Manual for SOA Exam MLC.

Chapter 5. Life annuities.

Section 5.7. Computing present values from a life table

Extract from:
"Arcones' Manual for the SOA Exam MLC. Spring 2010 Edition".
available at http://www.actexmadriver.com/

Recall that:

Theorem 1

Assuming a uniform distribution of deaths, we have that:

(i) $\overline{A}_x = \frac{i}{\delta} A_x$.

(ii) $\overline{A}^{1}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|}$.

(iii) ${}_{n|}\overline{A}_x = \frac{i}{\delta} \cdot {}_{n|}A_x$.

(iv) $\overline{A}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$.

Theorem 2

Assuming a uniform distribution of deaths, we have that:

(i) $A^{(m)}_x = \frac{i}{i^{(m)}} A_x$.

(ii) $A^{1\,(m)}_{x:\overline{n}|} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|}$.

(iii) ${}_{n|}A^{(m)}_x = \frac{i}{i^{(m)}} \cdot {}_{n|}A_x$.

(iv) $A^{(m)}_{x:\overline{n}|} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$.

Whole life annuities

Recall that:

$$\ddot{a}_x = \frac{1 - A_x}{d},$$

$$a_x = \frac{v - A_x}{d},$$

$$\overline{a}_x = \frac{1 - \overline{A}_x}{\delta},$$

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}},$$

$$a_x^{(m)} = \frac{v^{1/m} - A_x^{(m)}}{d^{(m)}}.$$

Theorem 3

Under a uniform distribution of deaths within each year,

$$\ddot{a}_x^{(m)} = \frac{1 - \frac{i}{i^{(m)}} A_x}{d^{(m)}} = \frac{id}{i^{(m)} d^{(m)}} \ddot{a}_x + \frac{i^{(m)} - i}{i^{(m)} d^{(m)}},$$

$$a_x^{(m)} = \frac{v^{1/m} - \frac{i}{i^{(m)}} A_x}{d^{(m)}} = \frac{id}{i^{(m)} d^{(m)}} a_x + \frac{d^{(m)} - d}{i^{(m)} d^{(m)}},$$

$$\overline{a}_x = \frac{1 - \frac{i}{\delta} A_x}{\delta} = \frac{id}{\delta^2} \ddot{a}_x + \frac{\delta - i}{\delta^2}.$$

Theorem 3

Under a uniform distribution of deaths within each year,

$$\ddot{a}_x^{(m)} = \frac{1 - \frac{i}{i^{(m)}} A_x}{d^{(m)}} = \frac{id}{i^{(m)} d^{(m)}} \ddot{a}_x + \frac{i^{(m)} - i}{i^{(m)} d^{(m)}},$$

$$a_x^{(m)} = \frac{v^{1/m} - \frac{i}{i^{(m)}} A_x}{d^{(m)}} = \frac{id}{i^{(m)} d^{(m)}} a_x + \frac{d^{(m)} - d}{i^{(m)} d^{(m)}},$$

$$\bar{a}_x = \frac{1 - \frac{i}{\delta} A_x}{\delta} = \frac{id}{\delta^2} \ddot{a}_x + \frac{\delta - i}{\delta^2}.$$

Proof: Using that $\ddot{a}_x^{(m)} = \frac{1-A_x^{(m)}}{d^{(m)}}$, $A_x^{(m)} = \frac{i}{i^{(m)}} A_x$ and $\ddot{a}_x = \frac{1-A_x}{d}$, we get that

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}} = \frac{1 - \frac{i}{i^{(m)}} A_x}{d^{(m)}} = \frac{1 - \frac{i}{i^{(m)}}(1 - d\ddot{a}_x)}{d^{(m)}}$$

$$= \frac{di}{d^{(m)} i^{(m)}} \ddot{a}_x + \frac{1 - \frac{i}{i^{(m)}}}{d^{(m)}} = \frac{id}{i^{(m)} d^{(m)}} \ddot{a}_x + \frac{i^{(m)} - i}{i^{(m)} d^{(m)}}.$$

Theorem 3

Under a uniform distribution of deaths within each year,

$$\ddot{a}_x^{(m)} = \frac{1 - \frac{i}{i^{(m)}} A_x}{d^{(m)}} = \frac{id}{i^{(m)} d^{(m)}} \ddot{a}_x + \frac{i^{(m)} - i}{i^{(m)} d^{(m)}},$$

$$a_x^{(m)} = \frac{v^{1/m} - \frac{i}{i^{(m)}} A_x}{d^{(m)}} = \frac{id}{i^{(m)} d^{(m)}} a_x + \frac{d^{(m)} - d}{i^{(m)} d^{(m)}},$$

$$\bar{a}_x = \frac{1 - \frac{i}{\delta} A_x}{\delta} = \frac{id}{\delta^2} \ddot{a}_x + \frac{\delta - i}{\delta^2}.$$

Proof: Using that $a_x^{(m)} = \frac{v^{1/m} - A_x^{(m)}}{d^{(m)}}$, $A_x^{(m)} = \frac{i}{i^{(m)}} A_x$ and $a_x = \frac{v - A_x}{d}$, we get that

$$a_x^{(m)} = \frac{v^{1/m} - A_x^{(m)}}{d^{(m)}} = \frac{v^{1/m} - \frac{i}{i^{(m)}} A_x}{d^{(m)}}$$

$$= \frac{v^{1/m} - \frac{i}{i^{(m)}} (v - d a_x)}{d^{(m)}} = \frac{d \frac{i}{i^{(m)}}}{d^{(m)}} a_x + \frac{v^{1/m} + v \frac{i}{i^{(m)}}}{d^{(m)}}.$$

Theorem 3

Under a uniform distribution of deaths within each year,

$$\ddot{a}_x^{(m)} = \frac{1 - \frac{i}{i^{(m)}} A_x}{d^{(m)}} = \frac{id}{i^{(m)} d^{(m)}} \ddot{a}_x + \frac{i^{(m)} - i}{i^{(m)} d^{(m)}},$$

$$a_x^{(m)} = \frac{v^{1/m} - \frac{i}{i^{(m)}} A_x}{d^{(m)}} = \frac{id}{i^{(m)} d^{(m)}} a_x + \frac{d^{(m)} - d}{i^{(m)} d^{(m)}},$$

$$\overline{a}_x = \frac{1 - \frac{i}{\delta} A_x}{\delta} = \frac{id}{\delta^2} \ddot{a}_x + \frac{\delta - i}{\delta^2}.$$

Proof: We have that

$$\frac{v^{1/m} + v \frac{i}{i^{(m)}}}{d^{(m)}} = \frac{v^{1/m} i^{(m)} - vi}{i^{(m)} d^{(m)}} = \frac{d^{(m)} - d}{i^{(m)} d^{(m)}}.$$

So,

$$a_x^{(m)} = \frac{id}{i^{(m)} d^{(m)}} a_x + \frac{d^{(m)} - d}{i^{(m)} d^{(m)}}.$$

Theorem 3

Under a uniform distribution of deaths within each year,

$$\ddot{a}_x^{(m)} = \frac{1 - \frac{i}{i^{(m)}} A_x}{d^{(m)}} = \frac{id}{i^{(m)} d^{(m)}} \ddot{a}_x + \frac{i^{(m)} - i}{i^{(m)} d^{(m)}},$$

$$a_x^{(m)} = \frac{v^{1/m} - \frac{i}{i^{(m)}} A_x}{d^{(m)}} = \frac{id}{i^{(m)} d^{(m)}} a_x + \frac{d^{(m)} - d}{i^{(m)} d^{(m)}},$$

$$\overline{a}_x = \frac{1 - \frac{i}{\delta} A_x}{\delta} = \frac{id}{\delta^2} \ddot{a}_x + \frac{\delta - i}{\delta^2}.$$

Proof: We know that $\overline{a}_x = \frac{1-\overline{A}_x}{\delta}$, $\overline{A}_x = \frac{i}{\delta} A_x$ and $\ddot{a}_x = \frac{1-A_x}{d}$. Hence,

$$\overline{a}_x = \frac{1 - \overline{A}_x}{\delta} = \frac{1 - \frac{i}{\delta} A_x}{\delta} = \frac{1 - \frac{i}{\delta}(1 - d\ddot{a}_x)}{\delta} = \frac{id}{\delta^2} \ddot{a}_x + \frac{\delta - i}{\delta^2}.$$

Example 1

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Suppose that $i = 6.5\%$.
(i) Calculate $\ddot{a}_{80}^{(12)}$, $a_{80}^{(12)}$ *and* $\bar{a}_{80}$ *using that* $A_{80} = 0.8161901166$.
(ii) Calculate $\ddot{a}_{80}^{(12)}$, $a_{80}^{(12)}$ *and* $\bar{a}_{80}$ *using that* $\ddot{a}_{80} = 3.011654244$.

Solution: (i) We have that $i = 6.5\%$,
$d = \frac{0.065}{1+0.065} = 6.103286385\%$,
$i^{(12)} = 12((1+0.065)^{\frac{1}{12}} - 1) = 6.314033132\%$,
$d^{(12)} = 12(1 - (1+0.065)^{-\frac{1}{12}}) = 6.280984512\%$. So,

$$\ddot{a}_{80}^{(12)} = (50000)\frac{1 - \frac{i}{i^{(12)}}A_x}{d^{(12)}} = \frac{1 - \frac{0.065}{0.06314033132}(0.8161901166)}{0.06280984512}$$

$=2.543720348$,

$$a_{80}^{(12)} = \ddot{a}_{80}^{(12)} - 1 = 2.543720348 - 1 = 1.543720348,$$

$$\overline{a}_{80} = \frac{1 - \frac{i}{\delta}A_x}{\delta} = \frac{1 - \frac{0.065}{\ln(1.065)}0.8161901166}{\ln(1.065)} = 2.501986537.$$

Solution: (ii)

$$\ddot{a}_{80}^{(12)} = \frac{id}{i^{(12)}d^{(12)}}\ddot{a}_x + \frac{i^{(12)} - i}{i^{(12)}d^{(12)}}$$
$$= \frac{(0.065)(0.06103286385)}{(0.06314033132)(0.06280984512)}(3.011654244)$$
$$+ \frac{0.06314033132 - 0.065}{(0.06314033132)(0.06280984512)}$$
$$=2.543720349,$$
$$a_{80}^{(12)} = \ddot{a}_{80}^{(12)} - 1 = 2.543720349 - 1 = 1.543720349,$$
$$\bar{a}_{80} = \frac{id}{\delta^2}\ddot{a}_x + \frac{\delta - i}{\delta^2} = \frac{(0.065)(0.06103286385)}{(\ln(1.065))^2}(3.011654244)$$
$$+ \frac{\ln(1.065) - (0.065)}{(\ln(1.065))^2}$$
$$=2.501986538.$$

Deferred annuities

Recall that:

$$
{}_{n|}\ddot{a}_x = {}_nE_x \ddot{a}_{x+n} = \frac{{}_nE_x - {}_{n|}A_x}{d},
$$

$$
{}_{n|}\overline{a}_x = {}_nE_x \overline{a}_{x+n} = \frac{{}_nE_x - {}_{n|}A_x}{d},
$$

$$
{}_{n|}\ddot{a}_x^{(m)} = {}_nE_x \ddot{a}_{x+n}^{(m)} = \frac{{}_nE_x - {}_{n|}A_x^{(m)}}{d^{(m)}},
$$

$$
{}_{n|}a_x^{(m)} = {}_{n|}\ddot{a}_x^{(m)} - \frac{1}{m} {}_nE_x,
$$

Theorem 4

Under a uniform distribution of deaths within each year,

$$
{}_{n|}\ddot{a}_x^{(m)} = \frac{{}_nE_x - \frac{i}{i^{(m)}} \cdot {}_{n|}A_x}{d^{(m)}} = \frac{id}{i^{(m)}d^{(m)}} \cdot {}_{n|}\ddot{a}_x + \frac{i^{(m)} - i}{i^{(m)}d^{(m)}} \cdot {}_nE_x,
$$

$$
{}_{n|}a_x^{(m)} = {}_{n|}\ddot{a}_x^{(m)} - \frac{1}{m}{}_nE_x = \frac{id}{i^{(m)}d^{(m)}} \cdot {}_{n|}a_x + \frac{d^{(m)} - d}{i^{(m)}d^{(m)}} \cdot {}_nE_x,
$$

$$
{}_{n|}\bar{a}_x = \frac{{}_nE_x - \frac{i}{\delta} \cdot {}_{n|}A_x}{\delta} = \frac{id}{\delta^2} \cdot {}_{n|}\ddot{a}_x + \frac{\delta - i}{\delta^2}{}_nE_x.
$$

Proof: For the deferred life annuity due, using that ${}_{n|}\ddot{a}_x^{(m)} = {}_nE_x \ddot{a}_{x+n}^{(m)}$, $\ddot{a}_{x+n}^{(m)} = \frac{1-A_{x+n}^{(m)}}{d^{(m)}}$, $A_x^{(m)} = \frac{i}{i^{(m)}} A_x$, ${}_nE_x A_{x+n} = {}_{n|}A_x$ and ${}_{n|}\ddot{a}_x = \frac{{}_nE_x - {}_{n|}A_x}{d}$, we get that

$$
\begin{aligned}
{}_{n|}\ddot{a}_x^{(m)} &= {}_nE_x \ddot{a}_{x+n}^{(m)} = {}_nE_x \frac{1 - A_{x+n}^{(m)}}{d^{(m)}} = {}_nE_x \frac{1 - \frac{i}{i^{(m)}} A_{x+n}}{d^{(m)}} \\
&= \frac{{}_nE_x - \frac{i}{i^{(m)}} \cdot {}_{n|}A_x}{d^{(m)}} = \frac{{}_nE_x - \frac{i}{i^{(m)}} \cdot ({}_nE_x - d \cdot {}_{n|}\ddot{a}_x)}{d^{(m)}} \\
&= \frac{id}{i^{(m)} d^{(m)}} \cdot {}_{n|}\ddot{a}_x + \frac{i^{(m)} - i}{i^{(m)} d^{(m)}} \cdot {}_nE_x .
\end{aligned}
$$

Proof: For a deferred life annuity immediate, using that ${}_{n|}a_x^{(m)} = {}_{n|}\ddot{a}_x^{(m)} - \frac{1}{m}{}_nE_x$, ${}_{n|}\ddot{a}_x^{(m)} = {}_nE_x\ddot{a}_{x+n}^{(m)}$, $\ddot{a}_x^{(m)} = \frac{1-A_x^{(m)}}{d^{(m)}}$, $A_x^{(m)} = \frac{i}{i^{(m)}}A_x$, we get that

$$
\begin{aligned}
&{}_{n|}a_x^{(m)} = {}_{n|}\ddot{a}_x^{(m)} - \frac{1}{m}{}_nE_x = {}_nE_x\ddot{a}_{x+n}^{(m)} - \frac{1}{m}{}_nE_x = {}_nE_x\frac{1-A_{x+n}^{(m)}}{d^{(m)}} - \frac{1}{m}{}_nE_x \\
=&{}_nE_x\frac{1-\frac{i}{i^{(m)}}A_{x+n}}{d^{(m)}} - \frac{1}{m}{}_nE_x = \frac{{}_nE_x(1-\frac{d^{(m)}}{m}) - {}_nE_x\frac{i}{i^{(m)}}A_{x+n}}{d^{(m)}} \\
=&\frac{v^{1/m}\cdot{}_nE_x - \frac{i}{i^{(m)}}\cdot{}_{n|}A_x}{d^{(m)}}
\end{aligned}
$$

and

$$
\begin{aligned}
&{}_{n|}a_x^{(m)} = {}_nE_xa_{x+n}^{(m)} = {}_nE_x\left(\frac{id}{i^{(m)}d^{(m)}}a_x + \frac{d^{(m)}-d}{i^{(m)}d^{(m)}}\right) \\
=&\frac{id}{i^{(m)}d^{(m)}}\cdot{}_{n|}a_x + \frac{d^{(m)}-d}{i^{(m)}d^{(m)}}\cdot{}_nE_x
\end{aligned}
$$

Proof: For a deferred continuous life annuity, using that ${}_{n}|\overline{a}_{x} = {}_{n}E_{x}\overline{a}_{x+n}$, $\overline{a}_{x} = \frac{1-\overline{A}_{x}}{\delta}$ $\overline{A}^{1}_{x:\overline{n}|} = \frac{i}{\delta}A^{1}_{x:\overline{n}|}$, ${}_{n}E_{x}A_{x+n} = {}_{n}|A_{x}$ and ${}_{n}|\ddot{a}_{x} = \frac{{}_{n}E_{x}-{}_{n}|A_{x}}{d}$, we get that

$$
{}_{n}|\overline{a}_{x} = {}_{n}E_{x}\overline{a}_{x+n} = {}_{n}E_{x}\frac{1-\overline{A}_{x+n}}{\delta} = {}_{n}E_{x}\frac{1-\frac{i}{\delta}A_{x+n}}{\delta} = \frac{{}_{n}E_{x} - \frac{i}{\delta}\cdot {}_{n}|A_{x}}{\delta}
$$

$$
= \frac{{}_{n}E_{x} - \frac{i}{\delta}\cdot({}_{n}E_{x} - d\cdot {}_{n}|\ddot{a}_{x})}{\delta} = \frac{id}{\delta^{2}}\cdot {}_{n}|\ddot{a}_{x} + \frac{\delta - i}{\delta^{2}}\cdot {}_{n}E_{x}.
$$

Temporary annuities

Recall that

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d},$$

$$\overline{a}_{x:\overline{n}|} = \frac{1 - \overline{A}_{x:\overline{n}|}}{\delta},$$

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1 - A^{(m)}_{x:\overline{n}|}}{d^{(m)}},$$

$$a^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{m} + \frac{1}{m}\,{}_nE_x,$$

Theorem 5

Under a uniform distribution of deaths within each year,

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1 - {}_nE_x - \frac{i}{i^{(m)}} A^1_{x:\overline{n}|}}{d^{(m)}} = \frac{id}{i^{(m)} d^{(m)}} \ddot{a}_{x:\overline{n}|} + \frac{i^{(m)} - i}{i^{(m)} d^{(m)}} (1 - {}_nE_x),$$

$$a^{(m)}_{x:\overline{n}|} = \frac{id}{i^{(m)} d^{(m)}} a_{x:\overline{n}|} + \frac{d^{(m)} - d}{i^{(m)} d^{(m)}} (1 - {}_nE_x),$$

$$\overline{a}_{x:\overline{n}|} = \frac{1 - {}_nE_x - \frac{i}{\delta} A^1_{x:\overline{n}|}}{\delta} = \frac{id}{\delta^2} \ddot{a}_{x:\overline{n}|} + \frac{\delta - i}{\delta^2} (1 - {}_nE_x).$$

Proof: For a n–year term life annuity due, using that
$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1-A^{(m)}_{x:\overline{n}|}}{d^{(m)}}$, $A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + {}_nE_x$, $A^{(m)}_x = \frac{i}{i^{(m)}} A_x$, $\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{d}$,
we get that

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1 - A^{(m)}_{x:\overline{n}|}}{d^{(m)}} = \frac{1 - {A^{1}_{x:\overline{n}|}}^{(m)} - {}_nE_x}{d^{(m)}} = \frac{1 - {}_nE_x - \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|}}{d^{(m)}}$$

$$= \frac{1 - {}_nE_x - \frac{i}{i^{(m)}} \left(1 - d\ddot{a}_{x:\overline{n}|} - {}_nE_x\right)}{d^{(m)}}$$

$$= \frac{id}{i^{(m)} d^{(m)}} \ddot{a}_{x:\overline{n}|} + \frac{i^{(m)} - i}{i^{(m)} d^{(m)}} (1 - {}_nE_x)$$

Proof: For a n–year term life annuity immediate, using that $a^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{m} + \frac{1}{m}{}_nE_x$, $\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1-A^{(m)}_{x:\overline{n}|}}{d^{(m)}}$, $A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + {}_nE_x$, $A^{(m)}_x = \frac{i}{i^{(m)}}A_x$, $\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{d}$, , we get that

$$
\begin{aligned}
&a^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{m} + \frac{1}{m}{}_nE_x \\
=&\frac{id}{i^{(m)}d^{(m)}}\ddot{a}_{x:\overline{n}|} + \frac{i^{(m)}-i}{i^{(m)}d^{(m)}}(1-{}_nE_x) - \frac{1}{m} + \frac{1}{m}{}_nE_x \\
=&\frac{id}{i^{(m)}d^{(m)}}(1+a_{x:\overline{n}|}-{}_nE_x) + \frac{i^{(m)}-i}{i^{(m)}d^{(m)}}(1-{}_nE_x) - \frac{1}{m}(1-{}_nE_x) \\
=&\frac{id}{i^{(m)}d^{(m)}}a_{x:\overline{n}|} + (1-{}_nE_x)\left(\frac{id}{i^{(m)}d^{(m)}} + \frac{i^{(m)}-i}{i^{(m)}d^{(m)}} - \frac{1}{m}\right).
\end{aligned}
$$

Proof: We have that

$$\frac{id}{i^{(m)}d^{(m)}} + \frac{i^{(m)} - i}{i^{(m)}d^{(m)}} - \frac{1}{m} = \frac{id + i^{(m)} - i + \frac{i^{(m)}d^{(m)}}{m}}{i^{(m)}d^{(m)}}$$

$$= \frac{i^{(m)}\left(1 - \frac{d^{(m)}}{m}\right) - i(1-d)}{i^{(m)}d^{(m)}}$$

$$= \frac{i^{(m)}v^{1/m} - iv}{i^{(m)}d^{(m)}} = \frac{d^{(m)} - d}{i^{(m)}d^{(m)}}.$$

Hence,

$$a^{(m)}_{x:\overline{n}|} = \frac{id}{i^{(m)}d^{(m)}} a_{x:\overline{n}|} + \frac{d^{(m)} - d}{i^{(m)}d^{(m)}}(1 - {}_nE_x).$$

For a n–year term life continuous annuity, using that $\overline{a}_{x:\overline{n}|} = \frac{1-\overline{A}_{x:\overline{n}|}}{\delta}$, $\overline{A}_{x:\overline{n}|} = \overline{A}^{1}_{x:\overline{n}|} + {}_nE_x$, $\overline{A}^{1}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|}$, $\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{d}$, we get that

$$\overline{a}_{x:\overline{n}|} = \frac{1-\overline{A}_{x:\overline{n}|}}{\delta} = \frac{1-\overline{A}^{1}_{x:\overline{n}|} - {}_nE_x}{\delta} = \frac{1 - {}_nE_x - \frac{i}{\delta} A^{1}_{x:\overline{n}|}}{\delta}$$

$$= \frac{1 - {}_nE_x - \frac{i}{\delta}\left(1 - d\ddot{a}_{x:\overline{n}|} - {}_nE_x\right)}{d^{(m)}} = \frac{id}{\delta^2}\ddot{a}_{x:\overline{n}|} + \frac{\delta - i}{\delta^2}(1 - {}_nE_x).$$

Linear interpolation of the actuarial discount factor.

Another way to interpolate is to assume that actuarial discount factor is linear. If we assume ${}_{k+\frac{j}{m}}E_x = v^{k+\frac{j}{m}} \cdot {}_{k+\frac{j}{m}}p_x$ is linear in j, then

$$ {}_{k+\frac{j}{m}}E_x = {}_kE_x - \frac{j}{m}({}_{k+1}E_x - {}_kE_x), j = 0, 1, \dots, m-1. $$

The actuarial discount factor appears in annuities computations. We have that

$$ \ddot{a}_x^{(m)} = \frac{1}{m}\sum_{k=0}^{\infty} v^{\frac{k}{m}} \cdot {}_{\frac{k}{m}}p_x = \frac{1}{m}\sum_{k=0}^{\infty}\sum_{j=0}^{m-1} v^{k+\frac{j}{m}} \cdot {}_{k+\frac{j}{m}}p_x = \frac{1}{m}\sum_{k=0}^{\infty}\sum_{j=0}^{m-1} {}_{k+\frac{j}{m}}E_x. $$

Theorem 6

Assuming that ${}_{k+\frac{j}{m}}E_x$ is linear in j, then

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m},$$

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x),$$

$$_{n|}\ddot{a}_x^{(m)} = {}_{n|}\ddot{a}_x - \frac{m-1}{2m} \cdot {}_nE_x.$$

Proof: Using that $\sum_{j=0}^{m-1} j = \frac{(m-1)m}{2}$, we have that

$$\ddot{a}_x^{(m)} = \frac{1}{m}\sum_{k=0}^{\infty}\sum_{j=0}^{m-1} {}_{k+\frac{j}{m}}E_x = \frac{1}{m}\sum_{k=0}^{\infty}\sum_{j=0}^{m-1}\left({}_kE_x - \frac{j}{m}\left({}_{k+1}E_x - {}_kE_x\right)\right)$$

$$=\frac{1}{m}\sum_{k=0}^{\infty}\left({}_kE_x - \frac{m-1}{2}\left({}_{k+1}E_x - {}_kE_x\right)\right)$$

$$=\frac{m+1}{2m}\sum_{k=0}^{\infty}{}_kE_x + \frac{m-1}{2m}\sum_{k=0}^{\infty}{}_{k+1}E_x$$

$$=\frac{m+1}{2m}\sum_{k=0}^{\infty}{}_kE_x + \frac{m-1}{2m}\sum_{k=1}^{\infty}{}_kE_x$$

$$=\frac{m+1}{2m}\sum_{k=0}^{\infty}{}_kE_x + \frac{m-1}{2m}\sum_{k=0}^{\infty}{}_kE_x - \frac{m-1}{2m} = \ddot{a}_x - \frac{m-1}{2m}.$$

We have that

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1}{m}\sum_{k=0}^{n-1}\sum_{j=0}^{m-1} {}_{k+\frac{j}{m}}E_x = \frac{1}{m}\sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\left(v^k \cdot {}_kE_x - \frac{j}{m}\left({}_{k+1}E_x - {}_kE_x\right)\right)$$

$$=\frac{1}{m}\sum_{k=0}^{n-1}\left(v^k \cdot {}_kE_x - \frac{m-1}{2}\left({}_{k+1}E_x - {}_kE_x\right)\right)$$

$$=\frac{m+1}{2m}\sum_{k=0}^{n-1} {}_kE_x + \frac{m-1}{2m}\sum_{k=0}^{n-1} {}_{k+1}E_x$$

$$=\frac{m+1}{2m}\sum_{k=0}^{n-1} {}_kE_x + \frac{m-1}{2m}\sum_{k=1}^{n} {}_kE_x = \frac{m+1}{2m}\sum_{k=0}^{n-1} {}_kE_x$$

$$+\frac{m-1}{2m}\sum_{k=0}^{n-1} {}_kE_x - \frac{m-1}{2m} + \frac{m-1}{2m}\,{}_nE_x$$

$$=\ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x).$$

Using that $\ddot{a}_x^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} + {}_{n}|\ddot{a}_x^{(m)}$ and $\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_{n}|\ddot{a}_x$, we can get the last formula.

Example 2

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

Suppose that $i = 6.5\%$. *Calculate* $\ddot{a}_{80}^{(12)}$ *assuming that the actuarial discount factor is linear.*

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Consider the life table

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Solution: Using that $\ddot{a}_{80} = 3.011654244$, we get that

$$\ddot{a}_{80}^{(12)} = 3.011654244 - \frac{12-1}{(2)(12)} = 2.553320911.$$

In the continuous case, we assume ${}_{k+t}E_x = v^{k+t} \cdot {}_{k+t}p_x$ is linear in t, $0 \le t \le 1$. In this case

$${}_{k+t}E_x = {}_kE_x - t({}_{k+1}E_x - {}_kE_x), 0 \le t \le 1.$$

Letting $m \to \infty$ in the previous theorem, we get that:

Theorem 7

Assuming that ${}_{k+t}E_x$ is linear in t, $0 \le t \le 1$,

$$\bar{a}_x = \ddot{a}_x - \frac{1}{2},$$

$$\bar{a}_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - \frac{1}{2}(1 - {}_nE_x),$$

$${}_{n|}\bar{a}_x = {}_{n|}\ddot{a}_x - \frac{1}{2} \cdot {}_nE_x.$$

