Manual for SOA Exam MLC.

Chapter 5. Life annuities.

Section 5.6. Non–level payments annuities.

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Non–level payments annuities

In general, we can consider the case of life annuities with varying payments and general discount rates. Recall that v_t is the t–year discount factor. The force of interest is $\delta_t = -\frac{d}{dt} \ln v_t = \frac{-\frac{d}{dt} v_t}{v_t}$. We have that $v_t = e^{-\int_0^t \delta_s \, ds}$. Under compound interest: $v_t = v^t = (1+i)^{-t}$.

Example 1

Suppose that a special 3–year life annuity due makes a payment of c_{k-1} *at the beginning of year* k, $k = 1, 2, 3$, *where* $c_0 = 10000$, $c_1 = 11000$ *and* $c_2 = 12000$. *The annual effective rate of interest earned in the first and second year are* 6.5% *and* 6%, *respectively.* $p_x = 0.98$ *and* $p_{x+1} = 0.95$. *Let* Y *the present value random variable. Calculate* $E[Y]$ *and* $\mathrm{Var}(Y)$.

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Solution: We have that

$$
\begin{aligned}
&\text{If } K_x = 1,\ Y = 10000,\\
&\text{If } K_x = 2,\ Y = (10000) + (11000)(1.065)^{-1} = 20328.6385,\\
&\text{If } K_x > 2,\ Y = (10000) + (11000)(1.065)^{-1}\\
&+ (12000)(1.065)^{-1}(1.06)^{-1} = 30958.45513,\\
&\mathbb{P}\{K_x = 1\} = (0.02),\ \mathbb{P}\{K_x = 2\} = (0.98)(0.05) = 0.049,\\
&\mathbb{P}\{K_x > 2\} = (0.98)(0.95) = 0.931.
\end{aligned}
$$

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$$\mathbb{P}\{K_x = 1\} = (0.02),\ \mathbb{P}\{K_x = 2\} = 0.049,\ \mathbb{P}\{K_x > 2\} = 0.931.$$

We have that

$$\begin{aligned} &E[Y] = (10000)(0.02) + (20328.6385)(0.049) \\ &+ (30958.45513)(0.931) = 30018.42501, \\ &E[Y^2] = (10000)^2(0.02) + (20328.6385)^2(0.049) \\ &+ (30958.45513)^2(0.931) = 914543977.5, \\ &\mathrm{Var}(Y) = 914543977.5 - (30018.42501)^2 = 13438137.42. \end{aligned}$$

Theorem 1

Assume that the t–year discount factor is v_t. The actuarial present value of a whole life annuity due with payments $c_0, c_1, \dots,$ is

$$\sum_{k=1}^{\infty} \sum_{j=0}^{k-1} c_j v_j \cdot {}_{k-1|}q_x = \sum_{k=0}^{\infty} c_k v_k \cdot {}_k p_x.$$

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Proof: If $K_x = k$, $k \geq 1$, then payments of $c_0, c_2, \dots, c_{k-1}$, are made at the beginning of year. The present value of these payments is $\sum_{j=0}^{k-1} c_j v_j$. The APV of this annuity is

$$\sum_{k=1}^{\infty} \sum_{j=0}^{k-1} c_j v_j \mathbb{P}\{K_x = k\} = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} c_j v_j \cdot {}_{k-1|}q_x.$$

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Proof: The present value of the k–th payment is $c_k v_k$ and the probability that this payment is made is ${}_k p_x$. Hence, the APV of this annuity is $\sum_{k=0}^{\infty} c_k v_k \cdot {}_k p_x$.

Corollary 1

A unit annually increasing due whole life annuity has payments $1, 2, \dots,$ *at the beginning of the year and actuarial present value*

$$(I\ddot{a})_x = \sum_{k=0}^{\infty}(k+1)v^k \cdot {}_kp_x.$$

Corollary 2

A unit annually increasing n–year term due life unit annuity has payments $1, 2, \dots, n$ *at the beginning of the year and actuarial present value*

$$(I\ddot{a})_{x:\overline{n}|} = \sum_{k=0}^{n-1} (k+1) v^k \cdot {}_k p_x.$$

Corollary 3

A unit annually decreasing n–year term due life annuity due has payments $n, n-1, \dots, 1$ *at the beginning of the year and actuarial present value*

$$(D\ddot{a})_{x:\overline{n}|} = \sum_{k=0}^{n-1} (n-k) v^k \cdot {}_k p_x.$$

Theorem 2

Assume that the t–year discount factor is v_t. The actuarial present value of a whole life annuity immediate with annual payments of $c_1, c_2, \dots,$

$$\sum_{k=1}^{\infty}\sum_{j=1}^{k} c_j v_j \cdot {}_{k|}q_x = \sum_{k=1}^{\infty} c_k v_k \cdot {}_k p_x.$$

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Proof: If $K_x = k + 1$, $k \geq 1$, then payments of $c_1, c_2, \ldots, c_k$, are made at the end of year. The present value of these payments is $\sum_{j=1}^{k} c_j v_j$. The APV of this annuity is

$$\sum_{k=1}^{\infty} \sum_{j=1}^{k} c_j v_j \mathbb{P}\{K_x = k + 1\} = \sum_{k=1}^{\infty} \sum_{j=1}^{k} c_j v_j \cdot {}_{k|}q_x.$$

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Proof: The present value of the k–th payment is $c_k v^k$ and the probability that this payment is made is ${}_k p_x$. Hence, the APV of this annuity is $\sum_{k=1}^{\infty} c_k v_k \cdot {}_k p_x$.

Corollary 4

A unit annually increasing immediate whole life annuity has payments $1, 2, \dots,$ *at the end of the year and actuarial present value*

$$(Ia)_x = \sum_{k=1}^{\infty} k v^k \cdot {}_k p_x.$$

Corollary 5

A unit annually increasing n–year term life immediate annuity has payments $1, 2, \dots, n$ *at the end of the year and actuarial present value*

$$(Ia)_{x:\overline{n}|} = \sum_{k=1}^{n} k v^k \cdot {}_k p_x.$$

Corollary 6

A unit annually decreasing n–year term life immediate annuity has payments $n, n-1, \dots, 1$ *at the end of the year and actuarial present value*

$$(Da)_{x:\overline{n}|} = \sum_{k=1}^{n} (n+1-k)v^k \cdot {}_kp_x.$$

Example 2

Suppose that $\mu_x(t) = 0.05$, $t \geq 0$, $\delta = 0.07$. *Find* $(Ia)_x$.

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Solution: Using that $\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$, $x \neq 0$,

$$(Ia)_x = \sum_{k=1}^{\infty} kv^k \cdot {}_kp_x = \sum_{k=1}^{\infty} ke^{-0.05k}e^{-0.07k} = \sum_{k=1}^{\infty} ke^{-0.12k}$$

$$= \frac{e^{-0.12}}{(1-e^{-0.12})^2} = 69.36117108.$$

Theorem 3

Assume that the t–year discount factor is v_t. A continuous whole life annuity with rate of payments $c(t)$ has an actuarial present value of

$$\int_0^\infty \int_0^t c(s)v_s\,ds\ {}_tp_x\mu_{x+t}\,dt = \int_0^\infty c(t)\cdot v_t\cdot {}_tp_x\,dt.$$

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Proof: If death happens at time t, the present value of the payment is $\int_0^t c(s)\cdot v_s\,ds$. The density of the age–at–death T_x is ${}_tp_x\mu_{x+t}$. So, the APV is

$$\int_0^\infty \int_0^t c(s)v_s\,ds\ {}_tp_x\mu_{x+t}\,dt = \int_0^\infty \int_s^\infty c(s)v_s\cdot {}_tp_x\mu_{x+t}\,dt\,ds$$
$$= \int_0^\infty c(s)\cdot v_s\cdot {}_sp_x\,ds.$$

Corollary 7

A continuously increasing whole life unit annuity paid at the time of death has an actuarial present value of

$$(\bar{I}\bar{a})_x = \int_0^\infty t \cdot v^t \cdot {}_tp_x \, dt.$$

Corollary 8

A continuously increasing n–year term life unit annuity paid at the time of death with rate of payments t has an actuarial present value of

$$(\bar{I}\bar{a})_{x:\overline{n}|} = \int_0^n t \cdot v^t \cdot {}_tp_x \, dt.$$

Corollary 9

A continuously increasing decreasing n–year term life unit annuity paid at the time of death has an actuarial present value of

$$(\overline{D}\overline{a})_{x:\overline{n}|} = \int_0^n (n-t) \cdot v^t \cdot {}_tp_x \, dt.$$

Corollary 10

An annually increasing whole life unit annuity paid at the time of death has an actuarial present value of

$$(I\overline{a})_x = \int_0^\infty \lceil t \rceil \cdot v^t \cdot {}_t p_x \, dt.$$

Corollary 11

An annually increasing n–year term life unit annuity paid at the time of death has an actuarial present value of

$$(I\overline{a})_{x:\overline{n}|} = \int_0^n \lceil t \rceil \cdot v^t \cdot {}_tp_x \, dt.$$

Corollary 12

An annually decreasing n–year term life unit annuity paid at the time of death has an actuarial present value of

$$(D\overline{a})_{x:\overline{n}|} = \int_0^n \lceil n - t \rceil \cdot v^t \cdot {}_t p_x \, dt.$$

Example 3

Suppose that $\mu_x(t) = 0.05$, $t \geq 0$, $\delta = 0.07$. *Find* $\left(\bar{I}\bar{a}\right)_{x:\overline{15}|}$.

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Solution: We have that

$$(\bar{I}\bar{a})_{x:\overline{15}|} = \int_0^{15} te^{-t(0.05)}e^{-t(0.07)}\,dt = \int_0^{(15)(0.12)} \frac{1}{(0.12)^2}te^{-t}\,dt$$

$$= \frac{1}{(0.12)^2}(-e^{-t}(1+t))\Big|_0^{1.8} = \frac{1}{(0.12)^2}\left(1 - e^{-1.8}(1+1.8)\right)$$

$$= 37.30299396.$$

Example 4

A pension plan pays continuous payments for the remaining lifetime of a life aged (65)*. The rate of payments is* 50000 *a year. Suppose that the force of mortality is* 0.01*. The force of interest is 0.08 for deposits made in the the next 10 years and 0.06 for deposits made after 10 years. Find the APV of the benefit payments. Find the actuarial present value of this pension plan.*

Solution 1: We have that

$$v_t = e^{-\int_0^t \delta_s\, ds} = \begin{cases} e^{-0.08t} & \text{if } 0 \le t \le 10, \\ e^{-(0.08)(10)-(0.06)(t-10)} & \text{if } 10 \le t \le 10, \end{cases}$$

The APV is

$$\int_0^\infty c(t) v_t \cdot {}_t p_x \, dt = \int_0^{10} (50000) e^{-0.08t} e^{-0.01t}\, dt$$
$$+ \int_{10}^\infty (50000) e^{-(0.08)(10)-(0.06)(t-10)} e^{-0.01t}\, dt$$
$$= \frac{(50000)(1 - e^{-(0.09)(10)})}{0.09}$$
$$+ \int_0^\infty (50000) e^{-(0.08)(10)} e^{-(0.06)t} e^{-(0.01)(t+10)}\, dt$$
$$= \frac{(50000)(1 - e^{-(0.09)(10)})}{0.09} + \frac{(50000) e^{-(0.08)(10)-(0.01)(10)}}{0.07}$$
$$= 329683.5 + 290406.9 = 620090.4.$$

Solution 2:

$$
\begin{aligned}
&(50000)\overline{a}_{x:\overline{10}|} + (50000)v^{10} \cdot {}_{10}p_x \overline{a}_{x+10} \\
=&(50000)\frac{1 - e^{-(10)(0.08)}e^{-(10)(0.01)}}{0.01 + 0.08} \\
&+ (50000)e^{-(10)(0.08)}e^{-(10)(0.01)}\frac{1}{0.01 + 0.06} \\
=&(50000)\frac{1 - e^{-0.9}}{0.09} + (50000)e^{-0.9}\frac{1}{0.07} = 620090.4.
\end{aligned}
$$

