Manual for SOA Exam MLC.

Chapter 5. Life annuities.
Section 5.3. Temporary annuities.

Extract from:
"Arcones' Manual for the SOA Exam MLC. Fall 2009 Edition".
available at http://www.actexmadriver.com/

Due n–year temporary annuity.

Definition 1

A **due** n**–year term annuity** *guarantees payments made at the beginning of the year while an individual is alive for at most n payments.*

The actuarial present value of an n-year term annuity due is denoted by $\ddot{Y}_{x:\overline{n}}$.

Definition 2

The actuarial present value of an n-year term annuity due for (x) *with unit payment is denoted by* $\ddot{a}_{x:\overline{n}|}$.

Theorem 1

(i)

$$\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{\min(K_x,n)}|} = \begin{cases} \ddot{a}_{\overline{K_x}|} & \text{if } K_x \le n, \\ \ddot{a}_{\overline{n}|} & \text{if } K_x > n, \end{cases}$$

(ii) If $i \neq 0$, $\ddot{Y}_{x:\overline{n}|} = \frac{1-Z_{x:\overline{n}|}}{d}$.

(iii) If $i = 0$, $\ddot{Y}_{x:\overline{n}|} = \min(K_x, n)$.

Theorem 1

(i)

$$\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{\min(K_x,n)}|} = \begin{cases} \ddot{a}_{\overline{K_x}|} & \text{if } K_x \le n, \\ \ddot{a}_{\overline{n}|} & \text{if } K_x > n, \end{cases}$$

(ii) If $i \neq 0$, $\ddot{Y}_{x:\overline{n}|} = \frac{1-Z_{x:\overline{n}|}}{d}$.
(iii) If $i = 0$, $\ddot{Y}_{x:\overline{n}|} = \min(K_x, n)$.

Proof: (i) If $K_x \le n$, benefit payments are made at times $0, 1, \dots, K_x - 1$ and $\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{k}|}$. If $K_x > n$, benefit payments are made at times $0, 1, \dots, n - 1$ and $\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{n}|}$. Therefore, $\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{\min(K_x,n)}|}$.

Theorem 1

(i)

$$\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{\min(K_x,n)}|} = \begin{cases} \ddot{a}_{\overline{K_x}|} & \text{if } K_x \le n, \\ \ddot{a}_{\overline{n}|} & \text{if } K_x > n, \end{cases}$$

(ii) If $i \neq 0$, $\ddot{Y}_{x:\overline{n}|} = \frac{1 - Z_{x:\overline{n}|}}{d}$.

(iii) If $i = 0$, $\ddot{Y}_{x:\overline{n}|} = \min(K_x, n)$.

Proof: (ii) If $i \neq 0$,

$$\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{\min(K_x,n)}|} = \frac{1 - v^{\min(K_x,n)}}{d} = \frac{1 - Z_{x:\overline{n}|}}{d}.$$

Theorem 1

(i)

$$\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{\min(K_x,n)}|} = \begin{cases} \ddot{a}_{\overline{K_x}|} & \text{if } K_x \le n, \\ \ddot{a}_{\overline{n}|} & \text{if } K_x > n, \end{cases}$$

(ii) If $i \neq 0$, $\ddot{Y}_{x:\overline{n}|} = \frac{1-Z_{x:\overline{n}|}}{d}$.

(iii) If $i = 0$, $\ddot{Y}_{x:\overline{n}|} = \min(K_x, n)$.

Proof: (iii) If $i = 0$, $\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{\min(K_x,n)}|} = \min(K_x, n)$.

Theorem 2

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=1}^{n} \ddot{a}_{\overline{k}|} \cdot {}_{k-1}|q_x + \ddot{a}_{\overline{n}|} \cdot {}_{n}p_x = \sum_{k=1}^{n-1} \ddot{a}_{\overline{k}|} \cdot {}_{k-1}|q_x + \ddot{a}_{\overline{n}|} \cdot {}_{n-1}p_x.$$

Theorem 2

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=1}^{n} \ddot{a}_{\overline{k}|} \cdot {}_{k-1}|q_x + \ddot{a}_{\overline{n}|} \cdot {}_{n}p_x = \sum_{k=1}^{n-1} \ddot{a}_{\overline{k}|} \cdot {}_{k-1}|q_x + \ddot{a}_{\overline{n}|} \cdot {}_{n-1}p_x.$$

Proof: Since $\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{\min(K_x,n)}|} = \begin{cases} \ddot{a}_{\overline{K_x}|} & \text{if } K_x \le n, \\ \ddot{a}_{\overline{n}|} & \text{if } K_x > n, \end{cases}$

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=1}^{n} \ddot{a}_{\overline{k}|} \mathbb{P}\{K_x = k\} + \ddot{a}_{\overline{n}|} \mathbb{P}\{K_x > n\} = \sum_{k=1}^{n} \ddot{a}_{\overline{k}|} \cdot {}_{k-1}|q_x + \ddot{a}_{\overline{n}|} \cdot {}_{n}p_x.$$

Since $\ddot{Y}_{x:\overline{n}|} = \ddot{a}_{\overline{\min(K_x,n)}|} = \begin{cases} \ddot{a}_{\overline{K_x}|} & \text{if } K_x < n, \\ \ddot{a}_{\overline{n}|} & \text{if } K_x \ge n, \end{cases}$

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=1}^{n-1} \ddot{a}_{\overline{k}|} \mathbb{P}\{K_x = k\} + \ddot{a}_{\overline{n}|} \mathbb{P}\{K_x \ge n\} = \sum_{k=1}^{n-1} \ddot{a}_{\overline{k}|} \cdot {}_{k-1}|q_x + \ddot{a}_{\overline{n}|} \cdot {}_{n-1}p_x.$$

Theorem 3

If $i \neq 0$,

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d} \quad \text{and } \mathrm{Var}(\ddot{Y}_{x:\overline{n}|}) = \frac{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}{d^2}.$$

Example 1

Suppose that $p_x = 0.98$, $p_{x+1} = 0.97$ *and* $v = 0.92$.
(i) Find $\ddot{a}_{x:\overline{3}|}$ *using Theorem 2.*
(ii) Find $\ddot{a}_{x:\overline{3}|}$ *using Theorem 3.*
(iii) Find $\mathrm{Var}(\ddot{Y}_{x:\overline{3}|})$.

Example 1

Suppose that $p_x = 0.98$, $p_{x+1} = 0.97$ *and* $v = 0.92$.
(i) Find $\ddot{a}_{x:\overline{3}|}$ *using Theorem 2.*
(ii) Find $\ddot{a}_{x:\overline{3}|}$ *using Theorem 3.*
(iii) Find $\mathrm{Var}(\ddot{Y}_{x:\overline{3}|})$.

Solution: (i)

$$\ddot{a}_{x:\overline{3}|} = (1)(0.02) + (1 + 0.92)(0.98)(0.03) + (1 + 0.92 + (0.92)^2)(0.98)(0.97) = 2.70618784,$$

Example 1

Suppose that $p_x = 0.98$, $p_{x+1} = 0.97$ *and* $v = 0.92$.
(i) Find $\ddot{a}_{x:\overline{3}|}$ *using Theorem 2.*
(ii) Find $\ddot{a}_{x:\overline{3}|}$ *using Theorem 3.*
(iii) Find $\mathrm{Var}(\ddot{Y}_{x:\overline{3}|})$.

Solution: (ii)

$$A_{x:\overline{3}|} = (0.92)(0.02) + (0.92)^2(0.98)(0.03) + (0.92)^3(0.98)(0.97)$$
$$=0.7835049728$$
$$\ddot{a}_{x:\overline{3}|} = \frac{1 - 0.7835049728}{1 - 0.92} = 2.70618784.$$

Example 1

Suppose that $p_x = 0.98$, $p_{x+1} = 0.97$ *and* $v = 0.92$.
(i) Find $\ddot{a}_{x:\overline{3}|}$ *using Theorem 2.*
(ii) Find $\ddot{a}_{x:\overline{3}|}$ *using Theorem 3.*
(iii) Find $\mathrm{Var}(\ddot{Y}_{x:\overline{3}|})$.

Solution: (iii)

$$^{2}A_{x:\overline{3}|} = (0.92)^2(0.02) + (0.92)^4(0.98)(0.03) + (0.92)^6(0.98)(0.97)$$

$=0.6143910173$

$$\mathrm{Var}(Z_{x:\overline{3}|}) = 0.6143910173 - (0.7835049728)^2 = 0.0005109748977$$

$$\mathrm{Var}(\ddot{Y}_{x:\overline{3}|}) = \frac{0.0005109748977}{(0.08)^2} = 0.07983982777$$

Example 2

Suppose that $v = 0.91$ *and De Moivre's model with terminal age 100. Find* $\ddot{a}_{40:\overline{20}|}$ *using Theorem 3.*

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Suppose that $v = 0.91$ *and De Moivre's model with terminal age 100. Find* $\ddot{a}_{40:\overline{20}|}$ *using Theorem 3.*

Solution: With $i = 9/91$, we have that

$$\begin{aligned}
A_{40:\overline{20}|} &= A^{1}_{40:\overline{20}|} + A_{40:\overset{\,1}{\overline{20}|}} \\
&= \frac{a_{\overline{20}|}}{60} + v^{20}\,{}_{20}p_{40} = \frac{a_{\overline{20}|}}{60} + (0.91)^{20}\frac{60-20}{60} = 0.2440601511, \\
\ddot{a}_{40:\overline{20}|} &= \frac{1 - A_{40:\overline{20}|}}{d} = \frac{1 - 0.2440601511}{0.09} = 8.399331654.
\end{aligned}$$

Theorem 4

If $i \neq 0$,

$$\ddot{Y}^2_{x:\overline{n}|} = \frac{2\ddot{Y}_{x:\overline{n}|} - (2-d)\cdot {}^2\ddot{Y}_{x:\overline{n}|}}{d}$$

and

$$E[\ddot{Y}^2_{x:\overline{n}|}] = \frac{2\ddot{a}_{x:\overline{n}|} - (2-d)\cdot {}^2\ddot{a}_{x:\overline{n}|}}{d}.$$

Theorem 4

If $i \neq 0$,

$$\ddot{Y}^2_{x:\overline{n}|} = \frac{2\ddot{Y}_{x:\overline{n}|} - (2-d)\cdot {}^2\ddot{Y}_{x:\overline{n}|}}{d}$$

and

$$E[\ddot{Y}^2_{x:\overline{n}|}] = \frac{2\ddot{a}_{x:\overline{n}|} - (2-d)\cdot {}^2\ddot{a}_{x:\overline{n}|}}{d}.$$

Proof: From $\ddot{Y}_{x:\overline{n}|} = \frac{1-Z_{x:\overline{n}|}}{d}$, we get that $Z_{x:\overline{n}|} = 1 - d\ddot{Y}_{x:\overline{n}|}$.
Hence,

$$\begin{aligned}\ddot{Y}^2_{x:\overline{n}|} &= \frac{1 - 2Z_{x:\overline{n}|} + {}^2Z_{x:\overline{n}|}}{d^2} \\ &= \frac{1 - 2(1 - d\ddot{Y}_{x:\overline{n}|}) + 1 - d(2-d)\cdot {}^2\ddot{Y}_{x:\overline{n}|}}{d^2} \\ &= \frac{2\ddot{Y}_{x:\overline{n}|} - (2-d)\cdot {}^2\ddot{Y}_{x:\overline{n}|}}{d}.\end{aligned}$$

Theorem 5

$$\ddot{Y}_{x:\overline{n}|} = \sum_{k=0}^{n-1} Z_{x:\overset{1}{\overline{k}|}}$$

and

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \, {}_kp_x.$$

Theorem 5

$$\ddot{Y}_{x:\overline{n}|} = \sum_{k=0}^{n-1} Z_{x:\overset{1}{\overline{k}|}}$$

and

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_k p_x.$$

Proof: The payment at time k, where $0 \le k \le n-1$, is made if $T_x > k$. Hence,

$$\ddot{Y}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k I(T_x > k) = \sum_{k=0}^{n-1} Z_{x:\overset{1}{\overline{k}|}}.$$

Example 3

Suppose that $p_x = 0.98$, $p_{x+1} = 0.97$ *and* $v = 0.92$. *Find* $\ddot{a}_{x:\overline{3}|}$ *using the previous theorem.*

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Suppose that $p_x = 0.98$, $p_{x+1} = 0.97$ *and* $v = 0.92$. *Find* $\ddot{a}_{x:\overline{3}|}$ *using the previous theorem.*

Solution:

$$\ddot{a}_{x:\overline{3}|} = (1) + (0.92)(0.98) + (0.92)^2(0.98)(0.97) = 2.70618784.$$

Theorem 6

If $i = 0$, $\ddot{a}_{x:\overline{n}|} = 1 + e_{x:\overline{n-1}|}$.

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Proof:

$$\ddot{a}_{x:\overline{n}|} = E[\min(K_x, n)] = E[\min(K(x) + 1, n)]$$
$$= 1 + E[\min(K(x), n-1)] = 1 + e_{x:\overline{n-1}|}.$$

Theorem 7

Under De Moivre's model,

$$\ddot{a}_{x:\overline{n}|} = \frac{(\omega - x - n)\ddot{a}_{\overline{n}|} + (D\ddot{a})_{\overline{n}|}}{\omega - x}.$$

Theorem 7

Under De Moivre's model,

$$\ddot{a}_{x:\overline{n}|} = \frac{(\omega - x - n)\ddot{a}_{\overline{n}|} + (D\ddot{a})_{\overline{n}|}}{\omega - x}.$$

Proof: We have that

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_x = \sum_{k=0}^{n-1} v^k \frac{\omega - x - k}{\omega - x} = \sum_{k=0}^{n-1} v^k \frac{\omega - x - n + n - k}{\omega - x}$$

$$= \sum_{k=0}^{n-1} v^k \frac{\omega - x - n}{\omega - x} + \sum_{k=0}^{n-1} v^k \frac{n - k}{\omega - x} = \frac{(\omega - x - n)\ddot{a}_{\overline{n}|} + (D\ddot{a})_{\overline{n}|}}{\omega - x}.$$

Example 4

Suppose that $v = 0.91$ *and De Moivre's model with terminal age 100. Find* $\ddot{a}_{40:\overline{20}|}$ *using previous theorem.*

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Suppose that $v = 0.91$ *and De Moivre's model with terminal age 100. Find* $\ddot{a}_{40:\overline{20}|}$ *using previous theorem.*

Solution: With $i = 9/91$, we have that

$$\ddot{a}_{\overline{20}|} = 9.426167633,$$

$$(D\ddot{a})_{\overline{20}|} = 126.9131939,$$

$$\ddot{a}_{40:\overline{20}|} = \frac{(40)\ddot{a}_{\overline{20}|} + (D\ddot{a})_{\overline{20}|}}{60} = \frac{(40)(9.426167633) + 126.9131939}{60}$$

$$= 8.399331654.$$

Theorem 8

We have that

$$\ddot{Y}_x = \ddot{Y}_{x:\overline{n}|} + {}_{n|}\ddot{Y}_x.$$

Hence,

$$\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_nE_x\ddot{a}_{x+n}.$$

Theorem 8

We have that

$$\ddot{Y}_x = \ddot{Y}_{x:\overline{n}|} + {}_{n|}\ddot{Y}_x.$$

Hence,

$$\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_nE_x \ddot{a}_{x+n}.$$

Proof: Using some previous theorems,

$$\ddot{Y}_{x:\overline{n}|} + {}_{n|}\ddot{Y}_x = \sum_{k=0}^{n-1} v^k I(K_x > k) + \sum_{k=n}^{\infty} v^k I(K_x > k)$$

$$= \sum_{k=0}^{\infty} v^k I(K_x > k) = \ddot{Y}_x.$$

Theorem 9

Under constant force of mortality, $\ddot{a}_{x:\overline{n}|} = \frac{1-v^n p_x^n}{1-vp_x}$.

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Under constant force of mortality, $\ddot{a}_{x:\overline{n}|} = \frac{1-v^n p_x^n}{1-vp_x}$.

Proof: From $\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_nE_x \ddot{a}_{x+n} = \ddot{a}_{x:\overline{n}|} + {}_nE_x \ddot{a}_x$, we get

$$\ddot{a}_{x:\overline{n}|} = (1 - {}_nE_x)\ddot{a}_x = \frac{1 - v^n p_x^n}{1 - vp_x}$$

Theorem 10

$\ddot{a}_{x:\overline{n}|} = 1 + vp_x \ddot{a}_{x+1:\overline{n-1}|}$.

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$\ddot{a}_{x:\overline{n}|} = 1 + vp_x \ddot{a}_{x+1:\overline{n-1}|}$.

Proof:

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \cdot {}_kp_x = 1 + \sum_{k=1}^{n-1} v^k \cdot {}_kp_x = 1 + vp_x \sum_{k=1}^{n-1} v^{k-1} \cdot {}_{k-1}p_{x+1}$$

$$= 1 + vp_x \sum_{k=0}^{n-2} v^k \cdot {}_kp_{x+1} = 1 + vp_x \ddot{a}_{x+1:\overline{n-1}|}.$$

For due annuities, we have that $\ddot{a}_{\overline{n}|} = v^n \ddot{s}_{\overline{n}|}$. $\ddot{a}_{\overline{n}|}$ is the present value of a due annuity. $\ddot{s}_{\overline{n}|}$ is the accumulated value of a due annuity. v^n is the n–year discount factor.

Definition 3

The **actuarial accumulated value at time n of n–year term temporary due annuity** *is defined by*

$$\ddot{s}_{x:\overline{n}|} = \frac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x}.$$

We have that $\ddot{a}_{x:\overline{n}|} = {}_nE_x \ddot{s}_{x:\overline{n}|}$. To take care that the number of living decreases over time, in actuarial computations, the n–year discount factor is ${}_nE_x = v^n {}_np_x$.

$\ddot{s}_{x:\overline{n}|}$ is the actuarial future value of a n–year due life annuity to (x).

We have that

$$\ddot{s}_{x:\overline{n}|} = \frac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x} = \frac{\ddot{a}_{x:\overline{n}|}}{v^n {}_np_x} = \frac{\sum_{k=0}^{n-1} v^k {}_kp_x}{v^n {}_np_x}$$

$$= \sum_{k=0}^{n-1} \frac{1}{v^{n-k} {}_{n-k}p_{x+k}} = \sum_{k=0}^{n-1} \frac{1}{{}_{n-k}E_{x+k}}.$$

Here, $\frac{1}{{}_{n-k}E_{x+k}}$ is the actuarial factor for time k to time n for a live age x.

Immediate n–year temporary annuity

Definition 4

A **immediate** n**–year term annuity** *guarantees payments made at the end of the year while an individual is alive for n years.*

The actuarial present value of an immediate n-year term annuity is denoted by $Y_{x:\overline{n}|}$.

Definition 5

The actuarial present value of an n-year term life immediate annuity for (x) *with unit payment is denoted by* $a_{x:\overline{n}|}$.

Theorem 11

$$Y_{x:\overline{n}|} = a_{\overline{\min(K_x-1,n)}|} = \begin{cases} a_{\overline{K_x-1}|} & \text{if } K_x \le n, \\ a_{\overline{n}|} & \text{if } K_x > n, \end{cases}$$

and

$$a_{x:\overline{n}|} = \sum_{k=2}^{n} a_{\overline{k-1}|} \cdot {}_{k-1|}q_x + a_{\overline{n}|} \cdot {}_{n}p_x.$$

Theorem 11

$$Y_{x:\overline{n}|} = a_{\overline{\min(K_x-1,n)}|} = \begin{cases} a_{\overline{K_x-1}|} & \text{if } K_x \le n, \\ a_{\overline{n}|} & \text{if } K_x > n, \end{cases}$$

and

$$a_{x:\overline{n}|} = \sum_{k=2}^{n} a_{\overline{k-1}|} \cdot {}_{k-1|}q_x + a_{\overline{n}|} \cdot {}_n p_x.$$

Proof: If $K_x = 1$, then no payment is made. If $2 \le K_x \le n+1$, then $T_x \in (K_x - 1, K_x]$ and payments are made at times $1, \dots, K_x - 1$. Hence, $Y_{x:\overline{n}|} = a_{\overline{k-1}|} = a_{\overline{K_x-1}|}$.
If $K_x > n$, then $T_x > n$ and the insurer makes payments at times $1, \dots, n$. Hence, $Y_{x:\overline{n}|} = a_{\overline{n}|}$.

Theorem 12

If $i \neq 0$,

$$Y_{x:\overline{n}|} = \frac{v - Z_{x:\overline{n+1}|}}{d}, \quad a_{x:\overline{n}|} = \frac{v - Z_{x:\overline{n+1}|}}{d}$$

and

$$\mathrm{Var}(Y_{x:\overline{n}|}) = \frac{{}^2A_{x:\overline{n+1}|} - (A_{x:\overline{n+1}|})^2}{d^2}.$$

Theorem 12

If $i \neq 0$,

$$Y_{x:\overline{n}|} = \frac{v - Z_{x:\overline{n+1}|}}{d}, \quad a_{x:\overline{n}|} = \frac{v - Z_{x:\overline{n+1}|}}{d}$$

and

$$\mathrm{Var}(Y_{x:\overline{n}|}) = \frac{{}^2A_{x:\overline{n+1}|} - (A_{x:\overline{n+1}|})^2}{d^2}.$$

Proof:

$$Y_{x:\overline{n}|} = a_{\overline{\min(K_x-1,n)}|} = \frac{1 - v^{\min(K_x-1,n)}}{i}$$
$$= \frac{v - v^{\min(K_x,n+1)}}{d} = \frac{v - Z_{x:\overline{n+1}|}}{d}.$$

Theorem 13

$Y_{x:\overline{n}|} = \sum_{k=1}^{n} Z_{x:\overline{k}|}^{\;\;1}$ *and* $a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \cdot {}_k p_x$.

Theorem 13

$Y_{x:\overline{n}|} = \sum_{k=1}^{n} Z_{x:\overset{1}{\overline{k}|}}$ *and* $a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \cdot {}_k p_x$.

Proof: For $1 \le k \le n$, a payment at the end of the k–th year is received if and only if $K_x > k$. Hence, the present value of this payment is $v^k I(K_x > k) = Z_{x:\overset{1}{\overline{k}|}}$. The total present value of the payments in an n-year term annuity is $\sum_{k=1}^{n} Z_{x:\overset{1}{\overline{k}|}}$.

Theorem 14

$Y_{x:\overline{n}|} = \ddot{Y}_{x:\overline{n+1}|} - 1$, $a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n+1}|} - 1$ *and* $\mathrm{Var}(Y_{x:\overline{n}|}) = \mathrm{Var}(\ddot{Y}_{x:\overline{n+1}|})$.

Theorem 14

$Y_{x:\overline{n}|} = \ddot{Y}_{x:\overline{n+1}|} - 1$, $a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n+1}|} - 1$ *and* $\mathrm{Var}(Y_{x:\overline{n}|}) = \mathrm{Var}(\ddot{Y}_{x:\overline{n+1}|})$.

Proof: We have that

$$\ddot{Y}_{x:\overline{n+1}|} - 1 = \sum_{k=0}^{n} Z_{x:\overset{1}{\overline{k}|}} - 1 = \sum_{k=1}^{n} Z_{x:\overset{1}{\overline{k}|}} = Y_{x:\overline{n}|}.$$

Example 5

Suppose that $p_x = 0.98$, $p_{x+1} = 0.97$, $p_{x+2} = 0.96$ *and* $v = 0.92$.
(i) Find $a_{x:\overline{3}|}$ *using each of the three previous theorems.*
(ii) Find $\mathrm{Var}(Y_{x:\overline{3}|})$.

Example 5

Suppose that $p_x = 0.98$, $p_{x+1} = 0.97$, $p_{x+2} = 0.96$ *and* $v = 0.92$.
(i) Find $a_{x:\overline{3}|}$ *using each of the three previous theorems.*
(ii) Find $\mathrm{Var}(Y_{x:\overline{3}|})$.

Solution: (i) Using the first theorem,

$$a_{x:\overline{3}|} = \sum_{k=2}^{3} a_{\overline{K_x-1}|}\mathbb{P}\{K_x = k\} + a_{\overline{3}|}\mathbb{P}\{K_x > 3\}$$

$$= a_{\overline{1}|} p_x q_{x+1} + a_{\overline{2}|} p_x p_{x+1} q_{x+2} + a_{\overline{3}|} p_x p_{x+1} p_{x+2}$$

$$= (0.92)(0.98)(0.03) + (0.92 + (0.92)^2)(0.98)(0.97)(0.04)$$

$$+ (0.92 + (0.92)^2 + (0.92)^3)(0.98)(0.97)(0.96)$$

$$= 2.41679982$$

Example 5

Suppose that $p_x = 0.98$, $p_{x+1} = 0.97$, $p_{x+2} = 0.96$ *and* $v = 0.92$.
(i) Find $a_{x:\overline{3}|}$ *using each of the three previous theorems.*
(ii) Find $\mathrm{Var}(Y_{x:\overline{3}|})$.

Solution: Using the second theorem,

$$A_{x:\overline{4}|} = (0.92)(0.02) + (0.92)^2(0.98)(0.03) + (0.92)^3(0.98)(0.97)(0.04)$$
$$+ (0.92)^4(0.98)(0.97)(0.96) = 0.7266560144,$$
$$a_{x:\overline{3}|} = \frac{0.92 - 0.7266560144}{0.08} = 2.41679982.$$

Example 5

Suppose that $p_x = 0.98$, $p_{x+1} = 0.97$, $p_{x+2} = 0.96$ *and* $v = 0.92$.
(i) Find $a_{x:\overline{3}|}$ *using each of the three previous theorems.*
(ii) Find $\mathrm{Var}(Y_{x:\overline{3}|})$.

Solution: Using the third theorem,

$$a_{x:\overline{3}|} = \sum_{k=1}^{3} v^k \cdot {}_kp_x$$
$$=(0.92)(0.98) + (0.92)^2(0.98)(0.97) + (0.92)^3(0.98)(0.97)(0.96)$$
$$=2.41679982.$$

Example 5

Suppose that $p_x = 0.98$, $p_{x+1} = 0.97$, $p_{x+2} = 0.96$ *and* $v = 0.92$.
(i) Find $a_{x:\overline{3}|}$ *using each of the three previous theorems.*
(ii) Find $\mathrm{Var}(Y_{x:\overline{3}|})$.

Solution: (ii)

$$\begin{aligned} {}^2A_{x:\overline{4}|} &= (0.92)^2(0.02) + (0.92)^4(0.98)(0.03) \\ &\quad + (0.92)^6(0.98)(0.97)(0.04) + (0.92)^8(0.98)(0.97)(0.96) \\ &= 0.529397222, \end{aligned}$$

$$\mathrm{Var}(Y_{x:\overline{3}|}) = \frac{0.529397222 - (0.7266560144)^2}{(0.08)^2} = 0.2137904275.$$

Theorem 15

$$Y_{x:\overline{n}|} = \ddot{Y}_{x:\overline{n}|} - 1 + Z_{x:\overline{n}|}^{\;\;\;1}$$

and $a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + A_{x:\overline{n}|}^{\;\;\;1}$.

Theorem 15

$$Y_{x:\overline{n}|} = \ddot{Y}_{x:\overline{n}|} - 1 + Z_{x:\overline{n}|}^{\;\;1}$$

and $a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + A_{x:\overline{n}|}^{\;\;1}$.

Proof: We have that

$$\ddot{Y}_{x:\overline{n}|} - 1 + Z_{x:\overline{n}|}^{\;\;1} = \sum_{k=0}^{n-1} Z_{x:\overline{k}|}^{\;\;1} - 1 + Z_{x:\overline{n}|}^{\;\;1} = \sum_{k=1}^{n} Z_{x:\overline{k}|}^{\;\;1} = Y_{x:\overline{n}|}.$$

Theorem 16

We have that

$$Y_x = {}_n|Y_x + Y_{x:\overline{n}|}$$

and

$$a_x = {}_n|a_x + a_{x:\overline{n}|} = {}_n|a_x + {}_nE_x a_{x+n}.$$

Theorem 16

We have that

$$Y_x = {}_n|Y_x + Y_{x:\overline{n}|}$$

and

$$a_x = {}_n|a_x + a_{x:\overline{n}|} = {}_n|a_x + {}_nE_x a_{x+n}.$$

Proof: We have that

$$Y_{x:\overline{n}|} + {}_n|Y_x = \sum_{k=1}^{n} v^k I(K_x > k) + \sum_{k=n+1}^{\infty} v^k I(K_x > k)$$

$$= \sum_{k=1}^{\infty} v^k I(K_x > k) = Y_x.$$

Theorem 17

If $i = 0$, $a_{x:\overline{n}|} = e_{x:\overline{n}|}$.

Theorem 17

If $i = 0$, $a_{x:\overline{n}|} = e_{x:\overline{n}|}$.

Proof: We have that

$a_{\overline{\min(K_x-1,n)}|} = \min(K_x - 1, n) = \min(K(x), n)$ and
$a_{x:\overline{n}|} = E[\min(K(x), n)] = e_{x:\overline{n}|}$.

Theorem 18

Under constant force of mortality,

$$a_{x:\overline{n}|} = \frac{(1 - v^n p_x^n) v p_x}{1 - v p_x}.$$

Theorem 18

Under constant force of mortality,

$$a_{x:\overline{n}|} = \frac{(1 - v^n p_x^n) v p_x}{1 - v p_x}.$$

Proof: We have that

$$a_{x:\overline{n}|} = a_x - {}_nE_x a_{x+n} = (1 - {}_nE_x) a_x = \frac{(1 - v^n p_x^n) v p_x}{1 - v p_x}.$$

Theorem 19

Under De Moivre's model,

$$a_{x:\overline{n}|} = \frac{(\omega - x - n - 1)a_{\overline{n}|} + (Da)_{\overline{n}|}}{\omega - x}.$$

Theorem 19

Under De Moivre's model,

$$a_{x:\overline{n}|} = \frac{(\omega - x - n - 1)a_{\overline{n}|} + (Da)_{\overline{n}|}}{\omega - x}.$$

Proof: We have that

$$\begin{aligned}
a_{x:\overline{n}|} &= \sum_{k=1}^{n} v^k {}_k p_x = \sum_{k=1}^{n} v^k \frac{\omega - x - k}{\omega - x} \\
&= \sum_{k=1}^{n} v^k \frac{\omega - x - n - 1 + n + 1 - k}{\omega - x} \\
&= \sum_{k=1}^{n} v^k \frac{\omega - x - n - 1}{\omega - x} + \sum_{k=1}^{n} v^k \frac{n + 1 - k}{\omega - x} \\
&= \frac{(\omega - x - n - 1)a_{\overline{n}|} + (Da)_{\overline{n}|}}{\omega - x}.
\end{aligned}$$

Example 6

Suppose that $v = 0.91$ *and the De Moivre's model with terminal age 100.*

(i) Find $a_{40:\overline{20}|}$ *using the previous theorem.*

(ii) Find $a_{40:\overline{20}|}$ *using that* $a_{40:\overline{20}|} = \frac{v - A_{40:\overline{21}|}}{d}$.

Example 6

Suppose that $v = 0.91$ *and the De Moivre's model with terminal age 100.*
(i) Find $a_{40:\overline{20}|}$ *using the previous theorem.*
(ii) Find $a_{40:\overline{20}|}$ *using that* $a_{40:\overline{20}|} = \frac{v - A_{40:\overline{21}|}}{d}$.

Solution: Since $i = (1 - v)/v = 9/91$,

$$a_{\overline{20}|9/91} = 8.577812546,$$

$$(Da)_{\overline{20}|9/91} = 115.4910065,$$

$$a_{40:\overline{20}|} = \frac{(39)(8.577812546) + 115.4910065}{60} = 7.500428262.$$

(ii) We have that

$$A_{40:\overline{21}|} = \frac{a_{\overline{21}|9/91}}{60} + (0.91)^{21}\frac{60 - 21}{60} = 0.2349614564,$$

$$a_{40:\overline{20}|} = \frac{0.91 - 0.2349614564}{0.09} = 7.500428262.$$

Example 7

Suppose that $p_x = 0.98$, $p_{x+1} = 0.97$ *and* $v = 0.92$. *Find* $\mathrm{Var}(Y_{x:\overline{3}|})$.

Example 7

Suppose that $p_x = 0.98$, $p_{x+1} = 0.97$ *and* $v = 0.92$. *Find* $\mathrm{Var}(Y_{x:\overline{3}|})$.

Solution: We have that

$$\begin{aligned}
&A_{x:\overline{4}|}\\
=&(0.92)(0.02) + (0.92)^2(0.98)(0.03) + (0.92)^3(0.98)(0.97)(0.04)\\
&+ (0.92)^4(0.98)(0.97)(0.96) = 0.7266560144,\\
&{}^2A_{x:\overline{4}|}\\
=&(0.92)^2(0.02) + (0.92)^4(0.98)(0.03) + (0.92)^6(0.98)(0.97)(0.04)\\
&+ (0.92)^8(0.98)(0.97)(0.96) = 0.529397222,\\
&\mathrm{Var}(Y_{x:\overline{4}|}) = \frac{0.529397222 - (0.7266560144)^2}{(0.08)^2} = 0.2137904275.
\end{aligned}$$

Definition 6

The **actuarial accumulated value** *at time n of an n–year term immediate annuity is*

$$s_{x:\overline{n}|} = \frac{a_{x:\overline{n}|}}{{}_nE_x}.$$

$s_{x:\overline{n}|}$ is the actuarial future value of a n–year immediate life insurance policy to (x).

We have that

$$s_{x:\overline{n}|} = \frac{a_{x:\overline{n}|}}{{}_nE_x} = \frac{a_{x:\overline{n}|}}{v^n {}_np_x} = \frac{\sum_{k=1}^{n} v^k {}_kp_x}{v^n {}_np_x} = \sum_{k=1}^{n} \frac{1}{v^{n-k} {}_{n-k}p_{x+k}}.$$

Theorem 20

$$a_{x:\overline{n}|} = vp_x \ddot{a}_{x+1:\overline{n}|} = vp_x(1 + a_{x+1:\overline{n-1}|}).$$

Theorem 20

$$a_{x:\overline{n}|} = vp_x \ddot{a}_{x+1:\overline{n}|} = vp_x(1 + a_{x+1:\overline{n-1}|}).$$

Proof: We have that

$$a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k {}_kp_x = vp_x \sum_{k=1}^{n} v^{k-1} {}_{k-1}p_{x+1} = vp_x \sum_{k=0}^{n-1} v^k {}_kp_{x+1}$$
$$= vp_x \ddot{a}_{x+1:\overline{n}|}.$$

Continuous *n*–year temporary annuity

Definition 7

*A n–***year term continuous annuity** *guarantees a continuous flow of payments at a constant rate for n years while the individual is alive.*

Definition 8

The present value of a n–year term continuous annuity is denoted by $\overline{Y}_{x:\overline{n}|}$.

Definition 9

The actuarial present value of an n–year term continuous annuity for (x) with unit payment is denoted by $\overline{a}_{x:\overline{n}|}$.

We have that $\overline{a}_{x:\overline{n}|} = E[\overline{Y}_{x:\overline{n}|}]$.

Recall that the the present value of the continuous annuity is $\overline{a}_{\overline{n}|i} = \int_0^n v^s\, ds = \frac{1-v^n}{\delta}$.

Theorem 21

$$\overline{Y}_{x:\overline{n}|} = \int_0^{\min(T_x,n)} v^t\, dt = \overline{a}_{\overline{\min(T_x,n)}|} = \begin{cases} \overline{a}_{\overline{T_x}|} & \text{if } T_x \le n, \\ \overline{a}_{\overline{n}|} & \text{if } T_x > n, \end{cases}$$

Hence,

$$\overline{a}_{x:\overline{n}|} = \int_0^n \overline{a}_{\overline{t}|} \cdot {}_tp_x \mu_{x+t}\, dt + \overline{a}_{\overline{n}|} \mathbb{P}\{T_x > n\}.$$

Recall that the the present value of the continuous annuity is $\overline{a}_{\overline{n}|i} = \int_0^n v^s\,ds = \frac{1-v^n}{\delta}$.

Theorem 21

$$\overline{Y}_{x:\overline{n}|} = \int_0^{\min(T_x,n)} v^t\,dt = \overline{a}_{\overline{\min(T_x,n)}|} = \begin{cases} \overline{a}_{\overline{T_x}|} & \text{if } T_x \le n, \\ \overline{a}_{\overline{n}|} & \text{if } T_x > n, \end{cases}$$

Hence,

$$\overline{a}_{x:\overline{n}|} = \int_0^n \overline{a}_{\overline{t}|} \cdot {}_t p_x \mu_{x+t}\,dt + \overline{a}_{\overline{n}|}\mathbb{P}\{T_x > n\}.$$

Proof: Notice that the continuous payments are paid until $\min(T_x, n)$.

Theorem 22

$$\overline{a}_{x:\overline{n}|} = E[\overline{a}_{\overline{\min(T_x,n)}|}] = \int_0^n v^t \cdot {}_tp_x \, dt.$$

Theorem 22

$$\overline{a}_{x:\overline{n}|} = E[\overline{a}_{\overline{\min(T_x,n)}|}] = \int_0^n v^t \cdot {}_tp_x \, dt.$$

Proof: Let $h(t) = v^t I(t \leq n)$ and let

$$H(t) = \int_0^t h(s)\, ds = \int_0^t v^s I(s \leq n)\, ds = \int_0^{\min(t,n)} v^s \, ds.$$

By a previous theorem,

$$\overline{a}_{x:\overline{n}|} = E[\overline{Y}_{x:\overline{n}|}] = E\left[\int_0^{\min(T_x,n)} v^s \, ds\right] = E[H(T_x)]$$

$$= \int_0^\infty h(t)\,{}_tp_x \, dt = \int_0^\infty v^t I(t \leq n)\,{}_tp_x \, dt = \int_0^n v^t \cdot {}_tp_x \, dt.$$

Theorem 23

If $\delta \neq 0$,

$$\overline{Y}_{x:\overline{n}|} = \frac{1 - \overline{Z}_{x:\overline{n}|}}{\delta}, \ \overline{a}_{x:\overline{n}|} = \frac{1 - \overline{A}_{x:\overline{n}|}}{\delta}$$

and

$$\mathrm{Var}(\overline{Y}_{x:\overline{n}|}) = \frac{{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2}{\delta^2}.$$

Theorem 23

If $\delta \neq 0$,

$$\overline{Y}_{x:\overline{n}|} = \frac{1 - \overline{Z}_{x:\overline{n}|}}{\delta}, \ \overline{a}_{x:\overline{n}|} = \frac{1 - \overline{A}_{x:\overline{n}|}}{\delta}$$

and

$$\mathrm{Var}(\overline{Y}_{x:\overline{n}|}) = \frac{{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2}{\delta^2}.$$

Proof: Using that $\overline{a}_{\overline{t}|} = \frac{1-v^t}{\delta}$, we get that

$$\overline{Y}_{x:\overline{n}|} = \overline{a}_{\overline{\min(T_x,n)}|} = \frac{1 - v^{\min(T_x,n)}}{\delta} = \frac{1 - \overline{Z}_{x:\overline{n}|}}{\delta}.$$

Theorem 24

If $\delta = 0$, $\overline{Y}_{x:\overline{n}|} = \min(T_x, n)$ *and* $\overline{Y}_{x:\overline{n}|} = \mathring{e}_{x:\overline{n}|}$.

Theorem 24

If $\delta = 0$, $\overline{Y}_{x:\overline{n}|} = \min(T_x, n)$ and $\overline{Y}_{x:\overline{n}|} = \mathring{e}_{x:\overline{n}|}$.

Proof: If $\delta = 0$, $\overline{Y}_{x:\overline{n}|} = \overline{a}_{\overline{\min(T_x,n)}|} = \min(T_x, n)$ and $\overline{Y}_{x:\overline{n}|} = E[\min(T_x, n)] = \mathring{e}_{x:\overline{n}|}$.

Theorem 25

Under De Moivre's model,

$$\bar{a}_{x:\overline{n}|} = \frac{(\omega - x - n)\bar{a}_{\overline{n}|} + \left(\overline{D}\bar{a}\right)_{\overline{n}|}}{\omega - x}.$$

Theorem 25

Under De Moivre's model,

$$\overline{a}_{x:\overline{n}|} = \frac{(\omega - x - n)\overline{a}_{\overline{n}|} + \left(\overline{D}\overline{a}\right)_{\overline{n}|}}{\omega - x}.$$

Proof: We have that

$$\begin{aligned}
\overline{a}_{x:\overline{n}|} &= \int_0^n v^t \cdot {}_t p_x \, dt = \int_0^n v^t \cdot \frac{\omega - x - t}{\omega - x} \, dt \\
&= \int_0^n v^t \cdot \frac{\omega - x - n + n - t}{\omega - x} \, dt \\
&= \int_0^n v^t \cdot \frac{\omega - x - n}{\omega - x} \, dt + \int_0^n v^t \cdot \frac{n - t}{\omega - x} \, dt \\
&= \frac{(\omega - x - n)\overline{a}_{\overline{n}|} + \left(\overline{D}\overline{a}\right)_{\overline{n}|}}{\omega - x}.
\end{aligned}$$

Example 8

Suppose that $\delta = 6\%$ and deaths are uniformly distributed with terminal age 105.

(i) Calculate $\overline{a}_{65:\overline{20}|}$ using that $\overline{a}_{x:\overline{n}|} = \frac{(\omega-x-n)\overline{a}_{\overline{n}|}+\left(\overline{D}\overline{a}\right)_{\overline{n}|}}{\omega-x}$.

(ii) Calculate $\overline{a}_{65:\overline{20}|}$ using that $\overline{a}_{x:\overline{n}|} = \frac{1-\overline{A}_{x:\overline{n}|}}{\delta}$.

Example 8

Suppose that $\delta = 6\%$ and deaths are uniformly distributed with terminal age 105.

(i) Calculate $\overline{a}_{65:\overline{20}|}$ using that $\overline{a}_{x:\overline{n}|} = \frac{(\omega-x-n)\overline{a}_{\overline{n}|}+(\overline{D}\overline{a})_{\overline{n}|}}{\omega-x}$.

(ii) Calculate $\overline{a}_{65:\overline{20}|}$ using that $\overline{a}_{x:\overline{n}|} = \frac{1-\overline{A}_{x:\overline{n}|}}{\delta}$.

Solution: (i) We have that

$$\overline{a}_{\overline{20}|} = \frac{1 - e^{-(20)(0.06)}}{0.06} = 11.64676313,$$

$$(\overline{D}\overline{a})_{\overline{20}|} = \frac{20 - \overline{a}_{\overline{20}|}}{0.06} = 139.2206144,$$

$$\overline{a}_{65:\overline{20}|} = \frac{(\omega - x - n)\overline{a}_{\overline{n}|} + (\overline{D}\overline{a})_{\overline{n}|}}{\omega - x}$$

$$= \frac{(20)(11.64676313) + 139.2206144}{40} = 9.303896928.$$

Example 8

Suppose that $\delta = 6\%$ and deaths are uniformly distributed with terminal age 105.

(i) Calculate $\overline{a}_{65:\overline{20}|}$ using that $\overline{a}_{x:\overline{n}|} = \frac{(\omega-x-n)\overline{a}_{\overline{n}|}+(\overline{D}\overline{a})_{\overline{n}|}}{\omega-x}$.

(ii) Calculate $\overline{a}_{65:\overline{20}|}$ using that $\overline{a}_{x:\overline{n}|} = \frac{1-\overline{A}_{x:\overline{n}|}}{\delta}$.

Solution: (ii) We have that

$$\overline{A}_{65:\overline{20}|} = \frac{\overline{a}_{\overline{20}|}}{40} + v^{20} \cdot {}_{20}p_{65} = \frac{1 - e^{-(20)(0.06)}}{(40)(0.06)} + e^{-(20)(0.06)}\frac{20}{40}$$

$$=0.4417661844,$$

$$\overline{a}_{65:\overline{20}|} = \frac{1 - \overline{A}_{65:\overline{20}|}}{\delta} = \frac{1 - 0.4417661843}{(0.06)} = 9.303896928.$$

Theorem 26

$$E[\overline{Y}^2_{x:\overline{n}|}] = \frac{2(\overline{a}_{x:\overline{n}|} - {}^2\overline{a}_{x:\overline{n}|})}{\delta}.$$

Theorem 26

$$E[\overline{Y}^2_{x:\overline{n}|}] = \frac{2(\overline{a}_{x:\overline{n}|} - {}^2\overline{a}_{x:\overline{n}|})}{\delta}.$$

Proof: From ${}^m\overline{a}_{x:\overline{n}|} = \frac{1-{}^m\overline{A}_{x:\overline{n}|}}{m\delta}$, we get that ${}^m\overline{A}_{x:\overline{n}|} = 1 - m\delta \cdot {}^m\overline{a}_{x:\overline{n}|}$. Hence,

$$E[\overline{Y}^2_{x:\overline{n}|}] = E\left[\left(\frac{1-\overline{Z}_{x:\overline{n}|}}{\delta}\right)^2\right] = E\left[\frac{1-2\overline{Z}_{x:\overline{n}|} + {}^2\overline{Z}_{x:\overline{n}|}}{\delta^2}\right]$$

$$= E\left[\frac{1-2(1-\delta\overline{a}_{x:\overline{n}|}) + 1 - 2\delta\cdot{}^2\overline{a}_{x:\overline{n}|}}{\delta^2}\right] = \frac{2(\overline{a}_{x:\overline{n}|} - {}^2\overline{a}_{x:\overline{n}|})}{\delta}.$$

Theorem 27

$$\overline{Y}_x = \overline{Y}_{x:\overline{n}|} + {}_{n|}\overline{Y}_x.$$

and

$$\overline{a}_x = \overline{a}_{x:\overline{n}|} + {}_{n|}\overline{a}_x = \overline{a}_{x:\overline{n}|} + {}_nE_x \overline{a}_{x+n}.$$

Theorem 27

$$\overline{Y}_x = \overline{Y}_{x:\overline{n}|} + {}_{n}|\overline{Y}_x.$$

and

$$\overline{a}_x = \overline{a}_{x:\overline{n}|} + {}_{n}|\overline{a}_x = \overline{a}_{x:\overline{n}|} + {}_{n}E_x\overline{a}_{x+n}.$$

Proof: Under a n–year term annuity consists of continuously payments in the interval $(0, n)$ until time of death. Under a n–year deferred annuity payments consists of continuously payments in the interval (n, ∞) until time of death. Hence, $\overline{Y}_x = \overline{Y}_{x:\overline{n}|} + {}_{n}|\overline{Y}_x$.

Taking $n = 1$, in previous theorem, we get the recurrence relation for $\overline{a}_x$:

Theorem 28

$$\overline{a}_x = \overline{a}_{x:\overline{1}|} + vp_x\overline{a}_{x+1}.$$

Example 9

Suppose that $\overline{a}_x = 10$, $q_x = 0.02$ *and* $\delta = 0.07$. *Deaths are uniformly distributed within each year of age. Find* $\overline{a}_{x+1}$.

Example 9

Suppose that $\overline{a}_x = 10$, $q_x = 0.02$ *and* $\delta = 0.07$. *Deaths are uniformly distributed within each year of age. Find* $\overline{a}_{x+1}$.

Solution: If deaths are uniformly distributed within each year of age, then $\ell_{x+t} = \ell_x - td_x$ and ${}_tp_x = 1 - tq_x$. Hence,

$$\begin{aligned}
10 =& \int_0^1 e^{-0.07t}(1-(0.02)t)\,dt + e^{-0.07}(0.98)\overline{a}_{x+1} \\
=& \frac{1-e^{-0.07}}{0.07} - \int_0^{0.07} \frac{(0.02)}{(0.07)^2} e^{-t} t\,dt + e^{-0.007}(0.98)\overline{a}_{x+1} \\
=& \frac{1-e^{-0.07}}{0.07} - \frac{(0.02)(1-e^{-0.07}(1+0.007))}{(0.07)^2} + e^{-0.007}(0.98)\overline{a}_{x+1} \\
=& 0.716498804 + 0.9731639541\overline{a}_{x+1}
\end{aligned}$$

and $\overline{a}_{x+1} = \frac{10-0.716498804}{0.9731639541} = 9.539503757$.

Theorem 29

Under constant force of mortality, $\overline{a}_{x:\overline{n}|} = \frac{1-e^{-n(\mu+\delta)}}{\mu+\delta}$.

Theorem 29

Under constant force of mortality, $\overline{a}_{x:\overline{n}|} = \frac{1-e^{-n(\mu+\delta)}}{\mu+\delta}$.

Proof:

$$\overline{a}_{x:\overline{n}|} = \overline{a}_x - {}_{n}|\overline{a}_x = \frac{1}{\mu+\delta} - \frac{e^{-n(\mu+\delta)}}{\mu+\delta} = \frac{1-e^{-n(\mu+\delta)}}{\mu+\delta}.$$

Example 10

Suppose that $\delta = 0.08$, *and the force of mortality is* $\mu_{x+t} = 0.01$, *for* $t \geq 0$. *Find* $\overline{a}_{x:\overline{10}|}$ *and* $\mathrm{Var}(\overline{Y}_{x:\overline{10}|})$.

Example 10

Suppose that $\delta = 0.08$, *and the force of mortality is* $\mu_{x+t} = 0.01$, *for* $t \geq 0$. *Find* $\bar{a}_{x:\overline{10}|}$ *and* $\mathrm{Var}(\overline{Y}_{x:\overline{10}|})$.

Solution: We have that

$$\bar{a}_{x:\overline{10}|} = \frac{1 - e^{-(10)(0.01+0.08)}}{0.01 + 0.08} = \frac{1 - e^{-0.9}}{0.09} = 6.593670447.$$

We have that

$$\overline{A}_{x:\overline{10}|} = \frac{(1 - e^{-(10)(0.01+0.08)})(0.01)}{0.01 + 0.08} + e^{-(10)(0.01+0.08)}$$

$$= \frac{1 - e^{-0.9}}{9} + e^{-0.9} = 0.4725063642,$$

$$^2\overline{A}_{x:\overline{10}|} = \frac{(1 - e^{-(10)(0.01+(2)0.08)})(0.01)}{(0.01 + (2)0.08)} + e^{-(10)(0.01+(2)0.08)}$$

$$= \frac{1 - e^{-1.7}}{17} + e^{-1.7} = 0.6634579217,$$

$$\frac{\mathrm{Var}(\overline{Z}_{x:\overline{10}|})}{\delta^2} = \frac{{}^2\overline{A}_{x:\overline{10}|} - \overline{A}_{x:\overline{10}|}^2}{\delta^2} = \frac{0.6634579217 - (0.4725063642)^2}{(0.08)^2}$$

$$= 68.78057148.$$

Theorem 30

$$\bar{a}_{x:\overline{m+n}|} = \bar{a}_{x:\overline{n}|} + {}_mE_x \cdot \bar{a}_{x+m:\overline{n}|}.$$

Proof:

$$\begin{aligned}
\bar{a}_{x:\overline{m+n}|} &= \int_0^{m+n} v^t \cdot {}_tp_x \, dt = \int_0^{m} v^t \cdot {}_tp_x \, dt + \int_m^{m+n} v^t \cdot {}_tp_x \, dt \\
&= \bar{a}_{x:\overline{m}|} + v^m {}_mp_x \int_m^{m+n} v^{t-m} \cdot {}_{t-m}p_{x+m} \, dt \\
&= \bar{a}_{x:\overline{m}|} + {}_mE_x \int_0^{n} v^t \cdot {}_tp_{x+m} \, dt \\
&= \bar{a}_{x:\overline{m}|} + {}_mE_x \cdot \bar{a}_{x+m:\overline{n}|}.
\end{aligned}$$

The actuarial accumulated value at time n of an n–year term temporary continuous annuity is

$$\bar{s}_{x:\overline{n}|} = \frac{\bar{a}_{x:\overline{n}|}}{{}_nE_x} = \frac{\bar{a}_{x:\overline{n}|}}{v^n \cdot {}_np_x} = \frac{\int_0^n v^t \cdot {}_tp_x \, dt}{v^n \cdot {}_np_x}$$

$$= \int_0^n \frac{1}{v^{n-t} \cdot {}_{n-t}p_{x+t}} \, dt = \int_0^n \frac{1}{{}_{n-t}E_{x+t}} \, dt.$$

$\frac{1}{{}_{n-t}E_{x+t}}$ is the actuarial factor from time t to time n for a live age x.

