Manual for SOA Exam MLC.

Chapter 5. Life annuities.
Section 5.1. Whole life annuities.

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"Arcones' Manual for the SOA Exam MLC. Spring 2010 Edition".
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Whole life annuity

A **whole life annuity** is a series of payments made while an individual is alive.
The payments can be made either at the beginning of the year, either at the end of the year, or continuously.

Whole life due annuity

Definition 1

A **whole life due annuity** *is a series payments made at the beginning of the year while an individual is alive.*

Definition 2

The present value of a whole life due annuity for (x) with unit payment is denoted by $\ddot{Y}_x$.

$\ddot{Y}_x$ is a random variable that depends on T_x. Recall that K_x is the interval of death of (x), i.e. K_x is the positive integer such that the death of (x) takes place in the interval $(K_x - 1, K_x]$.

Definition 3

The actuarial present value of a whole life due annuity for (x) with unit payment is denoted by $\ddot{a}_x$.

We have that $\ddot{a}_x = E[\ddot{Y}_x]$.

Theorem 1

(i) $\ddot{Y}_x = \ddot{a}_{\overline{K_x|}} = \sum_{k=0}^{K_x-1} v^k$.
(ii) If $i \neq 0$,

$$\ddot{Y}_x = \frac{1-v^{K_x}}{d} = \frac{1-Z_x}{d},$$
$$\ddot{a}_x = \sum_{k=1}^{\infty} \frac{1-v^k}{d} \cdot {}_{k-1|}q_x = \frac{1-A_x}{d}$$

and

$$\mathrm{Var}(\ddot{Y}_x) = \frac{\mathrm{Var}(Z_x)}{d^2} = \frac{{}^2A_x - A_x^2}{d^2},$$

where Z_x *is the present value of a unit life insurance paid at the end of the year of death.*
(iii) If $i = 0$, $\ddot{Y}_x = K_x = K(x) + 1$, $\ddot{a}_x = e_x + 1$ *and* $\mathrm{Var}(\ddot{Y}_x) = \mathrm{Var}(K(x))$.

Proof: (i) Since death happens in the interval $(K_x - 1, K_x]$, payments of one are made at times $0, 1, 2, \dots, K_x - 1$, i.e. the cashflow of payments is a K_x–year annuity due. Hence, $\ddot{Y}_x = \ddot{a}_{\overline{K_x}|} = \sum_{k=0}^{K_x-1} v^k$.

(ii) If $i > 0$,

$$\ddot{Y}_x = \ddot{a}_{\overline{K_x}|} = \frac{1 - v^{K_x}}{d} = \frac{1 - Z_x}{d}.$$

Therefore,

$$\ddot{a}_x = E[\ddot{a}_{\overline{K_x}|}] = \sum_{k=1}^{\infty} \ddot{a}_{\overline{k}|} \mathbb{P}\{K_x = k\} = \sum_{k=1}^{\infty} \frac{1 - v^k}{d} \cdot {}_{k-1}|q_x,$$

$$\ddot{a}_x = E\left[\frac{1 - Z_x}{d}\right] = \frac{1 - A_x}{d},$$

$$\mathrm{Var}(Y_x) = \mathrm{Var}\left(\frac{1 - Z_x}{d}\right) = \frac{\mathrm{Var}(Z_x)}{d^2} = \frac{{}^2A_x - A_x^2}{d^2}.$$

(iii) If $i = 0$, $\ddot{Y}_x = \ddot{a}_{\overline{K_x}|} = K_x = K(x) + 1$, $\ddot{a}_x = e_x + 1$ and $\mathrm{Var}(\ddot{Y}_x) = \mathrm{Var}(K(x))$.

Example 1

$i = 5\%$. K_x has probability mass function:

k	1	2	3
$\mathbb{P}\{K_x = k\}$	0.2	0.3	0.5

Find $\ddot{a}_x$ and $\mathrm{Var}(\ddot{Y}_x)$.

Example 1

$i = 5\%$. K_x has probability mass function:

k	1	2	3
$\mathbb{P}\{K_x = k\}$	0.2	0.3	0.5

Find $\ddot{a}_x$ and $\mathrm{Var}(\ddot{Y}_x)$.

Solution: We have that $\ddot{a}_{\overline{1}|} = 1$, $\ddot{a}_{\overline{2}|} = 1.952380952$ and $\ddot{a}_{\overline{3}|} = 2.859410431$. The probability mass of $\ddot{Y}_x$ is given by

k	1	1.952380952	2.859410431
$\mathbb{P}\{\ddot{Y}_x = k\}$	0.2	0.3	0.5

Hence,

$$\ddot{a}_x = (1)(0.2) + (1.952380952)(0.3) + (2.859410431)(0.5) = 2.2154195$$

$$E[\ddot{Y}_x^2] = (1)^2(0.2) + (1.952380952)^2(0.3) + (2.859410431)^2(0.5)$$

$$=5.431651421,$$

$$\mathrm{Var}(\ddot{Y}_x) = 5.431651421 - (2.215419501)^2 = 0.5235678556.$$

Example 2

Suppose that $p_{x+k} = 0.97$, *for each integer* $k \geq 0$, *and* $i = 6.5\%$. *Find* $\ddot{a}_x$ *and* $\mathrm{Var}(\ddot{Y}_x)$.

Example 2

Suppose that $p_{x+k} = 0.97$*, for each integer* $k \geq 0$*, and* $i = 6.5\%$*. Find* $\ddot{a}_x$ *and* $\mathrm{Var}(\ddot{Y}_x)$*.*

Solution: Recall that if $p_k = p$, for each $k \geq 1$, then $A_x = \frac{q_x}{q_x + i}$. Hence,

$$A_x = \frac{q_x}{q_x + i} = \frac{0.03}{0.03 + 0.065} = 0.3157894737,$$

$$\ddot{a}_x = \frac{1 - A_x}{d} = \frac{1 - 0.3157894737}{0.065/1.065} = 11.21052632,$$

$${}^2A_x = \frac{q_x}{q_x + i(2+i)} = \frac{0.03}{0.03 + (0.065)(2 + 0.065)} = 0.1826762064,$$

$$\mathrm{Var}(\ddot{Y}_x) = \frac{{}^2A_x - A_x^2}{d^2} = \frac{0.1826762064 - (0.3157894737)^2}{(0.065/1.065)^2} = 22.2692567$$

Example 3

Assuming $i = 6\%$ *and the life table in the manual, find* $\ddot{a}_{45}$.

Example 3

Assuming $i = 6\%$ and the life table in the manual, find $\ddot{a}_{45}$.

Solution: We have that

$$\ddot{a}_{45} = \frac{1 - A_{45}}{d} = \frac{1 - 0.16657}{0.06/(1.06)} = 14.72393.$$

Example 4

John, age 65, has \$750,000 in his retirement account. An insurance company offers a whole life due annuity to John which pays \$P at the beginning of the year while (65) *is alive for \$750,000. The annuity is priced assuming that* $i = 6\%$ *and the life table in the manual. The insurance company charges John 30% more of the APV of the annuity. Calculate* P.

Example 4

John, age 65, has $750,000 in his retirement account. An insurance company offers a whole life due annuity to John which pays $P at the beginning of the year while (65) is alive for $750,000. The annuity is priced assuming that $i = 6\%$ and the life table in the manual. The insurance company charges John 30% more of the APV of the annuity. Calculate P.

Solution: We have that

$$\ddot{a}_{65} = \frac{1 - A_{65}}{d} = \frac{1 - 0.37610}{0.06/(1.06)} = 11.0222333333333.$$

The APV of this is this annuity is $P\ddot{a}_{65} = (11.0222333333333)P$. We have that $750000 = (1.3)(11.0222333333333)P$ and $P = \frac{750000}{(1.3)(11.0222333333333)} = 52341.7586505226$.

Theorem 2

If $i \neq 0$,

$$\ddot{Y}_x^2 = \frac{2\ddot{Y}_x - (2-d)\cdot {}^2\ddot{Y}_x}{d}$$

and

$$E[\ddot{Y}_x^2] = \frac{2\ddot{a}_x - (2-d)\cdot {}^2\ddot{a}_x}{d}.$$

Proof.

From $\ddot{Y}_x = \frac{1-Z_x}{d}$, we get that $Z_x = 1 - d\ddot{Y}_x$. Hence,

$$\ddot{Y}_x^2 = \frac{1 - 2Z_x + {}^2Z_x}{d^2} = \frac{1 - 2(1 - d\ddot{Y}_x) + 1 - d(2-d)\cdot {}^2\ddot{Y}_x}{d^2}$$
$$= \frac{2\ddot{Y}_x - (2-d)\cdot {}^2\ddot{Y}_x}{d}.$$

□

Example 5

Suppose that $i = 0.075$, $\ddot{a}_x = 8.6$ *and* ${}^2\ddot{a}_x = 5.6$.

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(i) Calculate $\mathrm{Var}(\ddot{Y}_x)$ using the previous theorem.

Example 5

Suppose that $i = 0.075$, $\ddot{a}_x = 8.6$ *and* ${}^2\ddot{a}_x = 5.6$.

(i) Calculate $\mathrm{Var}(\ddot{Y}_x)$ using the previous theorem.

Solution: (i) We have that

$$E[\ddot{Y}_x^2] = \frac{2\ddot{a}_x - (2-d)\cdot {}^2\ddot{a}_x}{d} = \frac{2(8.6) - (2 - (0.075/1.075))(5.6)}{0.075/1.075}$$

$=91.6$,

$$\mathrm{Var}(Y_x) = 91.6 - (8.6)^2 = 17.64.$$

Example 5

Suppose that $i = 0.075$, $\ddot{a}_x = 8.6$ *and* ${}^2\ddot{a}_x = 5.6$.

(ii) Calculate $\mathrm{Var}(\ddot{Y}_x)$ using A_x and ${}^2A_x$.

Example 5

Suppose that $i = 0.075$, $\ddot{a}_x = 8.6$ *and* ${}^2\ddot{a}_x = 5.6$.

(ii) Calculate $\mathrm{Var}(\ddot{Y}_x)$ using A_x and ${}^2A_x$.

Solution: (ii) Using that $\ddot{a}_x = \frac{1-A_x}{d}$, we get that $A_x = 1 - d\ddot{a}_x = 1 - (0.075/1.075)(8.6) = 0.4$. Since ${}^2A_x$ uses a discount factor of v^2,

$$ {}^2A_x = 1-(1-v^2)\cdot {}^2\ddot{a}_x = 1-(1-(1.075)^{-2})(5.6) = 0.2458626284. $$

Hence,

$$ \mathrm{Var}(\ddot{Y}_x) = \frac{{}^2A_x - A_x^2}{d^2} = \frac{0.2458626284 - (0.4)^2}{(0.075/1.075)^2} = 17.63999999. $$

Theorem 3

(current payment method)

$$\ddot{Y}_x = \sum_{k=0}^{\infty} Z_{x:\overline{k}|}^{\;\;1},$$

$$\ddot{a}_x = \sum_{k=0}^{\infty} {}_kE_x = \sum_{k=0}^{\infty} v^k \cdot {}_kp_x.$$

Theorem 3

(current payment method)

$$\ddot{Y}_x = \sum_{k=0}^{\infty} Z_{x:\overset{1}{\overline{k|}}},$$

$$\ddot{a}_x = \sum_{k=0}^{\infty} {}_kE_x = \sum_{k=0}^{\infty} v^k \cdot {}_kp_x.$$

Proof: The payment at time k is made if and only if $k < T_x$.

$$\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k = \sum_{k=0}^{\infty} v^k I(k < T_x) = \sum_{k=0}^{\infty} Z_{x:\overset{1}{\overline{k|}}},$$

$$\ddot{a}_x = E[\ddot{Y}_x] = \sum_{k=0}^{\infty} {}_kE_x = \sum_{k=0}^{\infty} v^k \cdot {}_kp_x.$$

Example 6

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a due life annuity which will pay \$50000 at the end of the year. Suppose that $i = 6.5\%$. *Calculate the single benefit premium for this annuity.*

Example 6

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old buys a due life annuity which will pay \$50000 at the end of the year. Suppose that $i = 6.5\%$. Calculate the single benefit premium for this annuity.

Solution:

$$\ddot{a}_{80} = \sum_{k=0}^{\infty} v^k \frac{\ell_{80+k}}{\ell_{80}} = 1 + (1.065)^{-1}\frac{217}{250} + (1.065)^{-2}\frac{161}{250}$$

$$+ (1.065)^{-3}\frac{107}{250} + (1.065)^{-4}\frac{62}{250} + (1.065)^{-5}\frac{28}{250}$$

$$= 1 + 0.81502347418 + 0.5677885781 + 0.35431941129$$

$$+ 0.19277612654 + 0.08174665369 = 3.011654244,$$

$$(50000)\ddot{a}_{80} = (50000)(3.011654244) = 150582.7122.$$

Remember that a decreasing due annuity has payments of $n, n-1, \dots, 1$ at the beginning of the year, for n years. The present value of a decreasing due annuity is

$$(D\ddot{a})_{\overline{n}|i} = \sum_{k=0}^{n-1} v^k(n-k) = \frac{n - a_{\overline{n}|i}}{d}.$$

Theorem 4

Suppose that the mortality of (x) follows De Moivre's model with integer terminal age ω, where x and ω are nonnegative integers. Then,

(i) $\ddot{a}_x = \frac{(D\ddot{a})_{\overline{\omega-x}|}}{\omega-x}$.

(ii) If $i \neq 0$, $\ddot{a}_x = \frac{\omega-x-a_{\overline{\omega-x}|}}{d(\omega-x)}$.

(iii) If $i = 0$, $\ddot{a}_x = \frac{\omega-x+1}{2}$.

Proof: (i)

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x = \sum_{k=0}^{\omega-x-1} v^k \frac{\omega - x - k}{\omega - x} = \frac{(D\ddot{a})_{\overline{\omega-x}|}}{\omega - x}.$$

(ii) We have that

$$\ddot{a}_x = \frac{1 - A_x}{d} = \frac{1 - \frac{a_{\overline{\omega-x}|}}{\omega-x}}{d} = \frac{\omega - x - a_{\overline{\omega-x}|}}{d(\omega - x)}.$$

(iii) $\ddot{a}_x = e_x + 1 = \frac{\omega-x+1}{2}$.

Example 7

Suppose that $i = 6\%$ and De Moivre's model with terminal age 100.

(i) Find $\ddot{a}_{30}$ using A_{30}.

(ii) Find $\ddot{a}_{30}$ using $\ddot{a}_x = \frac{\omega - x - a_{\overline{\omega - x}|}}{d(\omega - x)}$.

Example 7

Suppose that $i = 6\%$ and De Moivre's model with terminal age 100.
(i) Find $\ddot{a}_{30}$ using A_{30}.
(ii) Find $\ddot{a}_{30}$ using $\ddot{a}_x = \frac{\omega - x - a_{\overline{\omega-x}|}}{d(\omega-x)}$.

Solution: (i) We have that

$$A_{30} = \frac{a_{\overline{\omega-x}|i}}{\omega - x} = \frac{a_{\overline{70}|0.06}}{70} = 0.2340649124.$$

Hence,

$$\ddot{a}_{30} = \frac{1 - A_{30}}{d} = \frac{1 - 0.2340649124}{\frac{0.06}{1.06}} = 13.53151988.$$

Example 7

Suppose that $i = 6\%$ and De Moivre's model with terminal age 100.
(i) Find $\ddot{a}_{30}$ using A_{30}.
(ii) Find $\ddot{a}_{30}$ using $\ddot{a}_x = \frac{\omega-x-a_{\overline{\omega-x}|}}{d(\omega-x)}$.

Solution: (i) We have that

$$A_{30} = \frac{a_{\overline{\omega-x}|i}}{\omega - x} = \frac{a_{\overline{70}|0.06}}{70} = 0.2340649124.$$

Hence,

$$\ddot{a}_{30} = \frac{1 - A_{30}}{d} = \frac{1 - 0.2340649124}{\frac{0.06}{1.06}} = 13.53151988.$$

(ii)

$$\ddot{a}_{30} = \frac{70 - a_{\overline{70}|0.06}}{\frac{0.06}{1.06}70} = \frac{70 - 16.38454387}{\frac{0.06}{1.06}70} = 13.53151988.$$

Theorem 5

(Iterative formula for the APV of a life annuity–due)

$$\ddot{a}_x = 1 + vp_x \ddot{a}_{x+1}.$$

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$$\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}.$$

Proof: We have that

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \cdot {}_kp_x = 1 + \sum_{k=1}^{\infty} v^k p_x \cdot {}_{k-1}p_{x+1}$$

$$= 1 + vp_x \sum_{k=1}^{\infty} v^{k-1}{}_{k-1}p_{x+1}$$

$$= 1 + vp_x \sum_{k=0}^{\infty} v^k{}_kp_{x+1} = 1 + vp_x\ddot{a}_{x+1}.$$

Example 8

Suppose that $\ddot{a}_x = \ddot{a}_{x+1} = 10$ *and* $q_x = 0.01$. *Find* i.

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Solution: Using that $\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$, we get that $10 = 1 + v(0.99)(10)$, $v = \frac{10-1}{(0.99)(10)}$ and $i = \frac{(0.99)(10)}{10-1} - 1 = 10\%$.

Theorem 6

If the probability function of time interval of failure is

$$\mathbb{P}\{K_x = k\} = p_x^{k-1}(1 - p_x), k = 1, 2, \ldots$$

then

$$\ddot{a}_x = E[\ddot{Z}_x] = \frac{1}{1 - vp_x} = \frac{1 + i}{i + q_x}.$$

Theorem 6

If the probability function of time interval of failure is

$$\mathbb{P}\{K_x = k\} = p_x^{k-1}(1-p_x), k = 1, 2, \dots$$

then

$$\ddot{a}_x = E[\ddot{Z}_x] = \frac{1}{1 - vp_x} = \frac{1+i}{i+q_x}.$$

Proof 1. We have that
${}_k p_x = \mathbb{P}\{K \geq k+1\} = \sum_{j=k+1}^{\infty} p_x^{j-1}(1-p_x) = (1-p_x)\frac{p_x^k}{1-p_x} = p_x^k$.
So,

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_k p_x = \sum_{k=0}^{\infty} v^k p_x^k = \frac{1}{1-vp_x} = \frac{1+i}{1+i-p_x} = \frac{1+i}{i+q_x}.$$

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If the probability function of time interval of failure is

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then

$$\ddot{a}_x = E[\ddot{Z}_x] = \frac{1}{1-vp_x} = \frac{1+i}{i+q_x}.$$

Proof 1. We have that
${}_kp_x = \mathbb{P}\{K \geq k+1\} = \sum_{j=k+1}^{\infty} p_x^{j-1}(1-p_x) = (1-p_x)\frac{p_x^k}{1-p_x} = p_x^k$.
So,

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x = \sum_{k=0}^{\infty} v^k p_x^k = \frac{1}{1-vp_x} = \frac{1+i}{1+i-p_x} = \frac{1+i}{i+q_x}.$$

Proof 2. Since K_x has a geometric distribution, K_x and K_{x+1} have the same distribution and $\ddot{a}_x = \ddot{a}_{x+1}$. So, $\ddot{a}_x = 1 + vp_x\ddot{a}_x$ and $\ddot{a}_x = \frac{1}{1-vp_x}$.

Recall that $\ddot{a}_{\overline{\infty}|} = \frac{1}{d} = \frac{1}{1-v}$. For a whole life annuity, we need to discount for interest and mortality and we get $\ddot{a}_x = \frac{1}{1-vp_x}$.

Example 9

An insurance company issues 800 identical due annuities to independent lives aged 65*. Each of of this annuities provides an annual payment of 30000. Suppose that* $p_{x+k} = 0.95$ *for each integer* $k \geq 0$*, and* $i = 7.5\%$*.*

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(i) Find $\ddot{a}_x$ and $\mathrm{Var}(\ddot{Y}_x)$.

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(i) Find $\ddot{a}_x$ and $\mathrm{Var}(\ddot{Y}_x)$.

Solution: (i) We have that

$$\ddot{a}_x = \frac{1}{1 - vp_x} = \frac{1}{1 - (1.075)^{-1}(0.95)} = \frac{1.075}{1.075 - 0.95} = 8.6,$$

$$A_x = \frac{q_x}{q_x + i} = \frac{0.05}{0.075 + 0.05} = 0.4,$$

$$^2A_x = \frac{q_x}{q_x + i(2 + i)} = \frac{0.05}{0.05 + (0.075)(2 + 0.075)} = 0.2444988,$$

$$\mathrm{Var}(\ddot{Y}_x) = \frac{^2A_x - A_x^2}{d^2} = \frac{0.2444988 - (0.4)^2}{(0.075/1.075)^2} = 17.35981.$$

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(ii) Using the central limit theorem, estimate the initial fund needed at time zero in order that the probability that the present value of the random loss for this block of policies exceeds this fund is 1%.

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An insurance company issues 800 identical due annuities to independent lives aged 65. Each of of this annuities provides an annual payment of 30000. Suppose that $p_{x+k} = 0.95$ for each integer $k \geq 0$, and $i = 7.5\%$.

(ii) Using the central limit theorem, estimate the initial fund needed at time zero in order that the probability that the present value of the random loss for this block of policies exceeds this fund is 1%.

Solution: (ii) Let $\ddot{Y}_{x,1}, \dots, \ddot{Y}_{x,800}$ be the present value per unit face value for the 800 due annuities. Let Q be the fund needed.

$$E[\sum_{j=1}^{800} 30000\ddot{Y}_{x,j}] = (30000)(800)(8.6) = 206400000,$$

$$\mathrm{Var}(\sum_{j=1}^{800} 30000\ddot{Y}_{x,j}) = (30000)^2(800)(17.35981) = 12499063200000,$$

$$Q = 206400000 + (2.326)\sqrt{12499063200000} = 214623343.702743.$$

Theorem 7

For the constant force of mortality model,

$$\ddot{a}_x = \frac{1}{1 - vp_x} = \frac{1+i}{i+q_x} = \frac{1}{1 - e^{-(\delta+\mu)}},$$

where $q_x = 1 - e^{-\mu}$.

Theorem 7

For the constant force of mortality model,

$$\ddot{a}_x = \frac{1}{1 - vp_x} = \frac{1+i}{i + q_x} = \frac{1}{1 - e^{-(\delta+\mu)}},$$

where $q_x = 1 - e^{-\mu}$.

Proof: We have that $\mathbb{P}\{K_x = k\} = e^{-\mu(k-1)}(1 - e^{-\mu}), k = 1, 2, \ldots$ Theorem 6 applies with $p_x = e^{-\mu}$. We have that $\ddot{a}_x = \frac{1}{1-vp_x} = \frac{1}{1-e^{-(\delta+\mu)}}$.

Whole life discrete immediate annuity

Definition 4

A **whole life discrete immediate annuity** *is a series payments made at the end of the year while an individual is alive.*

Definition 5

The present value of a whole life immediate annuity for (x) *with unit payment is denoted by* Y_x.

Notice that Y_x is a random variable. Y_x depends on T_x. It is easy to see that $Y_x = \ddot{Y}_x - 1$.

Definition 6

The actuarial present value of a whole life immediate annuity for (x) *with unit payment is denoted by* a_x.

We have that $a_x = \ddot{a}_x - 1$.

Theorem

(i) $Y_x = a_{\overline{K_x-1}|}$ and $a_x = \sum_{k=2}^{\infty} a_{\overline{k-1}|} \cdot {}_{k-1|}q_x$.

(ii) If $i \neq 0$,

$$Y_x = \frac{1-(1+i)Z_x}{i} = \frac{v - Z_x}{d}, \ a_x = \frac{v - A_x}{d} \text{ and } \mathrm{Var}(Y_x) = \frac{{}^2A_x - A_x^2}{d^2},$$

where Z_x is the present value of a life insurance paid at the end of the year of death.

(iii) If $i = 0$, $Y_x = K_x - 1 = K(x)$, $a_x = e_x$ and $\mathrm{Var}(Y_x) = \mathrm{Var}(K(x))$.

Theorem 8

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(i) $Y_x = a_{\overline{K_x-1}|}$ and $a_x = \sum_{k=2}^{\infty} a_{\overline{k-1}|} \cdot {}_{k-1|}q_x$.

Proof: (i) Since death happens in the interval $(K_x - 1, K_x]$, payments of one are made at times $1, 2, \dots, K_x - 1$, payments of one are made at times $1, 2, \dots, K_x - 1$, i.e. the cashflow of payments is an annuity immediate. Hence, $Y_x = a_{\overline{K_x-1}|}$. If $K_x = 1$, $Y_x = a_{\overline{0}|} = 0$. Therefore,

$$a_x = E[a_{\overline{K_x-1}|}] = \sum_{k=2}^{\infty} a_{\overline{k-1}|} \mathbb{P}\{K_x = k\} = \sum_{k=2}^{\infty} a_{\overline{k-1}|} \cdot {}_{k-1|}q_x.$$

Theorem 8

(ii) If $i \neq 0$,

$$Y_x = \frac{v - Z_x}{d}, \ a_x = \frac{v - A_x}{d} \text{ and } \mathrm{Var}(Y_x) = \frac{{}^2A_x - A_x^2}{d^2},$$

where Z_x is the present value of a life insurance paid at the end of the year of death.

Theorem 8

(ii) If $i \neq 0$,

$$Y_x = \frac{v - Z_x}{d}, \ a_x = \frac{v - A_x}{d} \text{ and } \mathrm{Var}(Y_x) = \frac{{}^2A_x - A_x^2}{d^2},$$

where Z_x is the present value of a life insurance paid at the end of the year of death.
(ii) If $i \neq 0$, $a_{\overline{k-1}|} = \frac{1-v^{k-1}}{i}$. Hence,

$$Y_x = a_{\overline{K_x-1}|} = \frac{1 - v^{K_x-1}}{i} = \frac{1-(1+i)Z_x}{i} = \frac{v - Z_x}{d},$$

$$a_x = E\left[\frac{v - Z_x}{d}\right] = \frac{v - A_x}{d},$$

$$\mathrm{Var}(Y_x) = \mathrm{Var}\left(\frac{v - Z_x}{d}\right) = \frac{\mathrm{Var}(Z_x)}{d^2} = \frac{{}^2A_x - A_x^2}{d^2}.$$

Theorem 8

(iii) If $i = 0$, $Y_x = K_x - 1 = K(x)$, $a_x = e_x$ and $\mathrm{Var}(Y_x) = \mathrm{Var}(K(x))$.

Theorem 8

(iii) If $i = 0$, $Y_x = K_x - 1 = K(x)$, $a_x = e_x$ and $\mathrm{Var}(Y_x) = \mathrm{Var}(K(x))$.
(iii) If $i \neq 0$, $Y_x = a_{\overline{K_x - 1}|} = K_x - 1 = K(x)$.

Example 10

Suppose that $p_{x+k} = 0.97$, *for each integer* $k \geq 0$, *and* $i = 6.5\%$. *Find* a_x *and* $\mathrm{Var}(Y_x)$.

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Solution: We have that

$$A_x = \frac{q_x}{q_x + i} = \frac{1 - 0.97}{1 - 0.97 + 0.065} = 0.3157894737,$$

$$a_x = \frac{v - A_x}{d} = \frac{(1.065)^{-1} - 0.3157894737}{0.065/1.065} = 10.21052632,$$

$$^2A_x = \frac{q_x}{q_x + (2+i)i} = \frac{1 - 0.97}{1 - 0.97 + (2 + 0.065)(0.065)} = 0.1826762064,$$

$$\mathrm{Var}(Y_x) = \frac{^2A_x - A_x^2}{d^2} = \frac{0.1826762064 - (0.3157894737)^2}{(0.065/1.065)^2}$$

$=22.26925679.$

Example 11

Suppose that $i = 6\%$ and the De Moivre model with terminal age 100. Find a_{30}.

Example 11

Suppose that $i = 6\%$ and the De Moivre model with terminal age 100. Find a_{30}.

Solution: We have that

$$A_{30} = \frac{a_{\overline{70}|0.06}}{70} = 0.2340649124.$$

Hence,

$$a_{30} = \frac{\nu - A_{30}}{d} = \frac{(1.06)^{-1} - 0.2340649124}{(0.06)(1.06)^{-1}} = 12.53151988.$$

Theorem 9

If $i \neq 0$,

$$Y_x^2 = \frac{2Y_x - (2+i) \cdot {}^2Y_x}{i}$$

and

$$E[Y_x^2] = \frac{2a_x - (2+i) \cdot {}^2a_x}{i}.$$

Theorem 9

If $i \neq 0$,

$$Y_x^2 = \frac{2Y_x - (2+i)\cdot {}^2Y_x}{i}$$

and

$$E[Y_x^2] = \frac{2a_x - (2+i)\cdot {}^2a_x}{i}.$$

Proof: We have that $Y_x = \frac{1-(1+i)Z_x}{i}$. Hence, $(1+i)Z_x = 1 - iY_x$. Similarly, ${}^2Y_x = \frac{1-(1+i)^2Z_x}{i(2+i)}$ and $(1+i)^2Z_x = 1 - i(2+i)\cdot {}^2Y_x$. Therefore,

$$Y_x^2 = \frac{1 - 2(1+i)Z_x + (1+i)^2\cdot {}^2Z_x}{i^2}$$

$$= \frac{1 - 2(1-iY_x) + 1 - i(2+i)\cdot {}^2Y_x}{i^2}$$

$$= \frac{2Y_x - (2+i)\cdot {}^2Y_x}{i}.$$

Example 12

Suppose that $i = 0.075$, $a_x = 7.6$ *and* ${}^2a_x = 4.6$.
(i) Calculate $\mathrm{Var}(Y_x)$ *using the previous theorem.*
(ii) Calculate $\mathrm{Var}(Y_x)$ *using* A_x *and* ${}^2A_x$.

Example 12

Suppose that $i = 0.075$, $a_x = 7.6$ *and* ${}^2a_x = 4.6$.
(i) Calculate $\mathrm{Var}(Y_x)$ *using the previous theorem.*
(ii) Calculate $\mathrm{Var}(Y_x)$ *using* A_x *and* ${}^2A_x$.

Solution: (i) We have that

$$E[Y_x^2] = \frac{2a_x - (2+i)\cdot {}^2a_x}{i} = \frac{2(7.6) - (2.075)(4.6)}{0.075} = 75.4,$$

$$\mathrm{Var}(Y_x) = 75.4 - (7.6)^2 = 17.64.$$

Example 12

Suppose that $i = 0.075$, $a_x = 7.6$ *and* ${}^2a_x = 4.6$.
(i) Calculate $\mathrm{Var}(Y_x)$ *using the previous theorem.*
(ii) Calculate $\mathrm{Var}(Y_x)$ *using* A_x *and* ${}^2A_x$.

Solution: (ii) Using that $a_x = \frac{1-(1+i)A_x}{i}$, we get that $A_x = \frac{1-ia_x}{1+i} = \frac{1-(0.075)(7.6)}{1.075} = 0.4$. Since ${}^2A_x$ uses a interest factor of $(1+i)^2$,

$$ {}^2A_x = \frac{1 - i(2+i)\cdot {}^2a_x}{(1+i)^2} = \frac{1-(0.075)(2+0.075)(4.6)}{(1.075)^2} $$
$$ = 0.2458626284. $$

Hence,

$$ \mathrm{Var}(Y_x) = \frac{{}^2A_x - A_x^2}{d^2} = \frac{0.2458626284 - (0.4)^2}{(0.075/1.075)^2} = 17.63999999. $$

Theorem 10

(current payment method) $Y_x = \sum_{k=1}^{\infty} Z_{x:\overset{1}{\overline{k|}}}$ *and*

$$a_x = \sum_{k=1}^{\infty} A_{x:\overset{1}{\overline{k|}}} = \sum_{k=1}^{\infty} {}_kE_x = \sum_{k=1}^{\infty} \nu^k \, {}_kp_x,$$

where $Z_{x:\overset{1}{\overline{k|}}} = \nu^n I(n < K_x)$ *is the present value of an n-year pure endowment life insurance*

Theorem 10

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where $Z_{x:\overset{1}{\overline{k|}}} = \nu^n I(n < K_x)$ *is the present value of an n-year pure endowment life insurance*

Proof: The payment at time k is made if and only if $k < T_x$. Hence,

$$Y_x = \sum_{k=1}^{\infty} \nu^k I(k < T_x) = \sum_{k=1}^{\infty} Z_{x:\overset{1}{\overline{k|}}}$$

and

$$a_x = \sum_{k=1}^{\infty} A_{x:\overset{1}{\overline{k|}}} = \sum_{k=1}^{\infty} {}_kE_x = \sum_{k=1}^{\infty} \nu^k {}_kp_x.$$

Theorem 11

If the probability function of time interval of failure is is

$$\mathbb{P}\{K = k\} = p_x^{k-1}(1 - p_x), k = 1, 2, \dots$$

then

$$a_x = E[Z_x] = \frac{vp_x}{1 - vp_x} = \frac{1 - q}{q + i}.$$

Theorem 11

If the probability function of time interval of failure is is

$$\mathbb{P}\{K = k\} = p_x^{k-1}(1 - p_x), k = 1, 2, \ldots$$

then

$$a_x = E[Z_x] = \frac{vp_x}{1 - vp_x} = \frac{1-q}{q+i}.$$

Proof: We have that ${}_kp_x = \mathbb{P}\{K \geq k+1\} = p_x^k$. So,

$$a = \sum_{k=1}^{\infty} v^k \cdot {}_kp_x = \sum_{k=1}^{\infty} v^k p_x^k = \frac{vp_x}{1 - vp_x} = \frac{p}{1 + i - p_x} = \frac{1 - q_x}{q_x + i}.$$

Example 13

Suppose that $p_{x+k} = 0.95$, *for each integer* $k \geq 0$, *and* $i = 7.5\%$. *Find* a_x.

Example 13

Suppose that $p_{x+k} = 0.95$*, for each integer* $k \geq 0$*, and* $i = 7.5\%$*. Find* a_x*.*

Solution: We have that

$$a_x = \frac{p}{1 + i - p_x} = \frac{0.95}{1.075 - 0.95} = 7.6.$$

Theorem 12

For the constant force of mortality model,

$$a_x = \frac{vp_x}{1 - vp_x} = \frac{1 - q_x}{q_x + i} = \frac{e^{-(\delta+\mu)}}{1 - e^{-(\delta+\mu)}}.$$

Theorem 12

For the constant force of mortality model,

$$a_x = \frac{vp_x}{1 - vp_x} = \frac{1 - q_x}{q_x + i} = \frac{e^{-(\delta+\mu)}}{1 - e^{-(\delta+\mu)}}.$$

Proof: Previous theorem applies with $p_x = e^{-\mu}$. So,

$$a_x = \frac{vp_x}{1 - vp_x} = \frac{1 - q_x}{q_x + i} = \frac{e^{-(\delta+\mu)}}{1 - e^{-(\delta+\mu)}}.$$

Theorem 13

$$a_x = \nu p_x (1 + a_{x+1}).$$

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$$a_x = \nu p_x(1 + a_{x+1}).$$

Proof: We have that

$$\begin{aligned} a_x &= \sum_{k=1}^{\infty} \nu^k \cdot {}_k p_x = \sum_{k=1}^{\infty} \nu^k p_x \cdot {}_{k-1}p_{x+1} \\ &= \nu p_x \left(1 + \sum_{k=2}^{\infty} \nu^{k-1} {}_{k-1}p_{x+1}\right) \\ &= \nu p_x(1 + \sum_{k=1}^{\infty} \nu^k {}_k p_{x+1}) = \nu p_x(1 + a_{x+1}). \end{aligned}$$

Example 14

Using $i = 0.05$ *and a certain life table* $a_{30} = 4.52$*. Suppose that an actuary revises this life table and changes* p_{30} *from* 0.95 *to* 0.96*. Other values in the life table are unchanged. Find* a_{30} *using the revised life table.*

Example 14

Using $i = 0.05$ and a certain life table $a_{30} = 4.52$. Suppose that an actuary revises this life table and changes p_{30} from 0.95 to 0.96. Other values in the life table are unchanged. Find a_{30} using the revised life table.

Solution: Using that
$4.52 = a_{30} = vp_{30}(1 + a_{31}) = (1.05)^{-1}(0.95)(1 + a_{31})$, we get that $a_{31} = \frac{(4.52)(1.05)}{0.95} - 1 = 3.995789474$. The revised value of a_{30} is $(1.05)^{-1}(0.96)(1 + 3.995789474) = 4.567578948$.

Whole life continuous annuity

Definition 7

A **whole life continuous annuity** *is a continuous flow of payments with constant rate made while an individual is alive.*

Definition 8

The present value random variable of a whole life continuous annuity for (x) *with unit rate is denoted by* $\overline{Y}_x$.

Definition 9

The actuarial present value of a whole life continuous annuity for (x) *with unit rate is denoted by* $\overline{a}_x$.

We have that $\overline{a}_x = E[\overline{Y}_x]$.

Theorem 14

(i) $\overline{Y}_x = \overline{a}_{\overline{T_x|}}$.

(ii) If $\delta \neq 0$,

$$\overline{Y}_x = \frac{1-\overline{Z}_x}{\delta},\ \overline{a}_x = \frac{1-\overline{A}_x}{\delta} \text{ and } \mathrm{Var}(\overline{Y}_x) = \frac{{}^2\overline{A}_x - \overline{A}_x^2}{\delta^2}.$$

(iii) If $\delta = 0$, $\overline{Y}_x = T_x$ *and* $\overline{a}_x = \overset{\circ}{e}_x$.

Theorem 14

(i) $\overline{Y}_x = \bar{a}_{\overline{T_x}|}$.

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(iii) If $\delta = 0$, $\overline{Y}_x = T_x$ *and* $\bar{a}_x = \mathring{e}_x$.

Proof: (i) Since payments are received at rate one until time T_x, $\overline{Y}_x = \bar{a}_{\overline{T_x}|}$.

Theorem 14

(i) $\overline{Y}_x = \overline{a}_{\overline{T_x}|}$.
(ii) If $\delta \neq 0$,

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(iii) If $\delta = 0$, $\overline{Y}_x = T_x$ *and* $\overline{a}_x = \overset{\circ}{e}_x$.

Proof: (ii) If $\delta \neq 0$, $\overline{a}_{\overline{n}|i} = \frac{1-e^{-n\delta}}{\delta}$,

$$\overline{Y}_x = \overline{a}_{\overline{T_x}|} = \frac{1-v^{T_x}}{\delta} = \frac{1-\overline{Z}_x}{\delta},\ \overline{a}_x = E\left[\frac{1-\overline{Z}_x}{\delta}\right] = \frac{1-\overline{A}_x}{\delta}$$

and

$$\operatorname{Var}(\overline{Y}_x) = \operatorname{Var}\left(\frac{1-\overline{Z}_x}{\delta}\right) = \frac{\operatorname{Var}(\overline{Z}_x)}{\delta^2} = \frac{{}^2\overline{A}_x - \overline{A}_x^2}{\delta^2}.$$

Theorem 14

(i) $\overline{Y}_x = \overline{a}_{\overline{T_x}|}$.
(ii) If $\delta \neq 0$,

$$\overline{Y}_x = \frac{1 - \overline{Z}_x}{\delta},\ \overline{a}_x = \frac{1 - \overline{A}_x}{\delta} \text{ and } \mathrm{Var}(\overline{Y}_x) = \frac{{}^2\overline{A}_x - \overline{A}_x^2}{\delta^2}.$$

(iii) If $\delta = 0$, $\overline{Y}_x = T_x$ *and* $\overline{a}_x = \overset{\circ}{e}_x$.

Proof: (iii) If $\delta = 0$, $\overline{a}_{\overline{t}|i} = t$. Hence, $\overline{Y}_x = T_x$ and $\overline{a}_x = \overset{\circ}{e}_x$.

Example 15

Suppose that $v = 0.92$, *and the force of mortality is* $\mu_{x+t} = 0.02$, *for* $t \geq 0$. *Find* $\overline{a}_x$ *and* $\mathrm{Var}(\overline{Y}_x)$.

Example 15

Suppose that $v = 0.92$, *and the force of mortality is* $\mu_{x+t} = 0.02$, *for* $t \geq 0$. *Find* $\overline{a}_x$ *and* $\mathrm{Var}(\overline{Y}_x)$.

Solution: Using previous theorem,

$$\overline{A}_x = \frac{0.02}{-\ln(0.92) + 0.02} = 0.1934580068,$$

$$\overline{a}_x = \frac{1 - \overline{A}_x}{\delta} = \frac{1 - 0.1934580068}{-\ln(0.92)} = 9.672900337,$$

$$^2\overline{A}_x = \frac{0.02}{(2)(-\ln(0.92)) + 0.02} = 0.1070874674,$$

$$\frac{\mathrm{Var}(\overline{Z}_x)}{\delta^2} = \frac{^2\overline{A}_x - \overline{A}_x^2}{\delta^2} = \frac{0.1070874674 - (0.1934580068)^2}{(-\ln(0.92))^2}$$

$$= 10.01963899.$$

We define ${}^{m}\overline{Y}_x$ as the present value of whole life continuous annuity with unit rate at a force of interest m times the original force, i.e. ${}^{m}\overline{Y}_x = \frac{1-e^{-T_x m\delta}}{m\delta}$. We define ${}^{m}\overline{a}_x = E\left[{}^{m}\overline{Y}_x\right]$.
It is not true that ${}^{m}\overline{a}_x = E\left[(\overline{Y}_x)^m\right]$.

Theorem 15

If $\delta \neq 0$,

$$E[\overline{Y}_x^2] = \frac{2(\bar{a}_x - {}^2\bar{a}_x)}{\delta}.$$

Theorem 15

If $\delta \neq 0$,

$$E[\overline{Y}_x^2] = \frac{2(\overline{a}_x - {}^2\overline{a}_x)}{\delta}.$$

Proof: From ${}^m\overline{a}_x = \frac{1 - {}^mA_x}{m\delta}$, we get that ${}^mA_x = 1 - m\delta \cdot {}^m\overline{a}_x$. Hence,

$$E[\overline{Y}_x^2] = E\left[\left(\frac{1 - Z_x}{\delta}\right)^2\right] = E\left[\frac{1 - 2Z_x + {}^2Z_x}{\delta^2}\right]$$

$$= E\left[\frac{1 - 2(1 - \delta\overline{a}_x) + 1 - 2\delta \cdot {}^2\overline{a}_x}{\delta^2}\right] = \frac{2(\overline{a}_x - {}^2\overline{a}_x)}{\delta}.$$

Example 16

Suppose that $\overline{a}_x = 12$, ${}^2\overline{a}_x = 7$ *and* $\delta = 0.05$.

(i) Find $\mathrm{Var}(\overline{Y}_x)$ *using that* $\mathrm{Var}(\overline{Y}_x) = \frac{{}^2\overline{A}_x - \overline{A}_x^2}{\delta^2}$.

(ii) Find $\mathrm{Var}(\overline{Y}_x)$ *using previous theorem.*

Example 16

Suppose that $\bar{a}_x = 12$, ${}^2\bar{a}_x = 7$ *and* $\delta = 0.05$.
(i) Find $\mathrm{Var}(\overline{Y}_x)$ *using that* $\mathrm{Var}(\overline{Y}_x) = \frac{{}^2\overline{A}_x - \overline{A}_x^2}{\delta^2}$.
(ii) Find $\mathrm{Var}(\overline{Y}_x)$ *using previous theorem.*

Solution: (i) We have that

$$\overline{A}_x = 1 - \bar{a}_x\delta = 1 - (12)(0.05) = 0.4,$$
$${}^2\overline{A}_x = 1 - {}^2\bar{a}_x 2\delta = 1 - (7)(2)(0.05) = 0.3.$$

Hence,

$$\mathrm{Var}(\overline{Y}_x) = \frac{{}^2\overline{A}_x - \overline{A}_x^2}{\delta^2} = \frac{0.3 - (0.4)^2}{(0.05)^2} = 56.$$

(ii) We have that

$$E[\overline{Y}_x^2] = \frac{2(\bar{a}_x - {}^2\bar{a}_x)}{\delta} = \frac{2(12-7)}{0.05} = 200,$$
$$\mathrm{Var}(\overline{Y}_x) = 200 - (12)^2 = 56.$$

Theorem 16

(current payment method)

$$\bar{a}_x = \int_0^\infty v^t \cdot {}_tp_x \, dt = \int_0^\infty {}_tE_x \, dt.$$

Proof: Let $h(x) = v^x$, $x \geq 0$. Let
$H(x) = \int_0^x h(t)\,dt = \int_0^x v^t\,dt = \bar{a}_{\overline{x}|}$, $x \geq 0$. By a previous theorem,

$$E[\overline{Y}_x] = E[\bar{a}_{\overline{T_x}|}] = E[H(T_x)] = \int_0^\infty h(t) s_{T_x}(t)\,dt = \int_0^\infty v^t \cdot {}_tp_x \, dt.$$

Example 17

Suppose that $\delta = 0.05$ *and* ${}_tp_x = 0.01te^{-01.t}$, $t \geq 0$. *Calculate* $\overline{a}_x$.

Example 17

Suppose that $\delta = 0.05$ *and* ${}_tp_x = 0.01te^{-01.t}$, $t \geq 0$. *Calculate* $\overline{a}_x$.

Solution: Using that $\int_0^\infty t^n e^{-t/\beta}\,dt = \frac{\beta^{n+1}}{n!}$,

$$\overline{a}_x = \int_0^\infty v^t \cdot {}_tp_x\,dt = \int_0^\infty e^{-(0.05)t}0.01te^{-0.1t}\,dt$$

$$= \int_0^\infty 0.01te^{-0.15t}\,dt = \frac{0.01}{(0.15)^2} = 0.444444444.$$

Theorem 17

The cumulative distribution function of $\overline{Y}_x = \bar{a}_{\overline{T_x}|}$ *is*

$$F_{\overline{Y}_x}(y) = \begin{cases} 0 & \text{if } y < 0, \\ F_{T_x}\left(-\frac{\ln(1-\delta y)}{\delta}\right) & \text{if } 0 < y \le \bar{a}_{\overline{\omega - x}|}, \\ 1 & \text{if } \bar{a}_{\overline{\omega - x}|} \le y. \end{cases}$$

Proof.

The function $y = H(x) = \bar{a}_{\overline{x}|} = \frac{1-e^{-\delta x}}{\delta}$ is increasing. If $y = H(x) = \frac{1-e^{-\delta x}}{\delta}$, then $\delta y = 1 - e^{-\delta x}$, $e^{-\delta x} = 1 - \delta y$ and $x = -\frac{\ln(1-\delta y)}{\delta}$. Hence, the inverse function of H is $H^{-1}(y) = -\frac{\ln(1-\delta y)}{\delta}$, $y \ge 0$. Hence, if $0 < y$,

$$F_{\overline{Y}_x}(y) = \mathbb{P}\{\overline{Y}_x \le y\} = \mathbb{P}\{H(T_x) \le y\} = \mathbb{P}\{T_x \le H^{-1}(y)\}$$
$$= \mathbb{P}\left\{T_x \le -\frac{\ln(1-\delta y)}{\delta}\right\} = F_{T_x}\left(-\frac{\ln(1-\delta y)}{\delta}\right).$$

Theorem 18

The probability density function of $\overline{Y}_x = \bar{a}_{\overline{T_x}|}$ *is*

$$f_{\overline{Y}_x}(z) = \begin{cases} \frac{f_{T_x}\left(-\frac{\ln(1-\delta y)}{\delta}\right)}{1-\delta y} & \text{if } 0 < y < \bar{a}_{\overline{\omega - x}|}, \\ 0 & \text{else.} \end{cases}$$

Proof: For $0 < y < \bar{a}_{\overline{\omega - x}|}$,

$$f_{\overline{Y}_x}(y) = \frac{d}{dy} F_{\overline{Y}_x}(y) = \frac{d}{dy} F_{T_x}\left(-\frac{\ln(1-\delta y)}{\delta}\right) = \frac{f_{T_x}\left(-\frac{\ln(1-\delta y)}{\delta}\right)}{1-\delta y}.$$

Corollary 1

Under the De Moivre model with terminal age ω, $\overline{Y}_x$ *is a continuous r.v. with cumulative distribution function*

$$F_{\overline{Y}_x}(y) = -\frac{\ln(1-\delta y)}{\delta(\omega - x)}, \quad \text{if } 0 < y \le \overline{a}_{\overline{\omega - x}|}.$$

and density function

$$f_{\overline{Y}_x}(y) = \frac{1}{(1-\delta y)(\omega - x)}, \quad \text{if } 0 < y \le \overline{a}_{\overline{\omega - x}|}.$$

Proof: The claims follows noticing that $F_{T_x}(t) = \frac{t}{\omega - x}$ and $f_{T_x}(t) = \frac{1}{\omega - x}$, if $0 \le t \le \omega - x$.

Corollary 2

Under constant force of mortality μ, $\overline{Y}_x$ is a continuous r.v. with cumulative distribution function

$$F_{\overline{Y}_x}(y) = 1 - (1-\delta y)^{\frac{\mu}{\delta}}, \quad 0 < y \le \frac{1}{\delta}.$$

and density function

$$f_{\overline{Y}_x}(y) = \mu(1-\delta y)^{\frac{\mu}{\delta}-1}, 0 < y < \frac{1}{\delta}.$$

Proof: Since $F_{T_x}(t) = 1 - e^{-\mu t}$, if $0 \le t < \infty$; we have that for $0 < y \le \frac{1}{\delta}$,

$$F_{\overline{Y}_x}(y) = F_{T_x}\left(-\frac{\ln(1-\delta y)}{\delta}\right) = 1 - \exp\left(-\mu\left(-\frac{\ln(1-\delta y)}{\delta}\right)\right)$$
$$= 1 - (1-\delta y)^{\frac{\mu}{\delta}}.$$

$\overline{Y}_x$ has density

$$f_{\overline{Y}_x}(y) = F'_{\overline{Y}_x}(y) = \mu(1-\delta y)^{\frac{\mu}{\delta}-1}, \quad 0 < y < \frac{1}{\delta}.$$

The present value of a continuously decreasing annuity is

$$(\overline{D}\overline{a})_{\overline{n}|i} = \int_0^n (n-t)v^t \, dt = \frac{n - \overline{a}_{\overline{n}|i}}{\delta}.$$

Theorem 19

Under De Moivre's model with terminal age ω,

$$\overline{a}_x = \frac{\omega - x - \overline{a}_{\overline{\omega-x}|}}{\delta(\omega - x)} = \frac{(\overline{D}\overline{a})_{\overline{\omega-x}|i}}{\omega - x}.$$

Theorem 19

Under De Moivre's model with terminal age ω,

$$\bar{a}_x = \frac{\omega - x - \bar{a}_{\overline{\omega-x}|}}{\delta(\omega - x)} = \frac{(\bar{D}\bar{a})_{\overline{\omega-x}|i}}{\omega - x}.$$

Proof: We have that

$$\bar{a}_x = \frac{1 - \bar{A}_x}{\delta} = \frac{1 - \frac{\bar{a}_{\overline{\omega-x}|}}{\omega-x}}{\delta} = \frac{\omega - x - \bar{a}_{\overline{\omega-x}|}}{\delta(\omega - x)}$$

and

$$\bar{a}_x = \int_0^\infty v^t \cdot {}_tp_x \, dt = \int_0^{\omega-x} v^t \cdot \frac{\omega - x - t}{\omega - x} \, dt = \frac{(\bar{D}\bar{a})_{\overline{\omega-x}|i}}{\omega - x}$$

Example 18

Suppose that $i = 6.5\%$ *and deaths are uniformly distributed with terminal age 100.*

(i) Calculate the density of $\overline{Y}_{40}$.

(ii) Calculate $\overline{a}_{40}$ *using that* $\overline{a}_x = \frac{1-\overline{A}_x}{\delta}$.

(iii) Calculate $\overline{a}_{40}$ *using that* $\overline{a}_x = \frac{\omega-x-\overline{a}_{\overline{\omega-x}|}}{\delta(\omega-x)}$.

Example 18

Suppose that $i = 6.5\%$ and deaths are uniformly distributed with terminal age 100.
(i) Calculate the density of $\overline{Y}_{40}$.
(ii) Calculate $\overline{a}_{40}$ using that $\overline{a}_x = \frac{1-\overline{A}_x}{\delta}$.
(iii) Calculate $\overline{a}_{40}$ using that $\overline{a}_x = \frac{\omega-x-\overline{a}_{\overline{\omega-x|}}}{\delta(\omega-x)}$.

Solution: (i) We have that

$$f_{\overline{Y}_{40}}(y) = \frac{1}{(1-\delta y)(\omega - x)} = \frac{1}{(60)(1 - y\ln(1.065))},$$
$$\text{if } 0 < y < \frac{1-(1.065)^{-60}}{\ln(1.065)} = 15.51640952.$$

Example 18

Suppose that $i = 6.5\%$ and deaths are uniformly distributed with terminal age 100.
(i) Calculate the density of $\overline{Y}_{40}$.
(ii) Calculate $\overline{a}_{40}$ using that $\overline{a}_x = \frac{1-\overline{A}_x}{\delta}$.
(iii) Calculate $\overline{a}_{40}$ using that $\overline{a}_x = \frac{\omega-x-\overline{a}_{\overline{\omega-x}|}}{\delta(\omega-x)}$.

Solution: (ii) We have that

$$\overline{A}_{40} = \frac{\overline{a}_{\overline{60}|0.065}}{60} = \frac{1-(1.065)^{-60}}{(60)\ln(1.065)} = 0.2586068254.$$

Hence,

$$\overline{a}_{40} = \frac{1-\overline{A}_{40}}{\delta} = \frac{1-0.2586068254}{\ln(1.065)} = 11.77285493.$$

Example 18

Suppose that $i = 6.5\%$ and deaths are uniformly distributed with terminal age 100.

(i) Calculate the density of $\overline{Y}_{40}$.

(ii) Calculate $\overline{a}_{40}$ using that $\overline{a}_x = \frac{1-\overline{A}_x}{\delta}$.

(iii) Calculate $\overline{a}_{40}$ using that $\overline{a}_x = \frac{\omega-x-\overline{a}_{\overline{\omega-x}|}}{\delta(\omega-x)}$.

Solution: (iii) We have that

$$\overline{a}_{\overline{60}|} = \frac{1-(1.065)^{-60}}{\ln(1.065)} = 15.51640952.$$

Hence,

$$\overline{a}_{40} = \frac{100-40-\overline{a}_{\overline{100-40}|}}{(\ln(1.065))(60)} = \frac{60-15.51640952}{(\ln(1.065))(60)} = 11.77285493.$$

Theorem 20

Under constant force of mortality μ, $\overline{a}_x = \frac{1}{\mu+\delta}$.

Proof.

We have that

$$\overline{a}_x = \frac{1 - A_x}{\delta} = \frac{1 - \frac{\mu}{\mu+\delta}}{\delta} = \frac{1}{\mu + \delta}.$$

□

Example 19

Suppose that $\delta = 0.07$ *and the force of mortality is* 0.02.

(i) Calculate the density of $\overline{Y}_{40}$.

(ii) Calculate $\overline{a}_{40}$.

(iii) Calculate the variance of $\overline{Y}_{40}$.

Example 19

Suppose that $\delta = 0.07$ *and the force of mortality is* 0.02.
(i) Calculate the density of $\overline{Y}_{40}$.
(ii) Calculate $\overline{a}_{40}$.
(iii) Calculate the variance of $\overline{Y}_{40}$.

Solution: (i) The density of $\overline{Y}_{40}$ is

$$f_{\overline{Y}_{40}}(t) = (0.02)(1 - 0.07y)^{-\frac{5}{7}}, 0 < y < \frac{1}{0.07}.$$

Example 19

Suppose that $\delta = 0.07$ *and the force of mortality is* 0.02.
(i) Calculate the density of $\overline{Y}_{40}$.
(ii) Calculate $\overline{a}_{40}$.
(iii) Calculate the variance of $\overline{Y}_{40}$.

Solution: (i) The density of $\overline{Y}_{40}$ is

$$f_{\overline{Y}_{40}}(t) = (0.02)(1 - 0.07y)^{-\frac{5}{7}}, 0 < y < \frac{1}{0.07}.$$

(ii) $\overline{a}_{40} = \frac{1}{0.07+0.02} = 11.1111111111111$

Example 19

Suppose that $\delta = 0.07$ and the force of mortality is 0.02.
(i) Calculate the density of $\overline{Y}_{40}$.
(ii) Calculate $\overline{a}_{40}$.
(iii) Calculate the variance of $\overline{Y}_{40}$.

Solution: (i) The density of $\overline{Y}_{40}$ is

$$f_{\overline{Y}_{40}}(t) = (0.02)(1 - 0.07y)^{-\frac{5}{7}}, 0 < y < \frac{1}{0.07}.$$

(ii) $\overline{a}_{40} = \frac{1}{0.07+0.02} = 11.1111111111111$
(iii)

$$\overline{A}_{40} = \frac{0.02}{0.02 + 0.07} = 0.222222222222222,$$

$$^2\overline{A}_{40} = \frac{0.02}{0.02 + (2)0.07} = 0.125,$$

$$\mathrm{Var}(\overline{Y}_{40}) = \frac{0.125 - (0.222222222222222)^2}{(0.07)^2} = 15.4320987654321.$$

Theorem 21

Suppose that T_x has a p–th quantile ξ_p such that

$$\mathbb{P}\{T_x < \xi_p\} = p = \mathbb{P}\{T_x \le \xi_p\}.$$

Given $b, \delta > 0$, the p–th quantile of $b\frac{1-e^{-\delta T_x}}{\delta}$ is $b\frac{1-e^{-\delta \xi_p}}{\delta}$.

Example 20

Suppose that $i = 6\%$ and De Moivre model with terminal age 100.
(i) Calculate the 30% percentile and the 70% percentiles of $\overline{Y}_{30}$.
(ii) Calculate $\overline{a}_{30}$.
(iii) Calculate the variance of $\overline{Y}_{30}$.

Example 20

Suppose that $i = 6\%$ and De Moivre model with terminal age 100.
(i) Calculate the 30% percentile and the 70% percentiles of $\overline{Y}_{30}$.
(ii) Calculate $\bar{a}_{30}$.
(iii) Calculate the variance of $\overline{Y}_{30}$.

Solution: (i) Let $\xi_{0.30}$ the 30% percentile of T_{30}. We have that $0.3 = F_{T_{30}}(\xi_{0.30}) = \frac{\xi_{0.30}}{70}$. So, $\xi_{0.30} = 21$. The 30% percentile of $\overline{Y}_{30} = \frac{1-(1.06)^{-T_{30}}}{\ln(1.06)}$ is $\frac{1-(1.06)^{-21}}{\ln(1.06)} = 12.11357171$.
Let $\xi_{0.70}$ the 70% percentile of T_{30}. We have that $0.7 = F_{T_{30}}(\xi_{0.70}) = \frac{\xi_{0.70}}{70}$. So, $\xi_{0.70} = 49$. The 70% percentile of $\overline{Y}_{30} = \frac{1-(1.06)^{-T_{30}}}{\ln(1.06)}$ is $\frac{1-(1.06)^{-49}}{\ln(1.06)} = 16.17422339$.

Example 20

Suppose that $i = 6\%$ and De Moivre model with terminal age 100.
(i) Calculate the 30% percentile and the 70% percentiles of $\overline{Y}_{30}$.
(ii) Calculate $\overline{a}_{30}$.
(iii) Calculate the variance of $\overline{Y}_{30}$.

Solution: (ii) We have that

$$\overline{A}_{30} = \frac{\overline{a}_{\overline{70}|i}}{\omega - x} = \frac{1 - (1.06)^{-70}}{(70)\ln(1.06)} = 0.2410186701$$

and

$$\overline{a}_{30} = \frac{1 - \overline{A}_{30}}{\delta} = \frac{1 - 0.2410186701}{\ln(1.06)} = 13.02549429.$$

Example 20

Suppose that $i = 6\%$ and De Moivre model with terminal age 100.
(i) Calculate the 30% percentile and the 70% percentiles of $\overline{Y}_{30}$.
(ii) Calculate $\overline{a}_{30}$.
(iii) Calculate the variance of $\overline{Y}_{30}$.

Solution: (iii) We have that

$$^2\overline{A}_{30} = \frac{\overline{a}_{\overline{70}|(1+i)^2-1}}{\omega - x} = \frac{1-(1.06)^{-(2)(70)}}{(70)(2)\ln(1.06)} = 0.1225492409$$

and

$$\mathrm{Var}(\overline{Y}_{30}) = \frac{^2\overline{A}_{30} - (\overline{A}_{30})^2}{\delta^2} = \frac{0.1225492409 - (0.2410186701)^2}{(\ln(1.06))^2}$$

$=207.8908307.$

Example 21

Suppose that $v = 0.92$ *and that the force of mortality is* $\mu_{x+t} = 0.02$, *for* $t \geq 0$.
(i) Calculate the density of $\overline{Y}_x$.
(ii) Calculate the first, second and third quartiles of $\overline{Y}_x$.
(iii) Calculate $\overline{a}_x$.

Example 21

Suppose that $v = 0.92$ and that the force of mortality is $\mu_{x+t} = 0.02$, for $t \geq 0$.
(i) Calculate the density of $\overline{Y}_x$.
(ii) Calculate the first, second and third quartiles of $\overline{Y}_x$.
(iii) Calculate $\overline{a}_x$.

Solution: (i) We have that

$$f_{\overline{Y}_x}(y) = \mu(1 - \delta y)^{\frac{\mu}{\delta} - 1} = (0.02)(1 + \ln(0.92)y)^{-\frac{0.02}{\ln(0.92)} - 1},$$

$$\text{if } 0 < y \leq \frac{1}{-\ln(0.92)}.$$

Example 21

Suppose that $v = 0.92$ *and that the force of mortality is* $\mu_{x+t} = 0.02$*, for* $t \geq 0$*.*
(i) Calculate the density of $\overline{Y}_x$*.*
(ii) Calculate the first, second and third quartiles of $\overline{Y}_x$*.*
(iii) Calculate $\overline{a}_x$*.*

Solution: (ii) Let ξ_p the $100(p)\%$ percentile of $\overline{Y}_x$. We have that

$$p = F_{Y_x}(\xi_p) = 1 - (1 - \delta\xi_p)^{\frac{\mu}{\delta}} = (1 + \ln(0.92)\xi_p)^{\frac{0.02}{-\ln(0.92)}}.$$

So, $\xi_p = \frac{1-(1-p)^{\frac{-\ln(0.92)}{0.02}}}{-\ln(0.92)}$. The first quartile of $\overline{Y}_x$ is

$$\xi_{0.25} = \frac{1 - (1 - 0.25)^{\frac{-\ln(0.92)}{0.02}}}{-\ln(0.92)} = 8.378536891.$$

Example 21

Suppose that $v = 0.92$ *and that the force of mortality is* $\mu_{x+t} = 0.02$, *for* $t \geq 0$.
(i) Calculate the density of $\overline{Y}_x$.
(ii) Calculate the first, second and third quartiles of $\overline{Y}_x$.
(iii) Calculate $\overline{a}_x$.

Solution: (ii)
The median of $\overline{Y}_x$ is

$$\xi_{0.5} = \frac{1 - (1 - 0.5)^{\frac{-\ln(0.92)}{0.02}}}{-\ln(0.92)} = 11.3263815.$$

The third quartile of $\overline{Y}_x$ is

$$\xi_{0.75} = \frac{1 - (1 - 0.75)^{\frac{-\ln(0.92)}{0.02}}}{-\ln(0.92)} = 11.95599338.$$

Example 21

Suppose that $v = 0.92$ *and that the force of mortality is* $\mu_{x+t} = 0.02$, *for* $t \geq 0$.
(i) Calculate the density of $\overline{Y}_x$.
(ii) Calculate the first, second and third quartiles of $\overline{Y}_x$.
(iii) Calculate $\overline{a}_x$.

Solution: (iii) We have that

$$\overline{a}_x = \frac{1}{\mu + \delta} = \frac{1}{0.02 - \ln(0.92)} = 9.672900338.$$

