of death at m times the original force of interest. We have that

$$ {}^m{}_{n|}\overline{A}_x = E[{}_{n|}\overline{Z}_x{}^m] = \int_n^\infty e^{-m\delta t} f_{T_x}(t)\, dt = \int_n^\infty e^{-m\delta t}\, {}_t p_x \mu_{x+t}\, dt $$

It is easy to see that

$$ \mathrm{Var}({}_{n|}\overline{Z}_x) = {}^2{}_{n|}\overline{A}_x - {}_{n|}\overline{A}_x{}^2. $$

Theorem 9

The cumulative distribution function of ${}_{n|}\overline{Z}_x = e^{-\delta T_x} I(n < T_x)$ *is*

$$F_{{}_{n|}\overline{Z}_x}(z) = \begin{cases} 0 & \text{if } z < 0, \\ {}_nq_x + S_{T_x}\left(\frac{-\ln z}{\delta}\right) & \text{if } 0 \le z \le e^{-n\delta}, \\ 1 & \text{if } 1 \le z. \end{cases}$$

Proof: If $0 \le z \le e^{-n\delta}$,

$$F_{{}_{n|}\overline{Z}_x}(z) = \mathbb{P}\{{}_{n|}\overline{Z}_x \le z\} = \mathbb{P}\{e^{-\delta T_x} I(n < T_x) \le z\}$$
$$= \mathbb{P}\{T_x \le n\} + \mathbb{P}\{n < T_x, e^{-\delta T_x} \le z\}$$
$$= \mathbb{P}\{T_x \le n\} + \mathbb{P}\{n < T_x, -\ln(z)/\delta \le T_x\}.$$

If $0 \le z \le e^{-n\delta}$,

$$F_{{}_{n|}\overline{Z}_x}(z) = \mathbb{P}\{T_x \le n\} + \mathbb{P}\{n < T_x, -\ln(z)/\delta \le T_x\}$$
$$= \mathbb{P}\{T_x \le n\} + \mathbb{P}\{-\ln(z)/\delta \le T_x\}$$
$$= {}_nq_x + S_{T_x}\left(\frac{-\ln z}{\delta}\right).$$

If $e^{-n\delta} \le z \le 1$,

$$F_{{}_{n|}\overline{Z}_x}(z) = \mathbb{P}\{T_x \le n\} + \mathbb{P}\{n < T_x, -\ln(z)/\delta \le T_x\}$$
$$= \mathbb{P}\{T_x \le n\} + \mathbb{P}\{n < T_x\} = 1.$$

Theorem 10

${}_{n|}\overline{Z}_x$ has a mixed distribution. The probability density function of the continuous part of ${}_{n|}\overline{Z}_x$ is

$$f_{{}_{n|}\overline{Z}_x}(z) = \begin{cases} \frac{f_{T_x}(-(\log z)/\delta)}{\delta z} & \text{if } e^{-(\omega-x)\delta} \le z \le e^{-n\delta}, \\ 0 & \text{else.} \end{cases}$$

The probability mass function of the discrete part of ${}_{n|}\overline{Z}_x$ is

$$p_{{}_{n|}\overline{Z}_x}(z) = \begin{cases} {}_nq_x & \text{if } z = 0, \\ 0 & \text{else.} \end{cases}$$

Theorem 11

Under De Moivre's model, if $n \le \omega - x$,

$$ {}_{n|}\overline{A}_x = e^{-n\delta} \frac{\overline{a}_{\overline{\omega-x-n}|i}}{\omega - x}. $$

Theorem 11

Under De Moivre's model, if $n \le \omega - x$,

$$_{n|}\overline{A}_x = e^{-n\delta}\frac{\overline{a}_{\overline{\omega-x-n}|i}}{\omega - x}.$$

Proof: The density T_x of is $f_{T_x}(t) = \frac{1}{\omega-x}$, $0 \le x \le \omega$. Hence,

$$_{n|}\overline{A}_x = \int_n^{\omega-x} e^{-t\delta}\frac{1}{\omega}\,dt = \int_0^{\omega-x-n} e^{-(t+n)\delta}\frac{1}{\omega}\,dt = e^{-n\delta}\frac{\overline{a}_{\overline{\omega-x-n}|i}}{\omega - x}.$$

Example 6

Julia is 40 year old. She buys a 15*-year term deferred life policy insurance which will pay \$50,000 at the time of her death. Suppose that the survival function is* $s(x) = 1 - \frac{x}{100}$, $0 \leq x \leq 100$*. Suppose that the continuously compounded rate of interest is* $\delta = 0.05$*.*

Example 6

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(i) Find the actuarial present value of the benefit of this life insurance.

Example 6

Julia is 40 year old. She buys a 15-year term deferred life policy insurance which will pay \$50,000 at the time of her death. Suppose that the survival function is $s(x) = 1 - \frac{x}{100}$, $0 \le x \le 100$. Suppose that the continuously compounded rate of interest is $\delta = 0.05$.

(i) Find the actuarial present value of the benefit of this life insurance.

Solution: (i) The density of T_{40} is $f_{T_{40}}(t) = \frac{1}{60}$, $0 \le t \le 60$. Hence,

$$
{}_{15|}\overline{A}_x = e^{-(15)(0.05)} \frac{\overline{a}_{\overline{60-15}|i}}{60} = e^{-(15)(0.05)} \frac{1 - e^{-(45)(0.05)}}{(0.05)(60)}
$$

$$
= \frac{e^{-0.75} - e^{-3}}{3} = 0.1408598281.
$$

The actuarial present value is $(50000)(0.1408598281) = 7042.991405$.

Example 6

Julia is 40 year old. She buys a 15*-year term deferred life policy insurance which will pay \$50,000 at the time of her death. Suppose that the survival function is* $s(x) = 1 - \frac{x}{100}$, $0 \leq x \leq 100$. *Suppose that the continuously compounded rate of interest is* $\delta = 0.05$.

(ii) Find the standard deviation of the present value of the benefit of this life insurance.

Example 6

Julia is 40 year old. She buys a 15-year term deferred life policy insurance which will pay \$50,000 at the time of her death. Suppose that the survival function is $s(x) = 1 - \frac{x}{100}$, $0 \le x \le 100$. Suppose that the continuously compounded rate of interest is $\delta = 0.05$.

(ii) Find the standard deviation of the present value of the benefit of this life insurance.

Solution: (ii) We have that

$$ {}^2{}_{15|}\overline{A}_x = e^{-(15)(2)(0.05)} \frac{\overline{a}_{\overline{60-15}|(1+i)^2-1}}{60} = e^{-(15)(0.05)} \frac{1 - e^{-(45)(2)(0.05)}}{(2)(0.05)(60)} $$

$$ = \frac{e^{-1.5} - e^{-6}}{6} = 0.03677523466, $$

$$ \mathrm{Var}(50000 \cdot {}_{15|}\overline{Z}_x) = (50000)^2(0.03677523466 - (0.1408598281)^2) $$

$$ = 42334358.72. $$

The standard deviation is $\sqrt{42334358.72} = 6506.485896$.

Theorem 12

Under constant force of mortality μ,

$$ {}_{n|}\overline{A}_x = e^{-n(\mu+\delta)} \frac{\mu}{\mu+\delta}. $$

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$$_{n|}\overline{A}_x = e^{-n(\mu+\delta)} \frac{\mu}{\mu+\delta}.$$

Proof:

$$_{n|}\overline{A}_x = \int_n^\infty e^{-t\delta} \mu e^{-t\mu}\, dt = e^{-n(\mu+\delta)} \frac{\mu}{\mu+\delta}.$$

Example 7

Find the APV of a 15*-year term deferred life insurance to* (x) *with unity payment if* $\delta = 0.06$ *and* (x) *has a constant force of mortality* $\mu = 0.05$.

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Find the APV of a 15*-year term deferred life insurance to* (x) *with unity payment if* $\delta = 0.06$ *and* (x) *has a constant force of mortality* $\mu = 0.05$.

Solution 1: We have that

$$_{15|}\overline{A}_x = e^{-(15)(0.05+0.06)} \frac{(0.05)}{0.05 + 0.06} = 0.08729541301.$$

Example 7

Find the APV of a 15*-year term deferred life insurance to* (x) *with unity payment if* $\delta = 0.06$ *and* (x) *has a constant force of mortality* $\mu = 0.05$.

Solution 1: We have that

$$_{15|}\overline{A}_x = e^{-(15)(0.05+0.06)}\frac{(0.05)}{0.05+0.06} = 0.08729541301.$$

Solution 2: We have that

$$_{15|}\overline{A}_x = \int_{15}^{\infty} \nu^t\,{}_tp_x\mu_{x+t}\,dt = \int_{15}^{\infty} e^{-(0.06)t}e^{-(0.05)t}(0.05)\,dt$$

$$=\frac{(0.05)e^{-(15)(0.11)}}{0.11} = 0.08729541301.$$

n–year pure endowment life insurance.

In the case of an n–year pure endowment life insurance, a payment at the end of n years is made if and only if the failure happens at least n years after issuing the policy. The present value of the benefit payment is

$$\overline{Z}_{x:\overline{n}|}^{\ \ 1} = \nu^n I(n < T_x) = \begin{cases} 0 & \text{if } T_x \le n, \\ \nu^n & \text{if } n < T_x. \end{cases}$$

Notice that this is exactly the insurance as in the discrete case, i.e. $Z_{x:\overline{n}|}^{\ \ 1} = \overline{Z}_{x:\overline{n}|}^{\ \ 1}$. There is no difference between the discrete and continuous case. We have that

$$\overline{A}_{x:\overline{n}|}^{\ \ 1} = E[\overline{Z}_{x:\overline{n}|}^{\ \ 1}] = \nu^n \mathbb{P}\{T_x > n\},$$

$${}^2\overline{A}_{x:\overline{n}|}^{\ \ 1} = E[\overline{Z}_{x:\overline{n}|}^{\ \ 1\ 2}] = \nu^{2n} \mathbb{P}\{T_x > n\},$$

and

$$\mathrm{Var}(\overline{Z}_{x:\overline{n}|}^{\ \ 1}) = {}^2\overline{A}_{x:\overline{n}|}^{\ \ 1} - A_{x:\overline{n}|}^{\ \ 1\ 2} = \nu^2 \cdot {}_n p_x \cdot {}_n q_x.$$

Example 8

Find the APV of a 15*-year pure endowment life insurance to* (x) *with unity payment if* $\delta = 0.06$ *and* (x) *has a constant force of mortality* $\mu = 0.05$.

Example 8

Find the APV of a 15-year pure endowment life insurance to (x) with unity payment if $\delta = 0.06$ and (x) has a constant force of mortality $\mu = 0.05$.

Solution: We have that

$$\overline{A}_{x:\overset{1}{\overline{15|}}} = \nu^{15}{}_{15}p_x = e^{-(0.06)(15)}e^{-(0.05)(15)} = 0.1920499086.$$

n–year endowment life insurance.

In the case of an n–year endowment life insurance, a payment is made at either the time of death or in n years, which ever comes first.

Definition 7

The present value of a unit payment n–year endowment life insurance paid at the moment of of death is denoted by $\overline{Z}^{1}_{x:\overline{n}|}$.

We have that

$$\overline{Z}_{x:\overline{n}|} = \nu^{\min(T_x, n)} = \begin{cases} \nu^{T_x} & \text{if } T_x \le n, \\ \nu^{n} & \text{if } n < T_x. \end{cases}$$

Definition 8

The actuarial present value of a unit payment n–year endowment life insurance paid at the moment of of death is denoted by $\overline{A}_{x:\overline{n}|}$.

We have that

$$\overline{A}_{x:\overline{n}|} = E[\overline{Z}_{x:\overline{n}|}] = \int_0^n \nu^t f_{T_x}(t)\,dt + \nu^n \mathbb{P}\{T_x > n\}$$

$$= \int_0^n \nu^t \, {}_tp_x \mu_{x+t}\,dt + \nu^n \, {}_np_x.$$

We denote by ${}^m\overline{A}_{x:\overline{n}|}$ to the APV of a unit payment n–year endowment life insurance paid at the moment of of death at m times the original force of interest. We have that

$${}^m\overline{A}_{x:\overline{n}|} = E[\overline{Z}^m_{x:\overline{n}|}] = \int_0^n \nu^{mt} f_{T_x}(t)\,dt + \nu^{mn} \mathbb{P}\{T_x > n\}$$

$$= \int_0^n \nu^{mt} \, {}_tp_x \mu_{x+t}\,dt + \nu^{mn} \, {}_np_x.$$

It is easy to see that $\mathrm{Var}(\overline{Z}_{x:\overline{n}|}) = {}^2\overline{A}_{x:\overline{n}|} - \overline{A}_{x:\overline{n}|}^{\,2}$.

Theorem 13

Suppose that $n < \omega - x$. The cumulative distribution function of $\overline{Z}_{x:\overline{n}|} = e^{-\delta \min(T_x, n)}$ is

$$F_{\overline{Z}_{x:\overline{n}|}}(z) = \begin{cases} 0 & \text{if } z < e^{-n\delta}, \\ S_{T_x}\left(\frac{-\ln z}{\delta}\right) & \text{if } e^{-n\delta} \le z \le 1, \\ 1 & \text{if } 1 \le z. \end{cases}$$

Proof: Since $e^{-n\delta} \le e^{-\delta \min(T_x, n)} \le 1$,

$$F_{\overline{Z}_{x:\overline{n}|}}(z) = 0, \text{if } z < e^{-n\delta},$$

and

$$F_{\overline{Z}_{x:\overline{n}|}}(z) = 1, \text{if } 1 \le z.$$

If $e^{-n\delta} \le z \le 1$, then

$$\begin{aligned} &F_{\overline{Z}_{x:\overline{n}|}}(z) = \mathbb{P}\{\overline{Z}_{x:\overline{n}|} \le z\} = \mathbb{P}\{e^{-\delta \min(T_x, n)} \le z\} \\ =&\mathbb{P}\{-\ln(z/\delta) \le \min(T_x, n)\} = \mathbb{P}\{-\ln(z/\delta) \le T_x\} \\ =&S_{T_x}\left(\frac{-\ln z}{\delta}\right) \end{aligned}$$

Theorem 14

Suppose that $n < \omega - x$. $\overline{Z}_{x:\overline{n}|}$ has a mixed distribution. The probability density function of the continuous part of $\overline{Z}_{x:\overline{n}|}$ is

$$f_{\overline{Z}_{x:\overline{n}|}}(z) = \begin{cases} \frac{f_{T_x}(-(\ln z)/\delta)}{\delta z} & \text{if } e^{-n\delta} \le z \le 1, \\ 0 & \text{else.} \end{cases}$$

The probability mass function of the discrete part of $\overline{Z}_{x:\overline{n}|}$ is

$$p_{\overline{Z}_{x:\overline{n}|}}(z) = \begin{cases} {}_n p_x & \text{if } z = e^{-n\delta}, \\ 0 & \text{else.} \end{cases}$$

Example 9

Julia is 40 year old. She buys a 15–year endowment policy insurance which will pay \$50,000 at the time of her death, or in 15 years whatever comes first. Suppose that the survival function is $s(x) = 1 - \frac{x}{100}$, $0 \le x \le 100$. *Suppose that* $\delta = 0.05$.

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(i) Find the actuarial present value of the benefit of this life insurance.

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(i) Find the actuarial present value of the benefit of this life insurance.

Solution: (i) The density of T_{40} is $f_{T_{40}}(t) = \frac{1}{60},\ 0 \le t \le 60$.

$$\overline{A}_{40:\overline{15}|} = \int_0^{15} e^{-t(0.05)} \frac{1}{60}\, dt + e^{-(15)(0.05)} \frac{60-15}{60}$$

$$= \frac{-e^{-t(0.05)}}{(60)(0.05)} \Big|_0^{15} + e^{-0.75}(0.75) = \frac{1 - e^{-(15)(0.05)}}{(60)(0.05)} + e^{-0.75}(0.75)$$

$$= \frac{1 - e^{-0.75}}{3} + e^{-0.75}(0.75) = 0.5301527303.$$

The APV is $(50000)(0.5301527303) = 26507.63652$.

Example 9

Julia is 40 year old. She buys a 15–year endowment policy insurance which will pay \$50,000 at the time of her death, or in 15 years whatever comes first. Suppose that the survival function is $s(x) = 1 - \frac{x}{100}$, $0 \leq x \leq 100$. *Suppose that* $\delta = 0.05$.

(ii) Find the standard deviation of the present value of the benefit of this life insurance.

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Julia is 40 year old. She buys a 15–year endowment policy insurance which will pay \$50,000 at the time of her death, or in 15 years whatever comes first. Suppose that the survival function is $s(x) = 1 - \frac{x}{100}$, $0 \le x \le 100$. *Suppose that* $\delta = 0.05$.

(ii) Find the standard deviation of the present value of the benefit of this life insurance.

Solution: (ii) We have that

$$
{}^{2}\overline{A}_{40:\overline{15}|} = \int_0^{15} e^{-t(2)(0.05)} \frac{1}{60}\,dt + e^{-(15)(2)(0.05)} \frac{60-15}{60}
$$

$$
= \frac{-e^{-t(2)(0.05)}}{(60)(2)(0.05)} \Big|_0^{15} + e^{-1.5}(0.75) = \frac{1 - e^{-(15)(2)(0.05)}}{(60)(2)(0.05)} + e^{-1.5}(0.75)
$$

$$
= \frac{1 - e^{-1.5}}{6} + e^{-1.5}(0.75) = 0.2968259268.
$$

Example 9

Julia is 40 year old. She buys a 15–year endowment policy insurance which will pay \$50,000 at the time of her death, or in 15 years whatever comes first. Suppose that the survival function is $s(x) = 1 - \frac{x}{100}$, $0 \le x \le 100$. *Suppose that* $\delta = 0.05$.

(ii) Find the standard deviation of the present value of the benefit of this life insurance.

Solution: (ii)

$$\mathrm{Var}(50000\overline{Z}_{40:\overline{15}|}) = (50000)^2(0.2968259268 - (0.5301527303)^2)$$

$$=39410023.39.$$

The standard deviation of $50000\overline{Z}_{40:\overline{15}|}$ is $\sqrt{39410023.39} = 6277.740309$.

Example 10

Find the APV of a 15*–year endowment life insurance to* (x) *with unity payment if* $\delta = 0.06$ *and* (x) *has a constant force of mortality* $\mu = 0.05$*.*

Example 10

Find the APV of a 15–year endowment life insurance to (x) with unity payment if $\delta = 0.06$ and (x) has a constant force of mortality $\mu = 0.05$.

Solution: We have that

$$\overline{A}_{x:\overline{15}|} = \int_0^{15} \nu^t \, {}_tp_x \mu_{x+t} \, dt + \nu^{15} \, {}_{15}p_x$$

$$= \int_0^{15} e^{-(0.06)t} e^{-(0.05)t} (0.05) \, dt + e^{-(0.06)(15)} \int_{15}^{\infty} e^{-(0.05)t} (0.05) \, dt$$

$$= \frac{(0.05)(1 - e^{-(0.11)(15)})}{(0.11)} + e^{-(0.06)(15)} e^{-(0.05)(15)} = 0.2793453216.$$

m–year deferred n–year term life insurance.

Definition 9

An m–year deferred n–year term life insurance if it makes a payment if death happens during the period of n years that starts m years from now.

Definition 10

The present value of an m–year deferred n–year term life insurance with unit payment paid at time of death is denoted by ${}_{m|n}\overline{Z}_x$.

We have that

$$ {}_{m|n}\overline{Z}_x = v^{T_x} I(m \le T_x < m+n) = \begin{cases} v^{T_x} & \text{if } m \le T_x < m+n, \\ 0 & \text{else.} \end{cases} $$

Definition 11

The actuarial present value of an m–year deferred n–year term life insurance with unit payment paid at end of year of death is denoted by ${}_{m|n}\overline{A}_x$.

We have that

$$ {}_{m|n}\overline{A}_x = E[{}_{m|n}\overline{Z}_x] = \int_m^{m+n} v^t f_{T_x}(t)\,dt = \int_m^{m+n} v^t\,{}_tp_x\mu_{x+t}\,dt, $$

$$ {}_{m|n}{}^2\overline{A}_x = E[({}_{m|n}\overline{Z}_x)^2] = \int_m^{m+n} v^{2t} f_{T_x}(t)\,dt = \int_m^{m+n} v^{2t}\,{}_tp_x\mu_{x+t}\,dt, $$

and

$$ \mathrm{Var}({}_{m|n}\overline{Z}_x) = {}_{m|n}{}^2\overline{A}_x - \left({}_{m|n}\overline{A}_x\right)^2. $$

Theorem 15

Under De Moivre's law,

$$ {}_{m|n}\overline{A}_x = \frac{v^m \overline{a}_{\overline{n}|}}{\omega - x}. $$

Theorem 15

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Proof:

$$ {}_{m|n}\overline{A}_x = \int_m^{m+n} v^t f_{T_x}(t)\, dt = \int_m^{m+n} v^t \frac{1}{\omega - x} dt $$
$$ = \frac{1}{\omega - x} \int_0^n v^{m+t} dt = \frac{v^m \overline{a}_{\overline{n}|}}{\omega - x}. $$

Theorem 16

$$ {}_{m|n}\overline{A}_x = {}_mE_x \cdot \overline{A}^{\,1}_{x+m:\overline{n}|}. $$

Theorem 16

$$ {}_{m|n}\overline{A}_x = {}_mE_x \cdot \overline{A}^{\,1}_{x+m:\overline{n}|}. $$

Proof: By a previous theorem, ${}_mp_x \cdot {}_{k-1|}q_{x+m} = {}_{m+k-1|}q_x$. Hence,

$$ {}_mE_x\overline{A}^{\,1}_{x+m:\overline{n}|} = {}_mp_x v^m \int_0^n v^t\, {}_tp_{x+m}\mu_{x+m+t}\, dt $$
$$ = \int_0^n v^{m+t}\, {}_{m+t}p_x\mu_{x+m+t}\, dt = \int_m^{m+n} v^t\, {}_tp_x\mu_{x+t}\, dt = {}_{m|n}\overline{A}_x. $$

Theorem 17

Under the constant force of mortality model,

$$ {}_{m|n}\overline{A}_x = e^{-m(\mu+\delta)} \left(1 - e^{-n(\mu+\delta)}\right) \frac{\mu}{\mu+\delta}. $$

Theorem 17

Under the constant force of mortality model,

$$ {}_{m|n}\overline{A}_x = e^{-m(\mu+\delta)}\left(1 - e^{-n(\mu+\delta)}\right)\frac{\mu}{\mu+\delta}. $$

Proof:

$$ {}_{m|n}\overline{A}_x \cdot {}_mE_x \overline{A}^{\,1}_{x+m:\overline{n}|} = e^{-m(\mu+\delta)}\left(1 - e^{-n(\mu+\delta)}\right)\frac{\mu}{\mu+\delta}. $$

Theorem 18

$$ {}_{m|n}\overline{A}_x = {}_{m|}\overline{A}_x - {}_{m+n|}\overline{A}_x. $$

Theorem 18

$$ {}_{m|n}\overline{A}_x = {}_{m|}\overline{A}_x - {}_{m+n|}\overline{A}_x. $$

Proof: We have that

$$ {}_{m|}\overline{A}_x - {}_{m+n|}\overline{A}_x = \int_m^{\infty} v^t\,{}_tp_x\mu_{x+t}\,dt - \int_{m+n}^{\infty} v^t\,{}_tp_x\mu_{x+t}\,dt $$

$$ = \int_m^{m+n} v^t\,{}_tp_x\mu_{x+t}\,dt = {}_{m|n}\overline{A}_x. $$

Theorem 19

$$\overline{A}_x = \overline{A}^{\,1}_{x:\overline{m}|} + {}_{m|n}\overline{A}_x + {}_{m+n|}\overline{A}_x.$$

Theorem 19

$$\overline{A}_x = \overline{A}^{1}_{x:\overline{m}|} + {}_{m|n}\overline{A}_x + {}_{m+n|}\overline{A}_x.$$

Proof: We have that

$$\overline{A}^{1}_{x:\overline{m}|} + {}_{m|n}\overline{A}_x + {}_{m+n|}\overline{A}_x = \overline{A}^{1}_{x:\overline{m}|} + {}_{m|}A_x - {}_{m+n|}\overline{A}_x + {}_{m+n|}\overline{A}_x$$
$$= \overline{A}^{1}_{x:\overline{m}|} + {}_{m|}\overline{A}_x = \overline{A}_x.$$

Theorem 20

$$ {}_{m|n}\overline{A}_x = \overline{A}^{\,1}_{x:\overline{m+n}|} - \overline{A}^{\,1}_{x:\overline{m}|}. $$

Theorem 20

$$ {}_{m|n}\overline{A}_x = \overline{A}^{\,1}_{x:\overline{m+n}|} - \overline{A}^{\,1}_{x:\overline{m}|}. $$

Proof: We have that

$$ \overline{A}^{\,1}_{x:\overline{m+n}|} - \overline{A}^{\,1}_{x:\overline{m}|} = \int_0^{m+n} v^t \, {}_tp_x \mu_{x+t} \, dt - \int_0^{m} v^{2t} \, {}_tp_x \mu_{x+t} \, dt $$

$$ = \int_m^{m+n} v^t \, {}_tp_x \mu_{x+t} \, dt = {}_{m|n}\overline{A}_x. $$

