Manual for SOA Exam MLC.

Chapter 5. Life annuities.

Section 5.2. Deferred life annuities.

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Due n–year deferred annuity

Definition 1

A **due** n**–year deferred annuity** *guarantees payments made at the beginning of the year while an individual is alive starting in n years.*

The present value of a due n–year deferred annuity is denoted by ${}_{n|}\ddot{Y}_x$.

Definition 2

The actuarial present value of a due n–year deferred annuity for (x) *with unit payment is denoted by* ${}_{n|}\ddot{a}_x$.

We have that ${}_{n|}\ddot{a}_x = E[{}_{n|}\ddot{Y}_x]$.

Theorem 1

$$ {}_{n|}\ddot{Y}_x = v^n \ddot{a}_{\overline{K_x - n|}} I(K_x > n) = \begin{cases} 0 & \text{if } K_x \le n, \\ v^n \ddot{a}_{\overline{K_x - n|}} & \text{if } K_x > n, \end{cases} $$

and

$$ {}_{n|}\ddot{a}_x = \sum_{k=n+1}^{\infty} v^n \ddot{a}_{\overline{k-n|}} \cdot {}_{k-1|}q_x. $$

Proof: If $K_x \le n$, then $T_x \le n$ and no payment is made. If $K_x \ge n+1$, then $T_x \in (K_x - 1, K_x]$ and unit payments at times $n, \cdots, K_x - 1$ are made. The present value of a unit annuity paid at times $n, \cdots, K_x - 1$ is $v^n \ddot{a}_{\overline{K_x - n|}}$. Hence,

$$ {}_{n|}\ddot{Y}_x = v^n \ddot{a}_{\overline{K_x - n|}} I(K_x > n) \text{ and} $$

$$ {}_{n|}\ddot{a}_x = \sum_{k=n+1}^{\infty} v^n \ddot{a}_{\overline{k-n|}} \mathbb{P}\{K_x = k\} = \sum_{k=n+1}^{\infty} v^n \ddot{a}_{\overline{k-n|}} \cdot {}_{k-1|}q_x. $$

Theorem 2

If $i \neq 0$,

$$_{n|}\ddot{Y}_x = \frac{Z_{x:\overline{n}|}^{\;1} - {}_{n|}Z_x}{d}$$

and

$$_{n|}\ddot{a}_x = \frac{A_{x:\overline{n}|}^{\;1} - {}_{n|}A_x}{d} = \frac{A_{x:\overline{n}|}^{\;1}(1 - A_{x+n})}{d}.$$

Theorem 2

If $i \neq 0$,

$$ {}_{n|}\ddot{Y}_x = \frac{Z_{x:\overline{n}|}^{\ \ 1} - {}_{n|}Z_x}{d} $$

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$$ {}_{n|}\ddot{a}_x = \frac{A_{x:\overline{n}|}^{\ \ 1} - {}_{n|}A_x}{d} = \frac{A_{x:\overline{n}|}^{\ \ 1}(1 - A_{x+n})}{d}. $$

Proof:

$$ {}_{n|}\ddot{Y}_x = v^n \ddot{a}_{\overline{K_x - n}|} I(K_x > n) = v^n \frac{1 - v^{K_x - n}}{d} I(K_x > n) $$
$$ = \frac{v^n - v^{K_x}}{d} I(K_x > n) = \frac{Z_{x:\overline{n}|}^{\ \ 1} - {}_{n|}Z_x}{d}. $$

So,

$$ {}_{n|}\ddot{a}_x = \frac{A_{x:\overline{n}|}^{\ \ 1} - {}_{n|}A_x}{d} = \frac{A_{x:\overline{n}|}^{\ \ 1} - A_{x:\overline{n}|}^{\ \ 1} A_{x+n}}{d} = \frac{A_{x:\overline{n}|}^{\ \ 1}(1 - A_{x+n})}{d}. $$

Example 1

Suppose that $A_{x:\overline{n}|}^{\;\;1} = 0.3$, $A_{x+n} = 0.6$ *and* $i = 0.05$. *Find* ${}_{n|}\ddot{a}_x$.

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Solution: We have that $d = \frac{i}{1+i} = \frac{0.05}{1.05} = \frac{1}{21}$ and

$$
\begin{aligned}
{}_{n|}A_x &= A_{x:\overline{n}|}^{\ \ 1} A_{x+n} = (0.3)(0.6) = 0.18, \\
{}_{n|}\ddot{a}_x &= \frac{A_{x:\overline{n}|}^{\ \ 1} - {}_{n|}A_x}{d} = \frac{0.3 - 0.18}{1/21} = 2.52.
\end{aligned}
$$

Theorem 3

(current payment method)

$$ {}_{n|}\ddot{Y}_x = \sum_{k=n}^{\infty} v^k I(K_x > k) = \sum_{k=n}^{\infty} Z_{x:\overset{\,1}{\overline{k|}}} $$

and

$$ {}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k \cdot {}_k p_x = {}_n E_x \ddot{a}_{x+n}. $$

Theorem 3

(current payment method)

$$ {}_{n|}\ddot{Y}_x = \sum_{k=n}^{\infty} v^k I(K_x > k) = \sum_{k=n}^{\infty} Z_{x:\overline{k|}}^{\;\;1} $$

and

$$ {}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k \cdot {}_k p_x = {}_n E_x \ddot{a}_{x+n}. $$

Proof: Given $k \geq n$, a payment at time k is made if and only if the individual is alive at time k. An individual is alive at time k if and only if $T_x > k$. Hence,

$$ {}_{n|}\ddot{Y}_x = \sum_{k=n}^{\infty} v^k I(T_x > k) = \sum_{k=n}^{\infty} Z_{x:\overline{k|}}^{\;\;1}. $$

Theorem 3

(current payment method)

$$ {}_{n|}\ddot{Y}_x = \sum_{k=n}^{\infty} v^k I(K_x > k) = \sum_{k=n}^{\infty} Z_{x:\overset{1}{\overline{k|}}} $$

and

$$ {}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k \cdot {}_k p_x = {}_n E_x \ddot{a}_{x+n}. $$

Proof: Thus,

$$ {}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k \cdot {}_k p_x = \sum_{k=0}^{\infty} v^{k+n} \cdot {}_{k+n} p_x = \sum_{k=0}^{\infty} v^{k+n} \cdot {}_n p_x \cdot {}_k p_{x+n} $$

$$ = v^n \cdot {}_n p_x \sum_{k=0}^{\infty} v^k \cdot {}_k p_{x+n} = {}_n E_x \cdot \ddot{a}_{x+n}. $$

Theorem 4

If $i = 0$, ${}_{n|}\ddot{a}_x = {}_n p_x(1 + e_{x+n})$.

Proof: ${}_{n|}\ddot{a}_x = {}_n E_x \ddot{a}_{x+n} = {}_n p_x(1 + e_{x+n})$.

Example 2

Suppose that $A_{x:\overline{n}|}^{\ \ 1} = 0.3$, $A_{x+n} = 0.6$ *and* $i = 0.05$. *Find* ${}_{n|}\ddot{a}_x$.

Example 2

Suppose that $A_{x:\overline{n}|}^{\ \ 1} = 0.3$, $A_{x+n} = 0.6$ *and* $i = 0.05$. *Find* ${}_{n|}\ddot{a}_x$.

Solution:

$$\ddot{a}_{x+n} = \frac{1 - 0.6}{1/21} = 8.4,$$

$$ {}_{n|}\ddot{a}_x = {}_nE_x \ddot{a}_{x+n} = (0.3)(8.4) = 2.52.$$

Theorem 5

$$E\left[\left({}_{n|}\ddot{Y}_x\right)^2\right] = v^n \cdot {}_n p_x E\left[\left(\ddot{Y}_{x+n}\right)^2\right]$$
$$=\frac{v^{2n} \cdot {}_n p_x (2\ddot{a}_{x+n} - (2-d) \cdot {}^2\ddot{a}_{x+n})}{d}.$$

Proof: Using that $K_x - n | K_x > n$ and K_{x+n} have the same distribution.

$$E\left[\left({}_{n|}\ddot{Y}_x\right)^2\right] = E\left[v^{2n}\left(\ddot{a}_{\overline{K_x-n}|}\right)^2 I(K_x > n)\right]$$
$$=v^{2n} \cdot {}_n p_x E\left[\left(\ddot{a}_{\overline{K_x-n}|}\right)^2 \,\middle|\, K_x > n\right]$$
$$=v^{2n} \cdot {}_n p_x E\left[\left(\ddot{a}_{\overline{K_{x+n}}|}\right)^2\right] = v^{2n} \cdot {}_n p_x E\left[\ddot{Y}^2_{x+n}\right]$$
$$=\frac{v^{2n} \cdot {}_n p_x (2\ddot{a}_{x+n} - (2-d) \cdot {}^2\ddot{a}_{x+n})}{d}.$$

Example 3

Suppose that $v = 0.91$ *and* $p_{x+k} = 0.97$ *for each integer* $k \geq 0$*. Find* ${}_{40|}\ddot{a}_x$ *and* $\mathrm{Var}({}_{40|}\ddot{Y}_x)$*.*

Example 3

Suppose that $v = 0.91$ and $p_{x+k} = 0.97$ for each integer $k \geq 0$. Find ${}_{40|}\ddot{a}_x$ and $\mathrm{Var}({}_{40|}\ddot{Y}_x)$.

Solution: We have that

$$
\begin{aligned}
{}_nE_x &= (0.97)^{40}(0.91)^{40} = 0.006800252887,\\
\ddot{a}_{x+n} &= \frac{1}{1-(0.97)(0.91)} = 8.52514919,\\
{}_{40|}\ddot{a}_x &= {}_nE_x \cdot \ddot{a}_{x+n} = (0.006800252887)(8.52514919)\\
=&0.05797317039.
\end{aligned}
$$

Example 3

Suppose that $v = 0.91$ *and* $p_{x+k} = 0.97$ *for each integer* $k \geq 0$. *Find* ${}_{40|}\ddot{a}_x$ *and* $\mathrm{Var}({}_{40|}\ddot{Y}_x)$.

Solution:

$$^2\ddot{a}_{x+n} = \frac{1}{1-(0.97)(0.91)^2} = 5.082772958,$$

$$E\left[\left({}_{40|}\ddot{Y}_x\right)^2\right]$$

$$=\frac{(0.91)^{80}(0.97)^{40}((2)(8.52514919)-(2-0.09)(5.082772958))}{0.09}$$

$$=0.01275747064$$

$$=\frac{(0.91)^{80}(0.97)^{40}((2)(8.52514919)-(2-0.09)(5.082772958))}{0.09}$$

$$=0.01275747064$$

$$\mathrm{Var}({}_{40|}\ddot{Y}_x) = 0.01275747064 - (0.05797317039)^2 = 0.009396582155.$$

Theorem 6

Under De Moivre's model and integers x and ω,

(i) ${}_{n|}\ddot{a}_x = \frac{v^n (D\ddot{a})_{\overline{\omega-x-n|}}}{\omega-x}$.

(ii) If $i \neq 0$, ${}_{n|}\ddot{a}_x = \frac{v^n(\omega-x-n-\ddot{a}_{\overline{\omega-x-n|}})}{(\omega-x)d}$.

(ii) If $i = 0$, ${}_{n|}\ddot{a}_x = \frac{(\omega-x-n)(\omega-x-n+1)}{2(\omega-x)}$.

Theorem 6

Under De Moivre's model and integers x and ω,

(i) ${}_{n|}\ddot{a}_x = \frac{v^n (D\ddot{a})_{\overline{\omega-x-n}|}}{\omega-x}$.

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(ii) If $i = 0$, ${}_{n|}\ddot{a}_x = \frac{(\omega-x-n)(\omega-x-n+1)}{2(\omega-x)}$.

Proof: (i) We have that

$${}_{n|}\ddot{a}_x = {}_nE_x \cdot \ddot{a}_{x+n} = v^n \frac{\omega - x - n}{\omega - x} \frac{(D\ddot{a})_{\overline{\omega-x-n}|}}{\omega - x - n} = \frac{v^n (D\ddot{a})_{\overline{\omega-x-n}|}}{\omega - x}.$$

Theorem 6

Under De Moivre's model and integers x and ω,

(i) $_{n|}\ddot{a}_x = \frac{v^n (D\ddot{a})_{\overline{\omega-x-n}|}}{\omega-x}$.

(ii) If $i \neq 0$, $_{n|}\ddot{a}_x = \frac{v^n(\omega-x-n-\ddot{a}_{\overline{\omega-x-n}|})}{(\omega-x)d}$.

(ii) If $i = 0$, $_{n|}\ddot{a}_x = \frac{(\omega-x-n)(\omega-x-n+1)}{2(\omega-x)}$.

Proof: (ii) If $i \neq 0$, $(D\ddot{a})_{\overline{n}|} = \frac{n-\ddot{a}_{\overline{n}|}}{d}$. So,

$$_{n|}\ddot{a}_x = \frac{v^n (D\ddot{a})_{\overline{\omega-x-n}|}}{\omega-x} = \frac{v^n \frac{\omega-x-n-\ddot{a}_{\overline{\omega-x-n}|})}{d}}{\omega-x}$$

$$= \frac{v^n(\omega-x-n-\ddot{a}_{\overline{\omega-x-n}|})}{(\omega-x)d}.$$

Theorem 6

Under De Moivre's model and integers x and ω,

(i) ${}_{n|}\ddot{a}_x = \frac{v^n (D\ddot{a})_{\overline{\omega-x-n}|}}{\omega-x}$.

(ii) If $i \neq 0$, ${}_{n|}\ddot{a}_x = \frac{v^n(\omega-x-n-\ddot{a}_{\overline{\omega-x-n}|})}{(\omega-x)d}$.

(ii) If $i = 0$, ${}_{n|}\ddot{a}_x = \frac{(\omega-x-n)(\omega-x-n+1)}{2(\omega-x)}$.

Proof: (iii) If $i = 0$, $(D\ddot{a})_{\overline{n}|} = \frac{n(n+1)}{2}$. So,

$$
{}_{n|}\ddot{a}_x = \frac{v^n \, (D\ddot{a})_{\overline{\omega-x-n}|}}{\omega - x} = \frac{\frac{(\omega-x-n)(\omega-x-n+1)}{2}}{\omega - x}
= \frac{(\omega - x - n)(\omega - x - n + 1)}{2(\omega - x)}.
$$

Example 4

Suppose that $v = 0.91$ *and De Moivre's model with terminal age 100. Find* ${}_{20|}\ddot{a}_{40}$.

Example 4

Suppose that $v = 0.91$ and De Moivre's model with terminal age 100. Find ${}_{20|}\ddot{a}_{40}$.

Solution 1: We have that $i = \frac{1-v}{v} = \frac{9}{91}$. Hence,

$$(D\ddot{a})_{\overline{40}|9/91} = \frac{40 - \ddot{a}_{\overline{40}|9/91}}{0.09} = 334.6822869,$$

$${}_{20|}\ddot{a}_{40} = \frac{(0.91)^{20}\,(D\ddot{a})_{\overline{40}|9/91}}{60} = \frac{(0.91)^{20}(334.6822869)}{60} = 0.8458811048.$$

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Solution 2: We have that

$${}_{20}E_{40} = v^{20}{}_{20}p_{40} = (0.91)^{20}\frac{60-20}{60} = 0.1010966087,$$

$$A_{60} = \frac{a_{\overline{40}|9/91}}{40} = 0.2469648546,$$

$$\ddot{a}_{60} = \frac{1 - A_{60}}{d} = \frac{1 - 0.2469648546}{0.09} = 8.367057171,$$

$${}_{20|}\ddot{a}_{40} = {}_{20}E_{40}\ddot{a}_{60} = (0.1010966087)(8.367057171) = 0.8458811048.$$

Theorem 7

Under constant force of mortality μ, ${}_{n|}\ddot{a}_x = \frac{v^n p_x^n}{1-vp_x} = \frac{e^{-n(\delta+\mu)}}{1-e^{-(\delta+\mu)}}$.

Proof.

$$ {}_{n|}\ddot{a}_x = {}_nE_x \ddot{a}_x = v^n p_x^n \frac{1}{1-vp_x} = \frac{v^n p_x^n}{1-vp_x} = \frac{e^{-n(\delta+\mu)}}{1-e^{-(\delta+\mu)}}. $$

□

Example 5

Suppose that $v = 0.91$ *and the force of mortality is* $\mu = 0.005$*.*
Find ${}_{25|}\ddot{a}_x$*.*

Example 5

Suppose that $v = 0.91$ *and the force of mortality is* $\mu = 0.005$. *Find* ${}_{25|}\ddot{a}_x$.

Solution:

$$ {}_{25|}\ddot{a}_x = \frac{(0.91)^{25} e^{-(25)(0.005)}}{1 - (0.91)e^{-0.005}} = 0.883361829627389. $$

Theorem 8

$${}_{n|}\ddot{a}_x = vp_x \cdot {}_{n-1|}\ddot{a}_{x+1}.$$

Proof:

$${}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k \cdot {}_k p_x = vp_x \sum_{k=n}^{\infty} v^{k-1} \cdot {}_{k-1}p_{x+1} = vp_x \sum_{k=n-1}^{\infty} v^k \cdot {}_k p_{x+1}$$

$$= vp_x \cdot {}_{n-1|}\ddot{a}_{x+1}.$$

Example 6

Using $i = 0.05$ *and a certain life table* ${}_{10|}\ddot{a}_{30} = 7.48$. *Suppose that an actuary revises this life table and changes* p_{30} *from* 0.95 *to* 0.96. *Other values in the life table are unchanged. Find* ${}_{10|}\ddot{a}_{30}$ *using the revised life table.*

Example 6

Using $i = 0.05$ *and a certain life table* ${}_{10|}\ddot{a}_{30} = 7.48$*. Suppose that an actuary revises this life table and changes* p_{30} *from* 0.95 *to* 0.96*. Other values in the life table are unchanged. Find* ${}_{10|}\ddot{a}_{30}$ *using the revised life table.*

Solution: We have that ${}_{10|}\ddot{a}_{30} = vp_x \cdot {}_{9|}\ddot{a}_{31}$. Hence, under the old table

$$ {}_{9|}\ddot{a}_{31} = \frac{(1.05)(7.48)}{0.95} = 8.267368421. $$

Since ${}_{9|}\ddot{a}_{31}$ does not depend on p_{30}, using the revised life table ${}_{9|}\ddot{a}_{31} = 8.267368421$. Hence, using the revised life table

$$ {}_{10|}\ddot{a}_{30} = (1.05)^{-1}(0.96)(8.267368421) = 7.558736842. $$

n-year deferred annuity immediate

Definition 3

An **immediate** n**–year deferred annuity** *guarantees payments made at the end of the year while an individual is alive starting n years from now.*

The present value of an immediate n-year term annuity is denoted by ${}_{n|}Y_x$.

Definition 4

The actuarial present value of an immediate n–year deferred annuity for (x) *with unit payment is denoted by* ${}_{n|}a_x$.

We have that ${}_{n|}a_x = E[{}_{n|}Y_x]$.

Theorem 9

$$ {}_{n|}Y_x = v^n a_{\overline{K_x-n-1}|} I(K_x > n+1) = \begin{cases} 0 & \text{if } K_x \le n+1, \\ v^n a_{\overline{K_x-n-1}|} & \text{if } K_x > n+1, \end{cases} $$

and

$$ {}_{n|}a_x = \sum_{k=n+2}^{\infty} v^n a_{\overline{k-n-1}|} \cdot {}_{k-1|}q_x. $$

Theorem 9

$$ {}_{n|}Y_x = v^n a_{\overline{K_x-n-1|}} I(K_x > n+1) = \begin{cases} 0 & \text{if } K_x \le n+1, \\ v^n a_{\overline{K_x-n-1|}} & \text{if } K_x > n+1, \end{cases} $$

and

$$ {}_{n|}a_x = \sum_{k=n+2}^{\infty} v^n a_{\overline{k-n-1|}} \cdot {}_{k-1|}q_x. $$

Proof: If $K_x \le n+1$, then $T_x \le n$ and no payment is made. If $K_x \ge n+2$, then $T_x \in (K_x - 1, K_x]$ and unit payments at times $n+1, \cdots, K_x - 1$ are made. The present value of a unit annuity paid at times $n+1, \cdots, K_x - 1$ is $v^n a_{\overline{K_x-n-1|}}$. Hence,

$$ {}_{n|}Y_x = v^n a_{\overline{K_x-n-1|}} I(K_x > n+1) = \begin{cases} 0 & \text{if } K_x \le n+1, \\ v^n a_{\overline{K_x-n-1|}} & \text{if } K_x > n+1, \end{cases} $$

Theorem 9

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and

$$_{n|}a_x = \sum_{k=n+2}^{\infty} v^n a_{\overline{k-n-1}|} \cdot {}_{k-1|}q_x.$$

Proof: and

$$_{n|}a_x = \sum_{k=n+2}^{\infty} v^n a_{\overline{k-n-1}|} \mathbb{P}\{K_x = k\} = \sum_{k=n+2}^{\infty} v^n a_{\overline{k-n-1}|} \cdot {}_{k-1|}q_x.$$

Theorem 10

$$
{}_{n|}Y_x = \sum_{k=n+1}^{\infty} Z_{x:\overline{k|}}^{\;\;1} = {}_{n+1|}\ddot{Y}_x \quad \text{and} \quad {}_{n|}a_x = \sum_{k=n+1}^{\infty} v^k \, {}_k p_x = {}_n E_x \cdot a_{x+n}.
$$

Theorem 10

$$ {}_{n|}Y_x = \sum_{k=n+1}^{\infty} Z_{x:\overset{1}{\overline{k|}}} = {}_{n+1|}\ddot{Y}_x \quad \text{and} \quad {}_{n|}a_x = \sum_{k=n+1}^{\infty} v^k\, {}_k p_x = {}_n E_x \cdot a_{x+n}. $$

Proof: Given $k \geq n+1$, a payment at time k is made if and only if the individual is alive at time k. An individual is alive at time k if and only if $T_x > k$. Hence,

$$ {}_{n|}Y_x = \sum_{k=n+1}^{\infty} v^k I(T_x > k) = \sum_{k=n+1}^{\infty} Z_{x:\overset{1}{\overline{k|}}}, $$

$$ {}_{n|}a_x = \sum_{k=n+1}^{\infty} v^k\, {}_k p_x = \sum_{k=1}^{\infty} v^{n+k} \cdot {}_{n+k} p_x = \sum_{k=1}^{\infty} v^n v^k \cdot {}_n p_x \cdot {}_k p_{x+n} $$

$$ = v^n \cdot {}_n p_x \sum_{k=1}^{\infty} v^k \cdot {}_k p_{x+n} = {}_n E_x \cdot a_{x+n}. $$

Theorem 11

If $i = 0$,

$$ {}_{n|}a_x = {}_n p_x e_x . $$

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$$ {}_{n|}a_x = {}_np_x e_x. $$

Proof: We have that ${}_{n|}a_x = {}_nE_x \cdot a_{x+n} = {}_np_x e_x$.

Theorem 12

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Proof: We have that

$$ {}_{n|}a_x = {}_{n+1|}\ddot{a}_x = \frac{e^{-(n+1)(\delta+\mu)}}{1 - e^{-(\delta+\mu)}}. $$

Example 7

Suppose that $v = 0.91$ *and* $p_x = 0.97$ *for each* $x \geq 0$. *Find* ${}_{40|}a_x$.

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Suppose that $v = 0.91$ *and* $p_x = 0.97$ *for each* $x \geq 0$. *Find* ${}_{40|}a_x$.

Solution: We have that

$$ {}_{40|}a_x = \frac{(0.91)^{41}(0.97)^{41}}{1-(0.91)(0.97)} = 0.0511729175. $$

Theorem 13

Under De Moivre model and integers x and ω,

$$ {}_{n|}a_x = \frac{v^n (Da)_{\overline{\omega-x-n-1}|}}{\omega - x}. $$

Proof: We have that

$$ {}_{n|}a_x = {}_{n|}\ddot{a}_x = \frac{v^{n+1} (D\ddot{a})_{\overline{\omega-x-n-1}|}}{\omega - x} = \frac{v^n (Da)_{\overline{\omega-x-n-1}|}}{\omega - x}. $$

Example 8

Suppose that $v = 0.91$ *and the De Moivre model with terminal age 100. Find* ${}_{20|}a_{40}$.

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Suppose that $v = 0.91$ *and the De Moivre model with terminal age 100. Find* ${}_{20|}a_{40}$.

Solution 1: We have that $i = \frac{1-v}{v} = \frac{9}{91}$. Hence,

$$ {}_{20|}a_{40} = \frac{(0.91)^{20}\,(Da)_{\overline{39}|\frac{9}{91}}}{60} = 0.7447844961. $$

Example 8

Suppose that $v = 0.91$ *and the De Moivre model with terminal age 100. Find* ${}_{20|}a_{40}$.

Solution 1: We have that $i = \frac{1-v}{v} = \frac{9}{91}$. Hence,

$$ {}_{20|}a_{40} = \frac{(0.91)^{20}\,(Da)_{\overline{39}|\frac{9}{91}}}{60} = 0.7447844961. $$

Solution 2:

$$ {}_{20}E_{40} = v^{20}{}_{20}p_{40} = (0.91)^{20}\frac{60-20}{60} = 0.1010966087, $$

$$ A_{60} = \frac{a_{\overline{40}|9/91}}{40} = 0.2469648546, $$

$$ a_{60} = \frac{v - A_{60}}{d} = \frac{0.91 - 0.2469648546}{1 - 0.91} = 8.522627307, $$

$$ {}_{20|}a_{40} = {}_{20}E_{40} \cdot a_{60} = (0.1010966087)(8.522627307) = 0.7447844961. $$

Theorem 14

$$E\left[({}_{n}|Y_x)^2\right] = v^n \cdot {}_n p_x E\left[Y_{x+n}^2\right] = \frac{v^{2n} \cdot {}_n p_x (2a_{x+n} - (2-d) \cdot {}^2a_{x+n})}{d}.$$

Theorem 14

$$E\left[({}_n|Y_x)^2\right] = v^n \cdot {}_np_x E\left[Y_{x+n}^2\right] = \frac{v^{2n} \cdot {}_np_x(2a_{x+n} - (2-d) \cdot {}^2a_{x+n})}{d}.$$

Proof: Using that $K_x - n | K_x > n$ has the distribution of K_{x+n},

$$\begin{aligned}
E\left[({}_n|Y_x)^2\right] &= E\left[v^{2n}\left(a_{\overline{K_x-n-1}|}\right)^2 I(K_x > n)\right] \\
&= v^{2n} \cdot {}_np_x E\left[\left(a_{\overline{K_x-n-1}|}\right)^2 | K_x > n\right] \\
&= v^{2n} \cdot {}_np_x E\left[\left(a_{\overline{K_{x+n}-1}|}\right)^2\right] = v^{2n} \cdot {}_np_x E\left[Y_{x+n}^2\right] \\
&= \frac{v^{2n} \cdot {}_np_x(2a_{x+n} - (2-d) \cdot {}^2a_{x+n})}{d}.
\end{aligned}$$

Theorem 15

$$ {}_{n|}a_x = v p_x \cdot {}_{n-1|}a_{x+1}. $$

Theorem 15

$$ {}_{n|}a_x = vp_x \cdot {}_{n-1|}a_{x+1}. $$

Proof:

$$ {}_{n|}a_x = \sum_{k=n+1}^{\infty} v^k \cdot {}_kp_x = vp_x \sum_{k=n+1}^{\infty} v^{k-1} \cdot {}_{k-1}p_{x+1} $$

$$ = vp_x \sum_{k=n}^{\infty} v^k \cdot {}_kp_{x+1} = vp_x \cdot {}_{n-1|}a_{x+1}. $$

n–year deferred annuity continuous

Definition 5

*A n–**year deferred continuous annuity** guarantees a continuous flow of payments while the individual is alive starting in n years.*

Definition 6

The present value of an immediate n–year term annuity is denoted by ${}_{n|}\overline{Y}_x$.

Definition 7

The actuarial present value of a whole life immediate annuity for (x) with unit payment is denoted by ${}_{n|}\overline{a}_x$.

We have that ${}_{n|}\overline{a}_x = E[{}_{n|}\overline{Y}_x]$.

Theorem 16

$$
{}_{n|}\overline{Y}_x = \int_n^{T_x} v^s \, ds I(T_x > n) = v^n \overline{a}_{\overline{T_x - n}|} I(T_x > n)
$$

$$
= \begin{cases} 0 & \text{if } T_x \le n, \\ v^n \overline{a}_{\overline{T_x - n}|} & \text{if } T_x > n. \end{cases}
$$

Theorem 16

$$
{}_{n|}\overline{Y}_x = \int_n^{T_x} v^s \, ds I(T_x > n) = v^n \overline{a}_{\overline{T_x - n}|} I(T_x > n)
$$

$$
= \begin{cases} 0 & \text{if } T_x \le n, \\ v^n \overline{a}_{\overline{T_x - n}|} & \text{if } T_x > n. \end{cases}
$$

Proof: If $T_x \le n$, ${}_{n|}\overline{Y}_x = 0$. If $T_x > n$, the unit continuous rate runs from n to T_x and

$$
{}_{n|}\overline{Y}_x = \int_n^{T_x} v^s \, ds = \int_0^{T_x - n} v^{n+s} \, ds = v^n \overline{a}_{\overline{T_x - n}|}.
$$

Hence, ${}_{n|}\overline{Y}_x = \int_n^{T_x} v^s \, ds I(T_x > n) = v^n \overline{a}_{\overline{T_x - n}|} I(T_x > n)$.

Theorem 17

$$ {}_{n|}\overline{Y}_x = \frac{Z_{x:\overline{n|}}^{\ 1} - {}_{n|}\overline{Z}_x}{\delta}. $$

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$$ {}_{n|}\overline{Y}_x = \frac{Z_{x:\overline{n}|}^{\ 1} - {}_{n|}\overline{Z}_x}{\delta}. $$

Proof: We have that

$$ v^n \overline{a}_{\overline{T_x - n}|} I(T_x > n) = v^n \frac{1 - v^{T_x - n}}{\delta} I(T_x > n) $$
$$ = \frac{e^{-n\delta} - e^{-T_x\delta}}{\delta} I(T_x > n) = \frac{Z_{x:\overline{n}|}^{\ 1} - {}_{n|}\overline{Z}_x}{\delta}. $$

Theorem 18

$$_{n|}\bar{a}_x = \int_n^\infty v^n \bar{a}_{\overline{t-n|}} \cdot {}_tp_x \cdot \mu_{x+t}\, dt.$$

Proof.

$$_{n|}\bar{a}_x = E[v^n \bar{a}_{\overline{T_x-n|}} I(T_x > n] = \int_n^\infty v^n \bar{a}_{\overline{t-n|}} f_{T_x}(t)\, dt$$
$$= \int_n^\infty v^n \bar{a}_{\overline{t-n|}} \cdot {}_tp_x \cdot \mu_{x+t}\, dt.$$

□

Theorem 19

(current payment method)

$$ {}_{n|}\overline{a}_x = \int_n^\infty v^t \cdot {}_tp_x \, dt. $$

Proof: Let

$$ h(t) = v^t I(t > n) = \begin{cases} 0 & \text{if } t \le n, \\ v^t & \text{if } t > n, \end{cases} $$

Then,

$$ H(t) = \int_0^t v^s I(s > n) \, ds = \int_n^t v^s \, ds I(t > n) = \begin{cases} 0 & \text{if } t \le n, \\ \int_n^t v^s \, ds & \text{if } t > n, \end{cases} $$

By a previous theorem,

$$ {}_{n|}\overline{a}_x = E[H(T_x)] = \int_0^\infty h(t) s_{T_x}(t) \, dt = \int_n^\infty v^t \cdot {}_tp_x \, dt. $$

Theorem 20

$$ {}_{n|}\overline{a}_x = v^n \cdot {}_n p_x \cdot \overline{a}_{x+n} = {}_n E_x \cdot \overline{a}_{x+n}. $$

Proof.

Using that $T_x - t | T_x > t$ and T_{x+t} have the same distribution,

$$ {}_{n|}\overline{a}_x = E[v^n \overline{a}_{\overline{T_x - n}|} I(T_x > n)] = v^n \cdot {}_n p_x E[\overline{a}_{\overline{T_x - n}|} | T_x > n] $$
$$ = v^n \cdot {}_n p_x E[\overline{a}_{\overline{T_{x+n}}|}] = v^n \cdot {}_n p_x \cdot \overline{a}_{x+n} = {}_n E_x \cdot \overline{a}_{x+n}. $$

□

Theorem 21

If $i = 0$, ${}_{n|}\overline{a}_x = {}_n p_x \mathring{e}_{x+n}$.

Proof.

$${}_{n|}\overline{a}_x = {}_n E_x \cdot \overline{a}_{x+n} = {}_n p_x \mathring{e}_{x+n}.$$

□

Theorem 22

$$E\left[\left({}_{n}|\overline{Y}_x\right)^2\right] = v^{2n}\cdot {}_np_x E[\left(\overline{Y}_{x+n}\right)^2] = v^{2n}\cdot {}_np_x \frac{2\left(\overline{a}_{x+n} - {}^2\overline{a}_{x+n}\right)}{\delta}.$$

Proof: Using that $T_x - t | T_x > t$ and T_{x+t} have the same distribution,

$$\begin{aligned} E\left[\left({}_{n}|\overline{Y}_x\right)^2\right] &= E\left[v^{2n}\left(\overline{a}_{\overline{T_x-n}|}\right)^2 I(T_x > n)\right] \\ =& v^{2n}\cdot {}_np_x E\left[\left(\overline{a}_{\overline{T_x-n}|}\right)^2 \,\middle|\, T_x > n)\right] \\ =& Ev^{2n}\cdot {}_np_x E\left[\left(\overline{a}_{\overline{T_{x+n}}|}\right)^2\right] = v^{2n}\cdot {}_np_x E[\left(\overline{Y}_{x+n}\right)^2] \\ =& v^{2n}\cdot {}_np_x \frac{2\left(\overline{a}_{x+n} - {}^2\overline{a}_{x+n}\right)}{\delta}. \end{aligned}$$

Theorem 23

Under De Moivre's model,

(i) ${}_{n|}\overline{a}_x = \frac{v^n(\overline{D}\overline{a})_{\overline{\omega-x-n|}}}{\omega-x}$.

(ii) If $i \neq 0$, ${}_{n|}\overline{a}_x = \frac{v^n\left(\omega-x-n-\overline{a}_{\overline{\omega-x-n|}}\right)}{(\omega-x)\delta}$.

(iii) If $i = 0$, ${}_{n|}\overline{a}_x = \frac{(\omega-x-n)^2}{2(\omega-x)}$.

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(iii) If $i = 0$, ${}_{n|}\overline{a}_x = \frac{(\omega-x-n)^2}{2(\omega-x)}$.

Proof: (i) We have that

$$ {}_{n|}\overline{a}_x = {}_nE_x\overline{a}_{x+n} = v^n\frac{\omega-x-n}{\omega-x}\frac{(\overline{D}\overline{a})_{\overline{\omega-x-n}|}}{\omega-x-n} = \frac{v^n\,(\overline{D}\overline{a})_{\overline{\omega-x-n}|}}{\omega-x}. $$

Theorem 23

Under De Moivre's model,

(i) ${}_{n|}\overline{a}_x = \frac{v^n\left(\overline{D}\overline{a}\right)_{\overline{\omega-x-n}|}}{\omega-x}$.

(ii) If $i \neq 0$, ${}_{n|}\overline{a}_x = \frac{v^n\left(\omega-x-n-\overline{a}_{\overline{\omega-x-n}|}\right)}{(\omega-x)\delta}$.

(iii) If $i = 0$, ${}_{n|}\overline{a}_x = \frac{(\omega-x-n)^2}{2(\omega-x)}$.

Proof: (ii) If $i \neq 0$, $\left(\overline{D}\overline{a}\right)_{\overline{n}|} = \frac{n-\overline{a}_n}{\delta}$. So,

$$ {}_{n|}\overline{a}_x = \frac{v^n \frac{\omega-x-n-\overline{a}_{\omega-x-n}}{\delta}}{\omega-x} = \frac{v^n\left(\omega-x-n-\overline{a}_{\overline{\omega-x-n}|}\right)}{(\omega-x)\delta}. $$

Theorem 23

Under De Moivre's model,

(i) ${}_{n|}\overline{a}_x = \frac{v^n (\overline{D}\overline{a})_{\overline{\omega-x-n}|}}{\omega-x}$.

(ii) If $i \neq 0$, ${}_{n|}\overline{a}_x = \frac{v^n\left(\omega-x-n-\overline{a}_{\overline{\omega-x-n}|}\right)}{(\omega-x)\delta}$.

(iii) If $i = 0$, ${}_{n|}\overline{a}_x = \frac{(\omega-x-n)^2}{2(\omega-x)}$.

Proof: (iii) If $i = 0$, $\left(\overline{D}\overline{a}\right)_{\overline{n}|} = \frac{n^2}{2}$. So,

$$ {}_{n|}\overline{a}_x = \frac{v^n \frac{(\omega-x-n)^2}{2}}{\omega - x} = \frac{(\omega - x - n)^2}{2(\omega - x)}. $$

Example 9

Suppose that $v = 0.91$ *and De Moivre's model with terminal age 100. Find* ${}_{20|}\overline{a}_{40}$.

Example 9

Suppose that $v = 0.91$ *and De Moivre's model with terminal age 100. Find* ${}_{20|}\overline{a}_{40}$.

Solution 1: We have that

$$\overline{a}_{\overline{40}|} = \frac{1-(0.91)^{40}}{-\ln(0.91)} = 10.35941874,$$

$${}_{20|}\overline{a}_{40} = \frac{v^n\left(\omega - x - n - \overline{a}_{\overline{\omega-x-n}|}\right)}{(\omega-x)\delta} = \frac{(0.91)^{20}(40-10.35941874)}{(60)(-\ln(0.91))}$$

$$=0.7943326944.$$

Example 9

Suppose that $v = 0.91$ and De Moivre's model with terminal age 100. Find ${}_{20|}\overline{a}_{40}$.

Solution 2: We have that

$$ {}_{20}E_{40} = v^{20}{}_{20}p_{40} = (0.91)^{20}\frac{60-20}{60} = 0.1010966087, $$

$$ \overline{A}_{60} = \frac{\overline{a}_{\overline{40}|}}{40} = \frac{1-(0.91)^{40}}{(40)(-\log(0.91))} = 0.2589854685, $$

$$ \overline{a}_{60} = \frac{1-\overline{A}_{60}}{\delta} = \frac{1-0.2589854685}{-\log(0.91)} = 7.857164593, $$

$$ {}_{20|}\overline{a}_{40} = {}_{20}E_{40}\overline{a}_{60} = (0.1010966087)(8.772527413) = 0.7943326944. $$

Theorem 24

Under constant force of mortality, ${}_{n|}\overline{a}_x = \frac{e^{-n(\mu+\delta)}}{\mu+\delta}$.

Proof.

We have that ${}_{n|}\overline{a}_x = {}_nE_x \overline{a}_{x+n} = \frac{e^{-n(\mu+\delta)}}{\mu+\delta}$. □

Example 10

Suppose that $v = 0.92$, *and the force of mortality is* $\mu_{x+t} = 0.02$, *for* $t \geq 0$. *Find* ${}_{20|}\overline{a}_x$ *and* $\mathrm{Var}({}_{20|}\overline{Y}_x)$.

Solution: We have that

$$\overline{a}_x = \frac{1}{-\log(0.92) + 0.02} = 9.672900338,$$

$$ {}_{20|}\overline{a}_x = (0.92)^{20} e^{-(20)(0.02)}(9.672900338) = 1.223476036,$$

$$ {}^2\overline{a}_x = \frac{1}{-2\log(0.92) + 0.02} = 5.354373368,$$

$$E[\left({}_{n|}\overline{Y}_x\right)^2] = v^{2n} \cdot {}_np_x \frac{2\left(\overline{a}_{x+n} - {}^2\overline{a}_{x+n}\right)}{\delta}$$

$$=(0.92)^{40} e^{-(20)(0.02)} \frac{2\,(9.672900338 - 5.354373368)}{-\log(0.92)} = 2.472240188,$$

$$\mathrm{Var}({}_{20|}\overline{Y}_x) = 2.472240188 - (1.223476036)^2 = 0.9753465773.$$

Theorem 25

$${}_{n|}\bar{a}_x = vp_x \cdot {}_{n-1|}\bar{a}_{x+1}.$$

Proof.

$$\begin{aligned} {}_{n|}\bar{a}_x &= \int_n^\infty v^t \cdot {}_tp_x \, dt = vp_x \int_n^\infty v^{t-1} \cdot {}_{t-1}p_{x+1} \, dt \\ &= vp_x \int_{n-1}^\infty v^t \cdot {}_tp_{x+1} \, dt = vp_x \cdot {}_{n-1|}\bar{a}_{x+1}. \end{aligned}$$

□

