Manual for SOA Exam MLC.

Chapter 5. Life annuities.

Section 5.5. Contingencies paid m times a year.

Extract from:
"Arcones' Manual for the SOA Exam MLC. Spring 2010 Edition".
available at http://www.actexmadriver.com/

Contingencies paid m times a year

In this section, we will consider the case of life insurance paid m times a year. In the unit case, a payment of $1 is made each year. Hence, each m–thly payment is $\frac{1}{m}$.

For a period of length $\frac{1}{m}$:

(i) the interest factor is $(1+i)^{1/m} = 1 + \frac{i^{(m)}}{m}$.

(ii) the effective rate of interest is $(1+i)^{1/m} - 1 = \frac{i^{(m)}}{m}$.

(iii) the discount factor is

$(1+i)^{-1/m} = v^{1/m} = (1-d)^{1/m} = 1 - \frac{d^{(m)}}{m}$.

(iv) the effective rate of discount is $1 - v^{1/m} = \frac{d^{(m)}}{m}$.

Whole life annuity due paid m times a year

Definition 1

A whole life due annuity with payments paid m times year is a series payments made at the beginning m–thly time interval while an individual is alive.

Let $J_x^{(m)} = \lceil m T_x \rceil$, i.e. $J_x^{(m)}$ is the integer such that if $T_x \in \left(\frac{J_x^{(m)}-1}{m}, \frac{J_x^{(m)}}{m} \right]$. A whole life due annuity paid m times a year makes payments at times $0, \frac{1}{m} \dots \frac{J_x^{(m)}-1}{m}$.

Definition 2

The present value of a whole life due annuity for (x) with unit annual payment paid m times a year is denoted by $\ddot{Y}_x^{(m)}$.

Definition 3

The actuarial present value of a whole life due annuity for (x) with unit annual payment paid m times a year is denoted by $\ddot{a}_x^{(m)}$.

Theorem 1

$$\ddot{Y}_x^{(m)} = \frac{1}{m}\ddot{a}_{\overline{J_x^{(m)}}|\frac{i^{(m)}}{m}} = \frac{1 - Z_x^{(m)}}{d^{(m)}} = \frac{1}{m}\sum_{k=0}^{\infty} Z_{x:\overline{\frac{1}{\frac{k}{m}}}|},$$

$$\ddot{a}_x^{(m)} = \frac{1}{m}\sum_{k=1}^{\infty} \ddot{a}_{\overline{k}|\frac{i^{(m)}}{m}} \cdot {}_{\frac{k-1}{m}|\frac{1}{m}}q_x = \frac{1 - A_x^{(m)}}{d^{(m)}} = \frac{1}{m}\sum_{k=0}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x,$$

$$\mathrm{Var}(\ddot{Y}_x^{(m)}) = \frac{{}^2A_x^{(m)} - (A_x^{(m)})^2}{(d^{(m)})^2}.$$

Theorem 1

$$\ddot{Y}_x^{(m)} = \frac{1}{m}\ddot{a}_{\overline{J_x^{(m)}}|\frac{i^{(m)}}{m}} = \frac{1 - Z_x^{(m)}}{d^{(m)}} = \frac{1}{m}\sum_{k=0}^{\infty} Z_{x:\overline{\frac{1}{\frac{k}{m}}}|},$$

$$\ddot{a}_x^{(m)} = \frac{1}{m}\sum_{k=1}^{\infty} \ddot{a}_{\overline{k}|\frac{i^{(m)}}{m}} \cdot {}_{\frac{k-1}{m}|\frac{1}{m}}q_x = \frac{1 - A_x^{(m)}}{d^{(m)}} = \frac{1}{m}\sum_{k=0}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x,$$

$$\mathrm{Var}(\ddot{Y}_x^{(m)}) = \frac{{}^2A_x^{(m)} - (A_x^{(m)})^2}{(d^{(m)})^2}.$$

Proof: The present value of level payments of $\frac{1}{m}$ at times $0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{J_x^{(m)}-1}{m}$ is

$$\ddot{Y}_x^{(m)} = \frac{1}{m}\ddot{a}_{\overline{J_x^{(m)}}|\frac{i^{(m)}}{m}} = \frac{1}{m}\frac{1 - v^{J_x^{(m)}/m}}{\frac{d^{(m)}}{m}} = \frac{1 - Z_x^{(m)}}{d^{(m)}}.$$

Theorem 1

$$\ddot{Y}_x^{(m)} = \frac{1}{m}\ddot{a}_{\overline{J_x^{(m)}}|\frac{i^{(m)}}{m}} = \frac{1 - Z_x^{(m)}}{d^{(m)}} = \frac{1}{m}\sum_{k=0}^{\infty} Z_{x:\overline{\frac{1}{\frac{k}{m}}}|},$$

$$\ddot{a}_x^{(m)} = \frac{1}{m}\sum_{k=1}^{\infty} \ddot{a}_{\overline{k}|\frac{i^{(m)}}{m}} \cdot {}_{\frac{k-1}{m}|\frac{1}{m}}q_x = \frac{1 - A_x^{(m)}}{d^{(m)}} = \frac{1}{m}\sum_{k=0}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x,$$

$$\mathrm{Var}(\ddot{Y}_x^{(m)}) = \frac{{}^2A_x^{(m)} - (A_x^{(m)})^2}{(d^{(m)})^2}.$$

Proof: Since the payment of $\frac{1}{m}$ at time $\frac{k}{m}$ is made if $T_x > \frac{k}{m}$, $\ddot{Y}_x^{(m)} = \frac{1}{m}\sum_{k=0}^{\infty} Z_{x:\overline{\frac{1}{\frac{k}{m}}}|}$.

Whole life annuity due paid m times a year.

Definition 4

A whole life immediate annuity with payments paid m times year is a series payments made at the end m–thly time interval while an individual is alive.

A whole life immediate annuity paid m times a year makes payments at times $\frac{1}{m}, \frac{2}{m}, \dots \frac{J_x^{(m)}-1}{m}$.

Definition 5

The present value of a whole life immediate annuity for (x) with unit annual payment paid m times a year is denoted by $Y_x^{(m)}$.

Definition 6

The actuarial present value of a whole life immediate annuity for (x) with unit annual payment paid m times a year is denoted by $a_x^{(m)}$.

Theorem 2

$$Y_x^{(m)} = \ddot{Y}_x^{(m)} - \frac{1}{m} = \frac{1}{m} a_{\overline{J_x^{(m)}-1}|\frac{i^{(m)}}{m}} = \frac{v^{1/m} - Z_x^{(m)}}{d^{(m)}}$$

$$= \frac{1}{m}\sum_{k=1}^{\infty} Z_{x:\overline{\frac{1}{\frac{k}{m}}}|},$$

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m} = \frac{1}{m}\sum_{k=2}^{\infty} a_{\overline{k-1}|\frac{i^{(m)}}{m}} \cdot {}_{\frac{k-1}{m}|\frac{1}{m}}q_x = \frac{v^{1/m} - A_x^{(m)}}{d^{(m)}}$$

$$= \frac{1}{m}\sum_{k=1}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x,$$

$$\mathrm{Var}(Y_x^{(m)}) = \frac{{}^2A_x^{(m)} - (A_x^{(m)})^2}{(d^{(m)})^2}.$$

Proof: The present value of level payments of $\frac{1}{m}$ at times $\frac{1}{m}, \frac{2}{m}, \dots, \frac{J_x^{(m)}-1}{m}$ is

$$Y_x^{(m)} = \frac{1}{m} a_{\overline{J_x^{(m)}-1}|\frac{i_m}{m}} = \frac{1}{m} \frac{1-(1+i)^{-\frac{(J_x^{(m)}-1)}{m}}}{i^{(m)}/m}$$
$$= \frac{1-(1+i)^{1/m} Z_x^{(m)}}{i^{(m)}} = \frac{v^{1/m} - Z_x^{(m)}}{d^{(m)}}.$$

Since the payment of $\frac{1}{m}$ at time $\frac{k}{m}$, $k \geq 1$, is made if $T_x > \frac{k}{m}$, $Y_x^{(m)} = \frac{1}{m} \sum_{k=1}^{\infty} Z_{x:\frac{k}{m}|}^{\;\;1}$.

n–year term life annuity due paid m times a year..

Definition 7

The present value of a n–year life due annuity for (x) with unit annual payment paid m times a year is denoted by $\ddot{Y}^{(m)}_{x:\overline{n}|}$.

Definition 8

The actuarial present value of a n–year life due annuity for (x) with unit annual payment paid m times a year is denoted by $\ddot{a}^{(m)}_{x:\overline{n}|}$.

Theorem 3

$$\ddot{Y}^{(m)}_{x:\overline{n}|} = \frac{1}{m}\ddot{a}_{\overline{\min(J^{(m)}_x, nm)}|\frac{i^{(m)}}{m}} = \frac{1 - Z^{(m)}_{x:\overline{n}|}}{d^{(m)}} = \frac{1}{m}\sum_{k=0}^{nm-1} Z_{x:\frac{1}{\frac{k}{m}|}},$$

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1}{m}\sum_{k=1}^{nm} \ddot{a}_{\overline{k}|\frac{i^{(m)}}{m}} \cdot {}_{\frac{k-1}{m}|\frac{1}{m}}q_x + \frac{1}{m}\ddot{a}_{\overline{nm}|\frac{i^{(m)}}{m}} \cdot {}_n p_x = \frac{1 - A^{(m)}_{x:\overline{n}|}}{d^{(m)}}$$

$$= \frac{1}{m}\sum_{k=0}^{nm-1} v^{\frac{1}{m}} \cdot {}_{\frac{k}{m}}p_x,$$

$$\mathrm{Var}(\ddot{Y}^{(m)}_{x:\overline{n}|}) = \frac{\mathrm{Var}(Z^{(m)}_{x:\overline{n}|})}{(d^{(m)})^2}.$$

Proof: The last beginning of the m–thly period of period where (x) is alive is $\frac{J_x^{(m)}-1}{m}$. The last beginning of the m–thly period of period before time n is $\frac{nm-1}{m}$. Payments are made at most until time $\min(\frac{J_x^{(m)}-1}{m}, \frac{nm-1}{m}) = \frac{\min(J_x^{(m)},nm)-1}{m}$. The present value of level payments of $\frac{1}{m}$ at times $0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{\min(J_x^{(m)},nm)-1}{m}$ is

$$Y_{x:\overline{n}|}^{(m)} = \frac{1}{m}\ddot{a}_{\overline{\min(J_x^{(m)},nm)}|\frac{i_m}{m}} = \frac{1}{m}\frac{1-(1+i)^{-\frac{\min(J_x^{(m)},nm)}{m}}}{d^{(m)}/m} = \frac{1-Z_{x:\overline{n}|}^{(m)}}{d^{(m)}}.$$

Since the payment of $\frac{1}{m}$ at time $\frac{k}{m}$, $0 \le k \le nm-1$, is made if $T_x > \frac{k}{m}$, $Y_{x:\overline{n}|}^{(m)} = \frac{1}{m}\sum_{k=0}^{nm-1} Z_{x:\overline{\frac{k}{m}}|}^{\;\;\,1}$.

n–year term annuity immediate paid m times a year..

Definition 9

The present value of an n–year life immediate annuity for (x) *with unit annual payment paid m times a year is denoted by* $Y^{(m)}_{x:\overline{n}|}$.

Definition 10

The actuarial present value of an n–year life immediate annuity for (x) *with unit annual payment paid m times a year is denoted by* $a^{(m)}_{x:\overline{n}|}$.

Theorem 4

$$Y^{(m)}_{x:\overline{n}|} = \frac{1}{m} a_{\overline{\min(J^{(m)}_x - 1, nm)}|\frac{i^{(m)}}{m}} = \frac{1}{m}\sum_{k=1}^{nm} Z_{x:\frac{1}{\frac{k}{m}}|} = \ddot{Y}^{(m)}_{x:\overline{n}|} - \frac{1}{m} + \frac{1}{m} Z_{x:\frac{1}{\overline{n}|}},$$

$$a^{(m)}_{x:\overline{n}|} = \frac{1}{m}\sum_{k=2}^{nm} a_{\overline{k-1}|} \cdot {}_{\frac{k-1}{m}|\frac{1}{m}}q_x + \frac{1}{m} a_{\overline{nm}|\frac{i^{(m)}}{m}} \cdot {}_n p_x = \frac{1}{m}\sum_{k=1}^{nm} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x$$

$$= \ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{m} + \frac{1}{m} \cdot {}_n E_x.$$

Proof: The last end of the m–thly period of period where (x) is alive is $\frac{J_x^{(m)}-1}{m}$. The last end of the m–thly period of period before time n is $\frac{nm}{m}$. Payments are made at most until time $\min(\frac{J_x^{(m)}-1}{m}, \frac{nm}{m}) = \frac{\min(J_x^{(m)}-1,nm)}{m}$. The present value of level payments of $\frac{1}{m}$ at times $\frac{1}{m}, \frac{2}{m}, \dots, \frac{\min(J_x^{(m)},nm)-1}{m}$ is

$$Y_x^{(m)} = \frac{1}{m} a_{\overline{\min(J_x^{(m)}-1,nm)}|\frac{i_m}{m}}.$$

Since the payment of $\frac{1}{m}$ at time $\frac{k}{m}$, $1 \le k \le nm$, is made if $T_x > \frac{k}{m}$,

$$Y_x^{(m)} = \frac{1}{m}\sum_{k=1}^{nm} Z_{x:\overline{\frac{k}{m}}|}^{\;\;1} = \frac{1}{m}\sum_{k=0}^{nm-1} Z_{x:\overline{\frac{k}{m}}|}^{\;\;1} - \frac{1}{m} + \frac{1}{m} Z_{x:\overline{n}|}^{\;\;1}$$
$$= \ddot{Y}_{x:\overline{n}|}^{(m)} - \frac{1}{m} + \frac{1}{m} Z_{x:\overline{n}|}^{\;\;1}.$$

Due n–year deferred annuity paid m times a year..

Definition 11

The present value of a due n–year deferred annuity for (x) with unit annual payment paid m times a year is denoted by ${}_{n|}\ddot{Y}_x^{(m)}$.

Definition 12

The actuarial present value of a due n–year deferred annuity for (x) with unit annual payment paid m times a year is denoted by ${}_{n|}\ddot{a}_x^{(m)}$.

Theorem 5

$$
{}_{n|}\ddot{Y}_x^{(m)} = \frac{1}{m} v^n \ddot{a}_{\overline{J_x^{(m)} - nm}|\frac{i^{(m)}}{m}} I(J_x^{(m)} > nm) = \frac{Z_{x:\overset{1}{\overline{n}|}} - {}_{n|}Z_x^{(m)}}{d^{(m)}}
$$

$$
= \frac{1}{m} \sum_{k=nm}^{\infty} Z_{x:\overset{1}{\overline{\frac{k}{m}}|}},
$$

$$
{}_{n|}\ddot{a}_x^{(m)} = \frac{1}{m} \sum_{k=nm+1}^{\infty} v^n \ddot{a}_{\overline{k-nm}|\frac{i^{(m)}}{m}} \cdot {}_{\frac{k-1}{m}|\frac{1}{m}} q_x = \frac{A_{x:\overset{1}{\overline{n}|}} - {}_{n|}A_x^{(m)}}{d^{(m)}}
$$

$$
= \frac{1}{m} \sum_{k=nm}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}} p_x = {}_nE_x \cdot \ddot{a}_{x+n}^{(m)},
$$

$$
\ddot{a}_x^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} + {}_{n|}\ddot{a}_x^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} + {}_nE_x \ddot{a}_{x+n}^{(m)}.
$$

Proof: The last beginning of the m–thly period of period where (x) is alive is $\frac{J_x^{(m)}-1}{m}$. The first beginning of the m–thly period of period after time n years is $\frac{nm}{m}$. If $J_x^{(m)} > nm$, payments of $\frac{1}{m}$ are made at times $\frac{nm}{m}, \frac{nm+1}{m}, \dots, \frac{J_x^{(m)}-1}{m}$. Hence,

$$
\begin{aligned}
{}_{n|}\ddot{Y}_x^{(m)} &= \frac{1}{m} v^n \ddot{a}_{\overline{J_x^{(m)}-nm}|\frac{i^{(m)}}{m}} I(J_x^{(m)} > nm) \\
=&\frac{1}{m} v^n \frac{1 - v^{(J_x^{(m)}-nm)/m}}{\frac{d^{(m)}}{m}} I(J_x^{(m)} > nm) \\
=&\frac{v^n - v^{J_x^{(m)}/m}}{d^{(m)}} I(J_x^{(m)} > nm) = \frac{Z_{x:\overset{1}{\overline{n}|}} - {}_{n|}Z_x^{(m)}}{d^{(m)}}.
\end{aligned}
$$

Since the payment of $\frac{1}{m}$ at time $\frac{k}{m}$, $k \geq nm$, is made if $T_x > \frac{k}{m}$, $Y_x^{(m)} = \frac{1}{m} \sum_{k=nm}^{\infty} Z_{x:\overset{1}{\overline{\frac{k}{m}}|}}$.

Immediate n–year deferred annuity paid m times a year..

Definition 13

The present value of an immediate n–year deferred annuity for (x) *with unit annual payment paid m times a year is denoted by* ${}_{n|}Y_x^{(m)}$.

Definition 14

The actuarial present value of an immediate n–year deferred annuity for (x) *with unit annual payment paid m times a year is denoted by* ${}_{n|}a_x^{(m)}$.

Theorem 6

$$
{}_{n|}Y_x^{(m)} = \frac{1}{m} v^n a_{\overline{J_x^{(m)} - nm - 1}|\frac{i^{(m)}}{m}} I(J_x^{(m)} > nm + 1) = \frac{1}{m} \sum_{k=nm+1}^{\infty} Z_{x:\frac{1}{\frac{k}{m}}|}
$$

$$
= {}_{n|}\ddot{Y}_x^{(m)} - \frac{1}{m} Z_{x:\frac{1}{\overline{n}|}},
$$

$$
{}_{n|}a_x^{(m)} = \frac{1}{m} \sum_{k=nm+2}^{\infty} v^n a_{\overline{k-nm-1}|\frac{i^{(m)}}{m}} \cdot {}_{\frac{k-1}{m}|\frac{1}{m}} q_x = \frac{1}{m} \sum_{k=nm+1}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}} p_x
$$

$$
= {}_nE_x \cdot a_{x+n}^{(m)} = {}_{n|}\ddot{a}_x^{(m)} - \frac{1}{m} {}_nE_x,
$$

$$
a_x^{(m)} = a_{x:\overline{n}|}^{(m)} + {}_{n|}a_x^{(m)} = a_{x:\overline{n}|}^{(m)} + {}_nE_x a_{x+n}^{(m)}.
$$

Proof: If $J_x^{(m)} \geq nm+2$, payments of $\frac{1}{m}$ are made at times $\frac{nm+1}{m}, \frac{nm+2}{m}, \dots, \frac{J_x^{(m)}-1}{m}$.

$$ {}_{n|}Y_x^{(m)} = \frac{1}{m} v^n a_{\overline{J_x^{(m)}-nm-1}|\frac{i^{(m)}}{m}} I(J_x^{(m)} > nm+1) = \frac{1}{m} \sum_{k=nm+1}^{\infty} Z_{x:\overline{\frac{k}{m}}|}^{\;\;1} $$

$$ = {}_{n|}\ddot{Y}_x^{(m)} - \frac{1}{m} Z_{x:\overline{n}|}^{\;\;1}, $$

$$ {}_{n|}a_x^{(m)} = \frac{1}{m} \sum_{k=nm+2}^{\infty} v^n a_{\overline{k-nm-1}|\frac{i^{(m)}}{m}} \cdot {}_{\frac{k-1}{m}|\frac{1}{m}}q_x = \frac{1}{m} \sum_{k=nm+1}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x $$

$$ = {}_nE_x \cdot a_{x+n}^{(m)} = {}_{n|}\ddot{a}_x^{(m)} - \frac{1}{m} {}_nE_x, $$

$$ a_x^{(m)} = a_{x:\overline{n}|}^{(m)} + {}_{n|}a_x^{(m)} = a_{x:\overline{n}|}^{(m)} + {}_nE_x a_{x+n}^{(m)}. $$

