Manual for SOA Exam MLC.

Chapter 6. Benefit premiums.
Section 6.10. Premiums found including expenses.

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Premiums found including expenses.

When finding the annual premium expenses and commissions have to be taken into in account. Possible costs are underwriting (making the policy) and maintaining the policy. The annual premium which an insurance company charges is called the **gross annual premium**. The gross annual premium is called the **contract premium** and the **loaded premium**. Usual, expenses are;

1. Issue cost.
2. Percentage of annual benefit premium.
3. Fixed amount per policy.
4. Percentage of (face value) contract amount.
5. Settlement cost.

Often the expenses related with the contract amount, are given as per thousand expenses, i.e. the per thousand expenses are the expenses made for each $1,000 of the face value of the insurance.

Theorem 1

Suppose that we have a whole life insurance on (x), with a death benefit of b paid at the end of the year of death. The fixed annual cost has an amount of e. In the the first year, there exists an additional cost of e_0^*. The percentage of the expense–augmented premium paid in expenses each year is r. During the first year, it is paid an additional percentage of the expense–augmented premium of r_0^*. The settlement cost is s. All cost except the settlement cost are paid at the beginning of the year. The insurance is funded by an expense–augmented premium of G paid at the beginning of the year while (x) is alive. If the equivalence principle is used, then

$$G = \frac{e_0^* + (b+s)A_x + e\ddot{a}_x}{(1-r)\ddot{a}_x - r_0^*}.$$

Proof.
The APV of benefits and expenses is

$$bA_x+e\ddot{a}_x+e_0^*+rG\ddot{a}_x+r_0^*G+sA_x = e_0^*+r_0^*G+(b+s)A_x+(rG+e)\ddot{a}_x.$$

The APV of benefit premiums is $G\ddot{a}_x$. Using the equivalence principle,

$$G\ddot{a}_x = e_0^* + r_0^*G + (b+s)A_x + (rG+e)\ddot{a}_x.$$

So,

$$G = \frac{e_0^* + (b+s)A_x + e\ddot{a}_x}{(1-r)\ddot{a}_x - r_0^*}.$$

□

Using that $P_x = \frac{A_x}{\ddot{a}_x}$ and $P_x + d = \frac{1}{\ddot{a}_x}$, we get that the expense–augmented annual benefit premium using the equivalence principle is

$$G = \frac{e_0^*(P_x + d) + (b + s)P_x + e}{1 - r - r_0^*(P_x + d)}.$$

Example 1

A fully discrete whole life insurance policy with face value of \$50,000 is made to (x). *The following costs are incurred:*
(i) \$800 for making the contract.
(ii) Percent of expense–loaded premium expenses are 6% in the first year and 2% thereafter.
(iii) Per thousand expenses are \$2 per year.
(iv) $P_x = 0.11$.
(iv) All expenses are paid at the beginning of the year.
(vi) $d = 5\%$.
Calculate the expected–augmented annual premium using the equivalence principle.

Solution: We have that

$$G\ddot{a}_x = (50000)A_x + 800 + (0.04)G + (0.02)G\ddot{a}_x + (2)(50)\ddot{a}_x$$

Hence,

$$G = \frac{(50000)A_x + 800 + (0.04)G + (0.02)G\ddot{a}_x + (2)(50)\ddot{a}_x}{(0.98)\ddot{a}_x - 0.04}$$

$$= \frac{(50000)P_x + 800(P_x + d) + (2)(50)}{(0.98) - (0.04)(P_x + d)}$$

$$= \frac{(50000)(0.11) + (800)(0.11 + 0.05) + (2)(50)}{(0.98) - (0.04)(0.11 + 0.05)} = 5883.319638.$$

The expense–augmented loss at issue random variable is the present value of expenses plus the present value of benefit minus the present value of premiums, i.e. it is

$$
\begin{aligned}
{}_0L_e &= e_0^* + r_0^* G + (b+s)Z_x + (rG+e)\ddot{Y}_x - G\ddot{Y}_x, \\
&= e_0^* + r_0^* G + (b+s)Z_x - ((1-r)G - e)\ddot{Y}_x.
\end{aligned}
$$

Theorem 2

Under the conditions in the previous theorem,
(i) The expense–augmented loss at issue random variable is

$$ {}_0L_e = (e_0^* + r_0^* G + b + s)(Z_x - P_x \ddot{Y}_x) = (e_0^* + r_0^* G + b + s)_0 L_x. $$

(ii) The variance of the expense–augmented loss is

$$ \mathrm{Var}({}_0L_e) = (e_0^* + r_0^* G + b + s)^2 \mathrm{Var}(L_x) $$

$$ =(e_0^* + r_0^* G + b + s)^2 \left(1 + \frac{P_x}{d}\right)^2 \mathrm{Var}(Z_x) $$

$$ =(e_0^* + r_0^* G + b + s)^2 \frac{{}^2A_x - A_x{}^2}{(1 - A_x)^2} = (e_0^* + r_0^* G + b + s)^2 \frac{{}^2A_x - A_x{}^2}{(d\ddot{a}_x)^2}. $$

Using that $1 = Z_x + d\ddot{Y}_x$, the expense–augmented loss at issue random variable is

$$\begin{aligned}
{}_0L_e &= e_0^* + r_0^* G + (b+s)Z_x - ((1-r)G - e)\ddot{Y}_x \\
&= (e_0^* + r_0^* G)(Z_x + d\ddot{Y}_x) + (b+s)Z_x - ((1-r)G - e)\ddot{Y}_x \\
&= (e_0^* + r_0^* G + b + s)Z_x - ((1-r)G - e - d(e_0^* + r_0^* G))\ddot{Y}_x \\
&= (e_0^* + r_0^* G + b + s)\left(Z_x - \frac{(1-r)G - e - d(e_0^* + r_0^* G)}{e_0^* + r_0^* G + b + s}\ddot{Y}_x\right).
\end{aligned}$$

Since G is found so that

$$0 = E[{}_0L_e] = E\left[Z_x - \frac{(1-r)G - e - d(e_0^* + r_0^* G)}{e_0^* + r_0^* G + b + s}\ddot{Y}_x\right],$$

we have that

$$P_x = \frac{(1-r)G - e - d(e_0^* + r_0^* G)}{e_0^* + r_0^* G + b + s}.$$

Hence, the expense–augmented loss at issue random variable is

$${}_0L_e = (e_0^* + r_0^* G + b + s)(Z_x - P_x\ddot{Y}_x) = (e_0^* + r_0^* G + b + s){}_0L_x.$$

When we compute the expense–augmented loss we get an expression of the type $c_1 + c_2 Z_x - c_2 \ddot{Y}_x$. The proof of the previous theorem gives that

$${}_0L_e = c_1 + c_2 Z_x - c_2 \ddot{Y}_x = (c_1 + c_2) L_x \text{ and } \mathrm{Var}({}_0L_e) = (c_1 + c_2) \mathrm{Var}(L_x).$$

Example 2

A whole life insurance policy with face value of \$40,000 payable at the end of the year of death is made to (45). The following costs are incurred:

(i) \$500 for making the contract.

(i) Percent of expense–loaded premium expenses are 5% in the first year and 1% thereafter.

(iii) Per policy expenses are \$20 per year.

(iv) Per thousand expenses are \$1.2 per year.

(v) \$600 for settlement.

All expenses, except the settlement expense, are paid at the beginning of the year.

The insurance is funded by annual gross premiums, paid at the beginning of the year.

Assume that $i = 4.5\%$ and death is modeled using the De Moivre model with terminal age 95.

(a) Calculate the gross annual premium.

Solution: (a) Using the equivalence principle,

$$G\ddot{a}_{45} = (40000)A_{45} + 500 + G(0.04) + G(0.01)\ddot{a}_{45} + 20\ddot{a}_{45} + (40)(1.2)\ddot{a}_{45} + (600)A_{45}, = (40600)A_{45} + 500 + G(0.04) + G(0.01)\ddot{a}_{45} + (68)\ddot{a}_{45}.$$

So, $G = \frac{500+(40600)A_{45}+(68)\ddot{a}_{45}}{(0.99)\ddot{a}_{45}-0.04}$. We have that

$$A_{45} = \frac{a_{\overline{50}|0.045}}{50} = 0.3952401556,$$

$$\ddot{a}_{45} = \frac{1 - A_{45}}{d} = \frac{1 - 0.3952401556}{0.045/1.045} = 14.0438675.$$

Hence,

$$G = \frac{500 + (40600)A_{45} + (68)\ddot{a}_{45}}{(0.99)\ddot{a}_{45} - 0.04} = \frac{500 + (40600)(0.3952401556) + (68)(14.0438675)}{(0.99)(14.0438675) - 0.04} = 1262.439006.$$

(b) Calculate the expected–augmented loss for an insuree that dies 7 years, 5 months and 10 days after the issue of this policy.

Solution: (b) The expense–augmented loss is

$$\begin{aligned} L_e = &(40000)Z_{45} + 500 + G(0.04) + G(0.01)\ddot{Y}_{45} \\ &+ 20\ddot{Y}_{45} + (40)(1.2)\ddot{Y}_{45} + (600)Z_{45} - G\ddot{Y}_{45}, \\ =&(40600)Z_{45} + 500 + (1262.439006)(0.04) \\ &+ (20 + 48 - (0.99)(1262.439006))\ddot{Y}_{45} \\ =&(40600)Z_{45} + 550.4975602 - 1181.814616\ddot{Y}_{45}. \end{aligned}$$

For an insuree that dies 7 years, 5 months and 10 days after the issue of this policy the expected–augmented loss is

$$L_e = (40600)v^8 + 550.4975602 - 1181.814616\ddot{a}_{\overline{8}|} = 18244.28524.$$

(c) Calculate the variance of the expense–augmented loss.

Solution: (c) The expense–augmented loss is

$$\begin{aligned} {}_0L_e &= (40600)Z_{45} + 550.4975602 - 1181.814616\ddot{Y}_{45} \\ &=(40600 + 550.4975602) \cdot {}_0L_{45} \\ &=(41150.49756) \cdot {}_0L_{45}. \end{aligned}$$

$$\begin{aligned} {}^2A_{45} &= \frac{a_{\overline{50}|0.045(2.045)}}{50} = 0.2146684865, \\ \mathrm{Var}({}_0L_e) &= (41150.49756)^2 \mathrm{Var}({}_0L_{45}) \\ &=(41150.49756)^2 \frac{0.2146684865 - (0.3952401556)^2}{(1 - 0.3952401556)^2} \\ &=270642713.1. \end{aligned}$$

Fully continuous case

Let b be the death benefit death paid at the time of the death. The fixed issue cost is e_0^*. The percentage of the expense–augmented premium paid in expenses at issue is r_0^*. There is an annual rate of contract expenses of e paid continuously while (x) is alive. The percentage of the expense–augmented premium paid continuously in expenses while (x) is alive is r. The settlement cost is s.
Let G be the expense–augmented premium rate using the equivalence principle. We have that

$$G\overline{a}_x = b\overline{A}_x + e_0^* + r_0^* G + e\overline{a}_x + rG\overline{a}_x + s\overline{A}_x$$
$$= e_0 + r_0 G + (b+s)\overline{A}_x + (rG + e)\overline{a}_x.$$

So,

$$G = \frac{e_0^* + (b+s)\overline{A}_x + e\overline{a}_x}{(1-r)\overline{a}_x - r_0^*}.$$

The expense–augmented loss at issue random variable is

$$
\begin{aligned}
{}_0\overline{L}_e &= e_0^* + r_0^* G + (b+s)\overline{Z}_x - ((1-r)G - e)\overline{Y}_x, \\
{}_0\overline{L}_e &= (e_0 + r_0 G + b + s)\overline{L}_x = (e_0 + r_0 G + b + s)(\overline{A}_x - \overline{P}_x \overline{a}_x)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Var}({}_0\overline{L}_e) = (e_0^* + r_0^* G + b + s)^2 \left(1 + \frac{\overline{P}_x}{\delta}\right)^2 \mathrm{Var}(\overline{Z}_x) \\
=&(e_0^* + r_0^* G + b + s)^2 \frac{{}^2\overline{A}_x - \overline{A}_x^2}{(1-\overline{A}_x)^2} = (e_0^* + r_0^* G + b + s)^2 \frac{{}^2\overline{A}_x - \overline{A}_x^2}{(\delta \overline{a}_x)^2}.
\end{aligned}
$$

Example 3

For a fully continuous whole life insurance of \$50,000 on (x)*:*
(i) The issuing expenses are \$1,000 and 5% of the expense–augmented annual premium rate.
(ii) The annual rate of continuous maintenance expense is 250*.*
(iii) There exists a continuous rate of expenses which is 10% of the benefit premium rate.
(iv) $\delta = 0.06$, $\overline{a}_x = 12$, $\mathrm{Var}(\overline{Z}_x) = 0.15$.

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(a) Calculate the expense–augmented annual premium rate using the equivalence principle.

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(a) Calculate the expense–augmented annual premium rate using the equivalence principle.

Solution: (a) We have that $\overline{A}_x = 1 - (0.06)(12) = 0.28$ and

$$G(12) = G\overline{a}_x = (50000)A_x + 1000 + (0.05)G + 250\overline{a}_x + (0.10)G\overline{a}_x$$
$$=(50000)(0.28) + 1000 + (0.05)G + (250)(12) + (0.10)(12)G$$
$$=18000 + 1.25G$$

and $G = \frac{18000}{12-1.25} = 1674.418605$.

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(ii) The annual rate of continuous maintenance expense is 250*.*
(iii) There exists a continuous rate of expenses which is 10% of the benefit premium rate.
(iv) $\delta = 0.06$, $\overline{a}_x = 12$, $\mathrm{Var}(\overline{Z}_x) = 0.15$.
(b) Calculate $\mathrm{Var}({}_0L_e)$.

Example 3

For a fully continuous whole life insurance of $50,000 on (x)*:*
(i) The issuing expenses are $1,000 and 5% of the expense–augmented annual premium rate.
(ii) The annual rate of continuous maintenance expense is 250*.*
(iii) There exists a continuous rate of expenses which is 10% of the benefit premium rate.
(iv) $\delta = 0.06$, $\overline{a}_x = 12$, $\mathrm{Var}(\overline{Z}_x) = 0.15$*.*
(b) Calculate $\mathrm{Var}({}_0L_e)$.

Solution: (b) The expense–augmented loss is

$$\begin{aligned}&(50000)Z_x + 1000 + (0.05)G + 250\overline{Y}_x + (0.10)G\overline{Y}_x - G\overline{Y}_x\\ =&(50000 + 1000 + (0.05)G)_0\overline{L}_x\\ =&(50000 + 1000 + (0.05)(1674.418605))\overline{L}_x = 51083.72093_0\overline{L}_x.\end{aligned}$$

Example 3

For a fully continuous whole life insurance of \$50,000 on (x):
(i) The issuing expenses are \$1,000 and 5% of the expense–augmented annual premium rate.
(ii) The annual rate of continuous maintenance expense is 250.
(iii) There exists a continuous rate of expenses which is 10% of the benefit premium rate.
(iv) $\delta = 0.06$, $\overline{a}_x = 12$, $\mathrm{Var}(\overline{Z}_x) = 0.15$.
(b) Calculate $\mathrm{Var}({}_0L_e)$.

Solution: (b) Using that $\mathrm{Var}({}_0\overline{L}_x) = \frac{{}^2\overline{A}_x - \overline{A}_x{}^2}{(\delta \overline{a}_x)^2}$, we get that

$$\mathrm{Var}({}_0\overline{L}_e) = (51083.72093)^2 \mathrm{Var}({}_0\overline{L}_x) = (51083.72093)^2 \frac{0.15}{(0.06)^2(12)^2}$$

$=755077125.$

Example 4

A 10–payment, fully discrete, 20–year term insurance policy with face value of 90000 payable at the time of death is made to (45). The following cost are incurred:
(i) 275 at the beginning of each year which the policy is active.
(ii) Per thousand expenses are $2.5 at the beginning of each year which the policy is active.
(iii) 1% for each annual premium received.
Assume that $i = 6\%$ and death follows the life table for the USA population in 2004. Find the gross annual premium using the equivalence principle.

Solution: Using the equivalence principle,

$$G\ddot{a}_{45:\overline{10}|} = (90000)A^1_{45:\overline{20}|} + 275\ddot{a}_{45:\overline{20}|} + (90)(2.5)\ddot{a}_{45:\overline{20}|}$$

$$+ G(0.01)\ddot{a}_{45:\overline{10}|} = (90000)A^1_{45:\overline{20}|} + 500\ddot{a}_{45:\overline{20}|} + G(0.01)\ddot{a}_{45:\overline{10}|},$$

$$G = \frac{(90000)A^1_{45:\overline{20}|} + 500\ddot{a}_{45:\overline{20}|}}{(0.99)\ddot{a}_{45:\overline{10}|}},$$

$$A^1_{45:\overline{20}|} = A_{45} - {}_{20}E_{45}A_{65} = 0.16656845 - (0.271632162)(0.37609614)$$

$$=0.06440864237,$$

$$\ddot{a}_{45:\overline{10}|} = \ddot{a}_{45} - {}_{10}E_{45}\ddot{a}_{55} = 14.723957 - (0.534696682)(13.160819)$$

$$=7.686910748,$$

$$\ddot{a}_{45:\overline{20}|} = \ddot{a}_{45} - {}_{20}E_{45}\ddot{a}_{65} = 14.723957 - (0.271632162)(11.022302)$$

$$=11.72994528,$$

$$G = \frac{(90000)(0.06440864237) + (500)(11.72994528)}{(0.99)(7.686910748)} = 1532.416116.$$

Example 5

For a 5–payment 20–year endowment insurance of $100,000 on (25), you are given:
(i) Percent of expense–loaded premium expenses are 10% in the first year and 2% thereafter.
(ii) Per active policy expenses are $200 in the first year and $80 in each year thereafter until death.
(iii) Expenses are paid at the beginning of each policy year.
(iv) Death benefits are payable at the end of the year of death.
(iv) The expense-loaded premium is determined using the equivalence principle.
(v) $i = 6\%$.
(vi) Mortality follows the life table for the USA population in 2004.
Calculate the expense-loaded premium using the equivalence principle.

Solution: Equating the APV of premiums and expenses, we get that

$$G\ddot{a}_{25:\overline{5}|} = (100000)\overline{A}_{25:\overline{20}|} + G(0.08) + G(0.02)\ddot{a}_{25:\overline{5}|} + 120 + 80\ddot{a}_{25:\overline{20}|}.$$

So,

$$G = \frac{(100000)A_{25:\overline{20}|} + 120 + 80\ddot{a}_{25:\overline{20}|}}{(0.98)\ddot{a}_{25:\overline{5}|} - (0.08)}.$$

We have that

$$A^1_{25:\overline{20}|} = A_{25} - {}_{20}E_{25}A_{45}$$
$$=0.065231113 - (0.302791379)(0.16656845) = 0.01479562233,$$
$$A_{25:\overline{20}|} = A^1_{25:\overline{20}|} + {}_{20}E_{25} = 0.01479562233 + 0.302791379$$
$$=0.3175870013,$$
$$\ddot{a}_{25:\overline{5}|} = \ddot{a}_{25} - {}_{5}E_{25}\ddot{a}_{30} = 16.51425 - (0.743683357)(16.212781)$$
$$=4.4570746,$$
$$\ddot{a}_{25:\overline{20}|} = \ddot{a}_{25} - {}_{20}E_{25}\ddot{a}_{45} = 16.51425 - (0.302791379)(14.723957)$$
$$=12.05596276,$$
$$G = \frac{(100000)A_{25:\overline{20}|} + 120 + 80\ddot{a}_{25:\overline{20}|}}{(0.98)\ddot{a}_{25:\overline{5}|} - (0.08)}$$
$$=\frac{(100000)(0.3175870013) + 120 + 80(12.05596276)}{(4.4570746)(0.98) - 0.08} = 7659.442515.$$

