Manual for SOA Exam MLC.

Chapter 6. Benefit premiums.
Section 6.2. Fully discrete benefit premiums.

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Fully discrete benefit premiums

In this section, we will consider the funding of insurance products paid at the end of the year of death with annual benefits premiums made at the beginning of the year. The funding is made as far as the individual is alive and the term of the insurance has not expired.

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Fully discrete whole life insurance.

Definition 1

The loss random random variable for a unit whole life insurance paid at the end of the year of death funded with an annual benefit premium at the beginning of the year while the individual is alive is denoted by L_x.

The insurance contract described in the previous definition is called a **fully discrete whole life insurance**.

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Theorem 1

For a fully discrete whole insurance,
(i) The loss random variable is

$$L_x = v^{K_x} - P\ddot{a}_{\overline{K_x|}} = Z_x - P\ddot{Y}_x$$
$$= Z_x - \frac{1 - Z_x}{d} = Z_x\left(1 + \frac{P}{d}\right) - \frac{P}{d}.$$

(ii) The actuarial present value of the loss L_x is

$$E[L_x] = A_x - P\ddot{a}_x = A_x\left(1 + \frac{P}{d}\right) - \frac{P}{d}.$$

(iii) The variance of the loss L_x is

$$\mathrm{Var}(L_x) = \left(1 + \frac{P}{d}\right)^2 \mathrm{Var}(Z_x) = \left(1 + \frac{P}{d}\right)^2 \left({}^2A_x - {A_x}^2\right).$$

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The loss L_x is a random variable. The loss is positive for some insurees and negative for another ones. The loss random variable L_x is a discrete random variable taking the values $v^k - P\ddot{a}_{\overline{k}|i}$, $k = 1, \dots, \omega - x$. Since $L_x = v^{K_x}\left(1 + \frac{P}{d}\right) - \frac{P}{d}$, the loss is decreasing with K_x. The biggest loss is attained when $K_x = 1$, and it is $v^1 - P\ddot{a}_{\overline{1}|} = v - P$. The smallest loss is attained when $K_x = \omega - x$, and it is $v^{\omega-x} - P\ddot{a}_{\overline{\omega-x}|}$. Hence,

$$v^{\omega-x} - P\ddot{a}_{\overline{\omega-x}|} \le L_x \le v - P.$$

To make a profit, an insurer would like that insurees will die as late as possible.

To find the probability that the loss is positive, we manipulate

$$\mathbb{P}\{L_x > 0\} = \mathbb{P}\{v^{K_x}\left(1 + \frac{P}{d}\right) - \frac{P}{d} > 0\}$$

to get $\mathbb{P}\{K_x \le k_0\} = {}_{k_0}q_x$.

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Example 1

Consider a fully discrete whole life insurance with face value 10000. The annual benefit premium paid at the beginning of each year which the insuree is alive is \$46. *Suppose that the force of mortality is* 0.005. $\delta = 0.075$

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Example 1

Consider a fully discrete whole life insurance with face value 10000. The annual benefit premium paid at the beginning of each year which the insuree is alive is \$46. *Suppose that the force of mortality is* 0.005. $\delta = 0.075$

(i) John entered this contract and died 10 years, 5 months and 5 days after the issue of this contract. Find the insurer's loss at the time of the issue of the policy.

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Example 1

Consider a fully discrete whole life insurance with face value 10000. The annual benefit premium paid at the beginning of each year which the insuree is alive is \$46. *Suppose that the force of mortality is* 0.005. $\delta = 0.075$

(i) John entered this contract and died 10 years, 5 months and 5 days after the issue of this contract. Find the insurer's loss at the time of the issue of the policy.

Solution: (i) John made annual benefit premiums of \$46 at times $0, 1, \dots, 10$ years. John's state received a death benefit of \$10000 at time 11 years. The insurer's loss at issue time is

$$(10000)e^{-(11)(0.075)} - (46)\ddot{a}_{\overline{11}|e^{0.075}-1} = 4024.718633.$$

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Example 1

Consider a fully discrete whole life insurance with face value 10000. The annual benefit premium paid at the beginning of each year which the insuree is alive is \$46. *Suppose that the force of mortality is* 0.005. $\delta = 0.075$

(ii) Peter entered this contract and died 42 years, 2 months and 20 days after the issue of this contract. Find the insurer's loss at the time of the issue of the policy.

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Example 1

Consider a fully discrete whole life insurance with face value 10000. The annual benefit premium paid at the beginning of each year which the insuree is alive is \$46. *Suppose that the force of mortality is* 0.005. $\delta = 0.075$

(ii) Peter entered this contract and died 42 years, 2 months and 20 days after the issue of this contract. Find the insurer's loss at the time of the issue of the policy.

Solution: (ii) Peter made annual benefit premiums of \$46 at times $0, 1, \dots, 42$ years. Peter's state received a death benefit of \$10000 at time 43 years. The insurer's loss at issue time is

$$(10000)e^{-(43)(0.075)} - (46)\ddot{a}_{\overline{43}|e^{0.075}-1} = -213.7536329.$$

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Example 1

Consider a fully discrete whole life insurance with face value 10000. The annual benefit premium paid at the beginning of each year which the insuree is alive is \$46. *Suppose that the force of mortality is* 0.005. $\delta = 0.075$

(iii) Calculate the probability that the loss at issue is positive.

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Example 1

Consider a fully discrete whole life insurance with face value 10000. The annual benefit premium paid at the beginning of each year which the insuree is alive is \$46. *Suppose that the force of mortality is* 0.005. $\delta = 0.075$

(iii) Calculate the probability that the loss at issue is positive.

Solution: (iii) The loss at issue random variable is

$$10000L_x = (10000)v^{K_x} - (46)\ddot{a}_{\overline{K_x}|} = (10000)v^{K_x} - (46)\frac{1 - v^{K_x}}{d}$$

$$= v^{K_x}\left(10000 + \frac{46}{1 - e^{-0.075}}\right) - \frac{46}{1 - e^{-0.075}}$$

$$= (10636.6208064)e^{-(0.075)K_x} - 636.6208064.$$

We need to find $\mathbb{P}\{10000L_x > 0\}$.

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$$10000L_x = (10636.6208064)e^{-(0.075)K_x} - 636.6208064 > 0$$

is equivalent to $e^{-(0.075)K_x} > \frac{636.6208064}{10636.6208064} = 0.05985179109$ and to

$$K_x < -\frac{\ln(0.05985179109)}{0.075} = 37.54511895.$$

The probability that the loss is positive is

$$\mathbb{P}\{K_x \le 37\} = \mathbb{P}\{T_x \le 37\} = 1 - e^{-(37)(0.005)} = 0.1688957161.$$

Notice that if $K_x = 37$, the loss is

$$(10000)e^{-(37)(0.075)} - (46)\ddot{a}_{\overline{37|}e^{0.075}-1} = 26.56693464.$$

If $K_x = 38$, the loss is

$$(10000)e^{-(38)(0.075)} - (46)\ddot{a}_{\overline{38|}e^{0.075}-1} = -21.35269944.$$

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Definition 2

The benefit premium of a fully discrete whole insurance funded under the equivalence principle is denoted by P_x.

If a whole insurance is funded under the equivalence principle, then $E[L_x] = 0$. P_x is the annual benefit premium at which the insurer expects to break even.

Theorem 2

If a fully discrete whole insurance is funded using the equivalence principle, then

$$P_x = \frac{A_x}{\ddot{a}_x} = \frac{dA_x}{1 - A_x} = \frac{1}{\ddot{a}_x} - d$$

and

$$\mathrm{Var}(L_x) = \frac{{}^2A_x - {A_x}^2}{(1 - A_x)^2} = \frac{{}^2A_x - {A_x}^2}{(d\ddot{a}_x)^2}.$$

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Theorem 3

If a fully discrete whole insurance is funded using the equivalence principle, then

$$P_x = \frac{A_x}{\ddot{a}_x} = \frac{dA_x}{1 - A_x} = \frac{1}{\ddot{a}_x} - d$$

and

$$\mathrm{Var}(L_x) = \frac{{}^2A_x - A_x{}^2}{(1 - A_x)^2} = \frac{{}^2A_x - A_x{}^2}{(d\ddot{a}_x)^2}.$$

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Theorem 3

If a fully discrete whole insurance is funded using the equivalence principle, then

$$P_x = \frac{A_x}{\ddot{a}_x} = \frac{dA_x}{1 - A_x} = \frac{1}{\ddot{a}_x} - d$$

and

$$\mathrm{Var}(L_x) = \frac{{}^2A_x - {A_x}^2}{(1 - A_x)^2} = \frac{{}^2A_x - {A_x}^2}{(d\ddot{a}_x)^2}.$$

Proof: Since $0 = E[L_x] = A_x - P\ddot{a}_x$, we have that $P_x = \frac{A_x}{\ddot{a}_x}$. Using that $\ddot{a}_x = \frac{1-A_x}{d}$, we get that

$$P_x = \frac{dA_x}{1 - A_x} = \frac{1}{\ddot{a}_x} - d.$$

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Theorem 3

If a fully discrete whole insurance is funded using the equivalence principle, then

$$P_x = \frac{A_x}{\ddot{a}_x} = \frac{dA_x}{1 - A_x} = \frac{1}{\ddot{a}_x} - d$$

and

$$\mathrm{Var}(L_x) = \frac{{}^2A_x - {A_x}^2}{(1 - A_x)^2} = \frac{{}^2A_x - {A_x}^2}{(d\ddot{a}_x)^2}.$$

Proof: We have that

$$1 + \frac{P_x}{d} = 1 + \frac{\frac{dA_x}{1-A_x}}{d} = \frac{1}{1 - A_x} = \frac{1}{d\ddot{a}_x}.$$

Hence,

$$\mathrm{Var}(L_x) = \left(1 + \frac{P_x}{d}\right)^2 \left({}^2A_x - {A_x}^2\right) = \frac{{}^2A_x - {A_x}^2}{(1 - A_x)^2} = \frac{{}^2A_x - {A_x}^2}{(d\ddot{a}_x)^2}.$$

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Example 2

Consider the life table

x	80	81	82	83	84	85	86
ℓ_x	250	217	161	107	62	28	0

An 80–year old individual signs a whole life policy insurance which will pay \$50000 at the end of the year of his death. The insuree will make level benefit premiums at the beginning of the year while he is alive. Suppose that $i = 6.5\%$.

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(i) Find the net single premium for this policy.

Solution: (i)

$$\begin{aligned} A_{80} =& (1.065)^{-1}\frac{250-217}{250} + (1.065)^{-2}\frac{217-161}{250} \\ &+ (1.065)^{-3}\frac{161-107}{250} + (1.065)^{-4}\frac{107-62}{250} \\ &+ (1.065)^{-5}\frac{62-28}{250} + (1.065)^{-6}\frac{28}{250} \\ =& 0.8161901166. \end{aligned}$$

The net single premium for this policy is

$$(50000)A_{80} = (50000)(0.8161901166) = 40809.50583.$$

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(ii) Find the benefit annual premium for this policy using the equivalence principle.

Solution: (ii) We have that

$$\ddot{a}_{80} = \frac{1 - 0.8161901166}{0.65/1.065} = 3.011654244,$$

$$P_{80} = \frac{A_{80}}{\ddot{a}_{80}} = \frac{0.8161901166}{3.011654244} = 0.2710105645.$$

The benefit annual premium for this policy using the equivalence principle is

$$(50000)P_{80} = (50000)(0.2710105645) = 13550.52822.$$

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(iii) Suppose that 250 80–year old individuals enter this insurance contract and they die according with the deterministic group. Suppose that the insurer makes a fund to paid for death benefits. Benefit annual premiums are deposited in this fund. Let BO_k be the balance in this fund at time k before making depostis/withdrwals of death benefits/benefit premiums. Let $\ell_{x+k}P_{80}$ be the total amount of annual benefit premiums received at time k. Let $d_{x+k-1}B$ be the total amount of death benefits paid at time k. Let B_k be the balance in this account at time k after making the deposits/withdrawals for death benefits/benefit premiums. Make a table with BO_k, $\ell_{x+k}P_{80}$, $d_{x+k-1}B$ and B_k, for $k = 0, 1, 2, \ldots, 6$.

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Solution: (iii) We have that $BO_0 = 0$ and $BO_k = (1+i)B_{k-1}$, for $k = 1, 2, \dots, 6$. We have that $B_k = BO_k + \ell_{x+k}P_{80} - d_{x+k-1}B$. We have that

k	BO_k	$\ell_{x+k}P$	$d_{x+k-1}B$	B_k
0	0	3387632.055	0	3387632.055
1	3607828.139	2940464.624	1650000	4898292.762
2	5216681.792	2181635.043	2800000	4598316.835
3	4897207.430	1449906.520	2700000	3647113.949
4	3884176.3558	840132.7496	2250000	2474309.1055
5	2635139.1973	379414.7902	1700000	1314553.9875
6	1399999.997	0	1400000	-0.003

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(iv) Find the probability mass function of the random loss when the benefit premium follows the equivalence principle.

Solution: (iv) The loss is

$$\begin{aligned} L_{80} &= (50000)Z_{80} - (13550.52822)\ddot{Y}_{80} \\ &=(50000)v^{K_{80}} - (13550.52822)\frac{1-v^{K_{80}}}{0.065/1.065} \\ &=(272020.1931)v^{K_{80}} - 222020.1931. \end{aligned}$$

We have that

$$\mathbb{P}\{L_{80} = (272020.1931)v^k - 222020.1931\} = \frac{\ell_{80+k-1} - \ell_{80+k}}{\ell_{80}},$$

$k = 1, 2, \dots, 6.$

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In particular,

$$\mathbb{P}\{L_{80} = 33397.8286\} = \frac{250 - 217}{250} = \frac{33}{250},$$

$$\mathbb{P}\{L_{80} = 3171.4767\} = \frac{217 - 161}{250} = \frac{56}{250},$$

$$\mathbb{P}\{L_{80} = 17808.9352\} = \frac{161 - 107}{250} = \frac{54}{250},$$

$$\mathbb{P}\{L_{80} = -10572.6158\} = \frac{107 - 62}{250} = \frac{45}{250},$$

$$\mathbb{P}\{L_{80} = -23477.8670\} = \frac{62 - 28}{250} = \frac{34}{250},$$

$$\mathbb{P}\{L_{80} = -35595.4738\} = \frac{28 - 0}{250} = \frac{28}{250}.$$

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(v) Find the probability that the loss is positive.

Solution: (v) The probability that the loss is positive is

$$\frac{250 - 107}{250} = 0.572.$$

Alternatively, we have that

$$\begin{aligned}
&\mathbb{P}\{(272020.1931)v^{K_{80}} - 222020.1931 > 0\} \\
=&\mathbb{P}\{K_{80} < \frac{\ln(222020.1931/272020.1931)}{\ln(1.065)}\} \\
=&\mathbb{P}\{K_{80} < 3.225226089\} = \mathbb{P}\{K_{80} \leq 3\} = \frac{250 - 107}{250} = 0.572.
\end{aligned}$$

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(vi) Find the variance of the loss.

Solution: (vi) We got that $A_{80} = 0.8161901166$. We have that

$$\begin{aligned}{}^2A_{80} =& (1.065)^{-2}\frac{250-217}{250} + (1.065)^{-4}\frac{217-161}{250} \\ &+ (1.065)^{-6}\frac{161-107}{250} + (1.065)^{-8}\frac{107-62}{250} \\ &+ (1.065)^{-10}\frac{62-28}{250} + (1.065)^{-12}\frac{28}{250} \\ =&0.6723484016.\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{Var}(L_{80}) =& (50000)^2 \ \frac{{}^2A_{80} - (A_{80})^2}{(1-A_{80})^2} \\ =&(50000)^2\frac{0.6723484016 - (0.8161901166)^2}{(1-(0.8161901166))^2} = 457444042.2.\end{aligned}$$

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Theorem 4

Under constant force of mortality μ for life insurance funded through the equivalence principle, $P_x = vq_x$.

Proof: Since $A_x = \frac{q_x}{q_x+i}$ and $\ddot{a}_x = \frac{1+i}{q_x+i}$, $P_x = \frac{\frac{q_x}{q_x+i}}{\frac{1+i}{q_x+i}} = vq_x$.

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Example 3

Maria is 30 years old and purchases a whole life insurance policy with face value of 70000 payable at the end of the year of death. This policy will be paid by a level benefit annual premium at the beginning of each year while Maria is alive. Assume that $i = 6\%$ *and death is modeled using the constant force of mortality* $\mu = 0.02$.

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Example 3

Maria is 30 years old and purchases a whole life insurance policy with face value of 70000 payable at the end of the year of death. This policy will be paid by a level benefit annual premium at the beginning of each year while Maria is alive. Assume that $i = 6\%$ *and death is modeled using the constant force of mortality* $\mu = 0.02$*.*

(i) Find the net single premium for this policy.

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Example 3

Maria is 30 years old and purchases a whole life insurance policy with face value of 70000 payable at the end of the year of death. This policy will be paid by a level benefit annual premium at the beginning of each year while Maria is alive. Assume that $i = 6\%$ and death is modeled using the constant force of mortality $\mu = 0.02$.

(i) Find the net single premium for this policy.

Solution: (i)

$$A_{30} = \frac{q_x}{q_x + i} = \frac{1 - e^{-0.02}}{1 - e^{-0.02} + 0.06} = 0.2481328007.$$

and the net single premium is $(70000)(0.2481328007) = 17369.29605$.

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Example 3

Maria is 30 years old and purchases a whole life insurance policy with face value of 70000 payable at the end of the year of death. This policy will be paid by a level benefit annual premium at the beginning of each year while Maria is alive. Assume that $i = 6\%$ *and death is modeled using the constant force of mortality* $\mu = 0.02$.

(ii) Find the benefit annual premium for this policy.

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Example 3

Maria is 30 years old and purchases a whole life insurance policy with face value of 70000 payable at the end of the year of death. This policy will be paid by a level benefit annual premium at the beginning of each year while Maria is alive. Assume that $i = 6\%$ *and death is modeled using the constant force of mortality* $\mu = 0.02$.

(ii) Find the benefit annual premium for this policy.

Solution: (ii) We have that

$$P_x = vq_x = (1.06)^{-1}(1 - e^{-0.02}) = 0.01868049688.$$

The benefit annual premium for this policy is $(70000)(0.01868049688) = 1307.634782$.

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Example 3

Maria is 30 years old and purchases a whole life insurance policy with face value of 70000 payable at the end of the year of death. This policy will be paid by a level benefit annual premium at the beginning of each year while Maria is alive. Assume that $i = 6\%$ *and death is modeled using the constant force of mortality* $\mu = 0.02$.

(iii) Find the variance of the present value of the loss for this insurance contract.

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Example 3

Maria is 30 years old and purchases a whole life insurance policy with face value of 70000 payable at the end of the year of death. This policy will be paid by a level benefit annual premium at the beginning of each year while Maria is alive. Assume that $i = 6\%$ *and death is modeled using the constant force of mortality* $\mu = 0.02$.

(iii) Find the variance of the present value of the loss for this insurance contract.

Solution: (iii) We have that

$$ {}^2A_{30} = \frac{1 - e^{-0.02}}{1 - e^{-0.02} + (1.06)^2 - 1} = 0.138083288, $$

$$ \mathrm{Var}(L_x) = (70000)^2 \frac{{}^2A_x - A_x^2}{(1 - A_x)^2} $$

$$ =(70000)^2 \frac{0.138083288 - (0.2481328007)^2}{(1 - 0.2481328007)^2} = 663210373. $$

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Besides the equivalence principle, there are other ways to determine annual benefit premiums. Unless said otherwise, we will assume that the equivalence principle is used. An insurer expects to break even if it uses the equivalence principle. However, an insurer should be compensated for facing losses. Often, the annual benefit premium in an insurance contract is bigger than the annual benefit premium obtained using the equivalence principle. The **risk charge** (or **security loading**) is the excess of the benefit annual premium over the benefit annual premium found using the equivalence principle.

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The 100α–th percentile annual premium for a fully discrete whole life insurance to (x) is the largest P_α such that

$$\alpha \geq \mathbb{P}\{Z_x - P_\alpha \ddot{Y}_x > 0\} = \mathbb{P}\{v^{K_x} - P_\alpha \ddot{a}_{\overline{K_x}|} > 0\} = \mathbb{P}\left\{\frac{1}{\ddot{s}_{\overline{K_x}|}} > P_\alpha\right\}.$$

P_α is a $(1-\alpha)$–th quantile of $\frac{1}{\ddot{s}_{\overline{K_x}|}}$. $P_\alpha = \frac{1}{\ddot{s}_{\overline{k_\alpha}|}}$, where k_α satisfies that

$$\mathbb{P}\{K_x < k_\alpha\} \leq \alpha < \mathbb{P}\{K_x < k_\alpha + 1\}.$$

Then, $P_\alpha = \frac{1}{\ddot{s}_{\overline{k_\alpha}|}}$ satisfies that

$$\mathbb{P}\{Z_x - P_\alpha \ddot{Y}_x > 0\} = \mathbb{P}\left\{\frac{1}{\ddot{s}_{\overline{K_x}|}} > \frac{1}{\ddot{s}_{\overline{k_\alpha}|}}\right\} = \mathbb{P}\{K_x < k_\alpha\} \leq \alpha$$

and for $P > P_\alpha$,

$$\mathbb{P}\{Z_x - P\ddot{Y}_x > 0\} = \mathbb{P}\left\{\frac{1}{\ddot{s}_{\overline{K_x}|}} > P\right\} \geq \mathbb{P}\left\{\frac{1}{\ddot{s}_{\overline{K_x}|}} \geq \frac{1}{\ddot{s}_{\overline{k_\alpha}|}}\right\}$$

$$= \mathbb{P}\{K_x \leq k_\alpha\} > \alpha.$$

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Example 4

Michael is 50 year old and purchases a whole life insurance policy with face value of 100000 payable at the end of the year of death. This policy will be paid by a level benefit annual premium at the beginning of each year while Michael is alive. Assume that $i = 6\%$ *and death is modeled using De Moivre's model with terminal age 100.*

(i) Calculate the benefit annual premium if the probability of a loss is at most 0.25.

(ii) Calculate the benefit annual premium using the equivalence principle.

(iii) Calculate the probability of a loss is positive for the benefit annual premium that uses the equivalence principle.

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Solution: (i) If

$$0.25 = \mathbb{P}\{T_{50} \leq t\} = \frac{t}{50}$$

then $t = 12.5$. Hence,

$$\mathbb{P}\{K_{50} < 13\} = \mathbb{P}\{K_{50} \leq 12\} < 0.25 < \mathbb{P}\{K_{50} \leq 13\} = \mathbb{P}\{K_{50} < 14\}.$$

We take $k_{0.25} = 13$ and $P_{0.25} = \frac{1}{\ddot{s}_{\overline{13}|0.06}} = 0.04996236353$. The benefit annual premium is
$(100000)(0.04996236353) = 4996.236353.$

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Solution: (ii)

$$A_{50} = \frac{a_{\overline{50}|0.06}}{50} = 0.3152372127,$$

$$P_{50} = \frac{dA_{50}}{1 - A_{50}} = \frac{(0.06/1.06)(0.3152372127)}{1 - 0.3152372127} = 0.02605809799.$$

The benefit annual premium is
$(100000)(0.02605809799) = 2605.809799.$

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Solution: (iii) The probability that the loss random variable is positive is

$$\mathbb{P}\left\{v^{K_{50}}\left(1+\frac{P_{50}}{d}\right)-\frac{P_{50}}{d}>0\right\}=\mathbb{P}\left\{v^{K_{50}}>\frac{P_{50}}{d+P_{50}}\right\}$$

$$=\mathbb{P}\left\{v^{K_{50}}>\frac{0.02605809799}{(0.06/1.06)+0.02605809799}\right\}$$

$$=\mathbb{P}\left\{v^{K_{50}}>0.3152372127\right\}=\mathbb{P}\left\{K_{50}<-\frac{\ln(0.3152372127)}{\ln(1.06)}\right\}$$

$$=\mathbb{P}\{K_{50}<19.81210743\}=\mathbb{P}\{K_{50}\le 19\}=\frac{19}{50}=0.38.$$

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Next, we discuss the percentile annual premium for an aggregate of policies. Suppose that an insurer offers a whole life insurance of B to n lives aged x paid at the end of the year of death. Each of these live insurances is funded by a level annual benefit premium of P paid at the beginning of the year. The loss for a policy is

$$BL_x = BZ_x - P\ddot{Y}_x = Bv^{K_x} - P\ddot{a}_{\overline{K_x}|} = BZ_x - P\frac{1-Z_x}{d}$$
$$= Z_x\left(B + \frac{P}{d}\right) - \frac{P}{d}.$$

Hence,

$$E[BL_x] = BA_x - P\ddot{a}_x \text{ and } \mathrm{Var}(BL_x) = \left(B + \frac{P}{d}\right)^2 \mathrm{Var}(Z_x).$$

Let $BL_{x,1}, \dots, BL_{x,n}$ be the losses for these n insurees. The aggregate loss is $L_{\mathrm{Agg}} = \sum_{j=1}^{n} BL_{x,1}$. Hence,

$$E[L_{\mathrm{Agg}}] = n(BA_x - P\ddot{a}_x) \text{ and } \mathrm{Var}(L_{\mathrm{Agg}}) = n\left(B + \frac{P}{d}\right)^2 \mathrm{Var}(Z_x).$$

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Suppose that P is chosen so that the aggregate loss is positive with probability α, where α is small. Then, zero is a $(1-\alpha)$–th quantile of L_{Agg} and

$$0 = E[L_{\mathrm{Agg}}] + z_{1-\alpha}\sqrt{\mathrm{Var}(L_{\mathrm{Agg}})}$$
$$= n(BA_x - P\ddot{a}_x) + z_{1-\alpha}\sqrt{n\mathrm{Var}(Z_x)}\left(B + \frac{P}{d}\right).$$

From this linear equation in P, we can find the aggregate percentile annual premium P.

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Example 5

An insurance company company offers a whole life insurance to lives aged 20 *paying 75000 at the end of the year of death. Each insuree will make an annual premium of P at the beginning of year while he is alive to fund this insurance. Suppose that* 1000 *policyholders enter this insurance product. The insurance company sets a fund with the proceeds earning a rate of* 6% *to pay future benefit claims. Use $i = 6\%$ and the life table for the USA population in 2004 to calculate P so that the probability that the aggregate loss is positive is less than or equal to* 0.99.

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We have that $A_x = 0.05246$ and ${}^2A_x = 0.01078$. Hence,

$$\ddot{a}_x = \frac{1 - 0.05246}{0.06/1.06} = 16.73987333,$$
$$\mathrm{Var}(Z_x) = 0.01078 - (0.05246)^2 = 0.0080279484,$$

Each loss is

$$(75000)L_x = (75000)Z_x - P\ddot{Y}_x = (75000)Z_x - P\frac{1 - Z_x}{d}$$
$$= Z_x\left(75000 + \frac{1.06P}{0.06}\right) - \frac{1.06P}{0.06}.$$

We have that

$$E[(75000)L_x] = (75000)A_x - P\ddot{a}_x = 3934.5 - 16.73987333P,$$
$$\mathrm{Var}((75000)L_x) = \left(75000 + \frac{1.06P}{0.06}\right)^2 (0.0080279484).$$

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For the aggregate model,

$$E\left[\sum_{j=1}^{1000}(75000)L_x\right] = 3934500 - 16739.87333P,$$

$$\mathrm{Var}\left(\sum_{j=1}^{1000}(75000)L_x\right) = (1000)\left(75000 + \frac{1.06P}{0.06}\right)^2 (0.0080279484)$$

$$=(8.0279484)\left(75000 + \frac{1.06P}{0.06}\right)^2.$$

We have that

$$\begin{aligned} 0 =& 3934500 - 16739.87333P \\ &+ (2.3263479)\sqrt{8.0279484}\left(75000 + \frac{1.06P}{0.06}\right) \\ =& 4428854.105 - 16623.42547P \end{aligned}$$

Hence, $P = \frac{4428854.105}{16623.42547} = 266.4224719.$

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Example 6

An insurer offers a fully discrete whole life insurances of 10000 on independent lives age 30, *you are given:*
(i) $i = 0.06$
(ii) Mortality follows the life table for USA population in 2004.
(iii) The annual contract premium for each policy is $1.25P_x$.
Using the normal approximation, calculate the minimum number of policies the insurer must issue so that the probability that the aggregate loss for the issued policies is less than 0.05.

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Solution: Let π be the annual contract premium. We have that

$$\pi = (1.25)(10000)\frac{A_x}{\ddot{a}_x} = (1.25)(10000)\frac{0.082295}{16.213} = 63.44831308,$$

$$E[(10000)L_x] = (10000)A_x - \pi\ddot{a}_x$$

$$=(10000)(0.082295) - (63.44831308)(16.213) = -205.7375,$$

$$E[L_{\mathrm{Aggreg}}] = -205.7375n,$$

$$\mathrm{Var}((10000)L_x) = \left(10000 + \frac{\pi}{d}\right)^2 ({}^2A_x - A_x^2)$$

$$= \left(10000 + \frac{63.44831308}{6/106}\right)^2 (0.017969 - (0.082295)^2) = 1384729.716,$$

$$\mathrm{Var}(L_{\mathrm{Aggreg}}) = n(1384729.716)$$

We have that $0 = -205.7375n + (1.6449)\sqrt{n(1384729.716)}$. So,

$$n = \frac{(1.6449)^2(1384729.716)}{(205.7375)^2} = 88.51504549$$

Since n is a positive integer, $n = 89$.

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Suppose that the funding scheme is limited to the first t years. The present value of the loss is

$$L = v^{K_x} - P\ddot{a}_{\overline{\min(K_x,t)}|} = Z_x - P\ddot{Y}_{x:\overline{t}|}.$$

The present value of the loss is

$$A_x - P\ddot{a}_{x:\overline{t}|}.$$

Definition 3

The benefit premium for a fully discrete whole life insurance funded for the first t years that satisfies the equivalence principle is denoted by

$${}_tP_x = \frac{A_x}{\ddot{a}_{x:\overline{t}|}}.$$

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Example 7

Ethan is 30 years old and purchases a whole life insurance policy with face value of 50000 payable at the end of the year of death. This policy will be paid by a level benefit annual premium at the beginning of the next 30 years while Ethan is alive. Assume that $\delta = 0.05$ *and death is modeled using the constant force of mortality* $\mu = 0.03$*. Find the benefit annual premium for this policy.*

Solution: We have that

$$A_{30} = \frac{q_x}{q_x + i} = \frac{1 - e^{-0.03}}{1 - e^{-0.03} + e^{0.05} - 1} = 0.365657416,$$

$$\ddot{a}_{30:\overline{30}|} = \frac{1 - e^{-(30)(0.08)}}{1 - e^{-0.08}} = (11.82672784)$$

The benefit annual premium for this policy is

$${}_{30}P_{30} = \frac{(50000)(0.365657416)}{11.82672784} = 1545.89427.$$

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n–year term insurance.

An n–year term insurance paid at the end of the year of death funded at the beginning of the year while the insuree is alive is a called a fully discrete n–year term insurance.

Definition 4

The loss random variable for a fully discrete n–year term insurance is denoted by $L^1_{x:\overline{n}|}$.

Theorem 5

$$L^1_{x:\overline{n}|} = v^{K_x} I(K_x \le n) - P\ddot{a}_{\overline{\min(K_x,n)}|} = Z^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|}$$

$$= Z^1_{x:\overline{n}|} - P\frac{1 - Z_{x:\overline{n}|}}{d},$$

$$E[L^1_{x:\overline{n}|}] = A^1_{x:\overline{n}|} - P\ddot{a}_{x:\overline{n}|}.$$

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We got that

$$L^1_{x:\overline{n}|} = Z^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|}.$$

Definition 5

The benefit premium for a fully discrete n–year term insurance obtained using the equivalence principle is denoted by $P^1_{x:\overline{n}|}$ *and by* $P(A^1_{x:\overline{n}|})$.

We have that

$$P^1_{x:\overline{n}|} = P(A^1_{x:\overline{n}|}) = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

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Theorem 6

We have that

$$\mathrm{Var}(L^1_{x:\overline{n}|}) = \left(1 + \frac{P}{d}\right)^2 \cdot {}^2A^1_{x:\overline{n}|} + \frac{P^2}{d^2} \cdot {}^2A_{x:\overset{1}{\overline{n}|}} - \left(E[L^1_{x:\overline{n}|}] + \frac{P}{d}\right)^2.$$

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Proof: Using that $Z_{x:\overline{n}|} = Z^1_{x:\overline{n}|} + Z_{x:\overset{1}{\overline{n}|}}$, we get that

$$L^1_{x:\overline{n}|} = Z^1_{x:\overline{n}|} - P\frac{1 - Z_{x:\overline{n}|}}{d} = -\frac{P}{d} + \left(1 + \frac{P}{d}\right) Z^1_{x:\overline{n}|} + \frac{P}{d} Z_{x:\overset{1}{\overline{n}|}}.$$

Since $Z^1_{x:\overline{n}|} Z_{x:\overset{1}{\overline{n}|}} = v^{K_x} I(K_x \le n) v^n I(n < K_x) = 0$,

$$E\left[\left(E[L^1_{x:\overline{n}|}] + \frac{P}{d}\right)^2\right] = E\left[\left(\left(1 + \frac{P}{d}\right) Z^1_{x:\overline{n}|} + \frac{P}{d} Z_{x:\overset{1}{\overline{n}|}}\right)^2\right]$$

$$= \left(1 + \frac{P}{d}\right)^2 \cdot {}^2A^1_{x:\overline{n}|} + \frac{P^2}{d^2} \cdot {}^2A_{x:\overset{1}{\overline{n}|}},$$

$$\mathrm{Var}(L^1_{x:\overline{n}|}) = \mathrm{Var}\left(L^1_{x:\overline{n}|}] + \frac{P}{d}\right)$$

$$= E\left[\left(E[L^1_{x:\overline{n}|}] + \frac{P}{d}\right)^2\right] - \left(E[L^1_{x:\overline{n}|}] + \frac{P}{d}\right)^2$$

$$= \left(1 + \frac{P}{d}\right)^2 \cdot {}^2A^1_{x:\overline{n}|} + \frac{P^2}{d^2} \cdot {}^2A_{x:\overset{1}{\overline{n}|}} - \left(E[L^1_{x:\overline{n}|}] + \frac{P}{d}\right)^2.$$

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Example 8

An insurer offers a four–year life insurance of 25 years old. Mortality is given by the table:

x	25	26	27	28
q_x	0.01	0.02	0.03	0.04

This life insurance is funded by benefit payments at the beginning of the year. The benefit payment is 10000. $i = 5\%$. *Calculate the annual benefit premium using the equivalence principle.*

Solution: We have to find $(10000)P^1_{25:\overline{4}|} = (10000)\frac{A^1_{25:\overline{4}|}}{\ddot{a}_{25:\overline{4}|}}$.

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Solution: We have that

$$A^1_{25:\overline{4}|} = (0.01)(1.05)^{-1} + (0.99)(0.02)(1.05)^{-2}$$
$$+ (0.99)(0.98)(0.03)(1.05)^{-3} + (0.99)(0.98)(0.97)(0.04)(1.05)^{-4}$$
$$=0.08359546485,$$
$$A_{25:\overset{1}{\overline{4}|}} = (0.99)(0.98)(0.97)(0.96)(1.05)^{-4} = 0.7432707483,$$
$$A_{25:\overline{4}|} = 0.08359546485 + 0.7432707483 = 0.8268662132,$$
$$\ddot{a}_{25:\overline{4}|} = \frac{1 - 0.8268662132}{0.05/1.05} = 3.635809523,$$
$$(10000)P^1_{25:\overline{4}|} = (10000)\frac{0.08359546485}{3.635809523} = 229.9225641.$$

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The present value of the loss for a t–year funded n–year term insurance is

$$v^{K_x} I(K_x \le n) - P\ddot{a}_{\overline{\min(K_x,t)}|} = Z^1_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{t}|}.$$

The actuarial present value of the loss for a t–year funded n–year term insurance is

$$A^1_{x:\overline{n}|} - P\ddot{a}_{x:\overline{t}|}.$$

The benefit premium which satisfies the equivalence principle is

$${}_tP^1_{x:\overline{n}|} = P({}_tA^1_{x:\overline{n}|}) = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{t}|}}.$$

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Example 9

A 20–year term life insurance policy to (x) with face value of 10000 payable at the end of the year of death is funded by a level benefit annual premium at the beginning of the next 10 years while (x) is alive. Assume that $\delta = 0.05$ and the constant force of mortality is $\mu = 0.02$. Find the benefit annual premium for this policy.

Solution: We have that

$$A^1_{x:\overline{20}|} = (1 - e^{-(20)(\delta+\mu)})\frac{q_x}{q_x + i}$$

$$=(1 - e^{-(20)(0.05+0.02)})\frac{1 - e^{-0.02}}{1 - e^{-0.02} + e^{0.05} - 1} = 0.2099039122,$$

$$\ddot{a}_{x:\overline{10}|} = (1 - e^{-(10)(0.05+0.02)})\frac{1}{1 - e^{-(0.05+0.02)}} = 7.446282211,$$

$$P = (10000) \cdot {}_{10}P^1_{x:\overline{20}|} = (10000)\frac{A^1_{x:\overline{20}|}}{\ddot{a}_{x:\overline{10}|}} = (10000)\frac{0.2099039122}{7.446282211}$$

$$=281.8908903.$$

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n–year pure endowment.

The loss for a fully discrete n–year pure endowment is

$$L_{x:\overset{1}{\overline{n}|}} = v^n I(K_x > n) - P\ddot{a}_{\overline{\min(K_x,n)}|} = Z_{x:\overset{1}{\overline{n}|}} - P\ddot{Y}_{x:\overline{n}|} = Z_{x:\overset{1}{\overline{n}|}} - P\frac{1 - Z_{x:\overline{n}|}}{d}.$$

The actuarial present value of the loss for a fully discrete n–year term insurance is

$$E[L_{x:\overset{1}{\overline{n}|}}] = A_{x:\overset{1}{\overline{n}|}} - P\ddot{a}_{x:\overline{n}|}.$$

The benefit premium which satisfies the equivalence principle is

$$P_{x:\overset{1}{\overline{n}|}} = P(A_{x:\overset{1}{\overline{n}|}}) = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}}.$$

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Theorem 7

We have that

$$\mathrm{Var}(L_{x:\overline{n}|}^{\;1}) = \frac{P^2}{d^2} \cdot {}^2A_{x:\overline{n}|}^{1} + \left(1 + \frac{P}{d}\right)^2 \cdot {}^2A_{x:\overline{n}|}^{\;1} - \left(E\left[L_{x:\overline{n}|}^{\;1}\right] + \frac{P}{d}\right)^2.$$

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Proof: We have that

$$L_{x:\overset{1}{\overline{n}|}} = Z_{x:\overset{1}{\overline{n}|}} - P\ddot{Y}_{x:\overline{n}|} = Z_{x:\overset{1}{\overline{n}|}} - P\frac{1-Z_{x:\overline{n}|}}{d} = -\frac{P}{d} + \frac{P}{d}Z^{1}_{x:\overline{n}|} + \left(1+\frac{P}{d}\right)Z_{x:\overset{1}{\overline{n}|}}$$

Hence,

$$E\left[\left(L_{x:\overset{1}{\overline{n}|}} + \frac{P}{d}\right)^2\right] = E\left[\left(\frac{P}{d}Z^{1}_{x:\overline{n}|}\right)^2\right] + E\left[\left(1+\frac{P}{d}\right)^2\left(Z_{x:\overset{1}{\overline{n}|}}\right)^2\right]$$

$$= \frac{P^2}{d^2}\cdot {}^2A^{1}_{x:\overline{n}|} + \left(1+\frac{P}{d}\right)^2\cdot {}^2A_{x:\overset{1}{\overline{n}|}}$$

and

$$\mathrm{Var}(L_{x:\overset{1}{\overline{n}|}}) = \mathrm{Var}\left(L_{x:\overset{1}{\overline{n}|}} + \frac{P}{d}\right)$$

$$= \frac{P^2}{d^2}\cdot {}^2A^{1}_{x:\overline{n}|} + \left(1+\frac{P}{d}\right)^2\cdot {}^2A_{x:\overset{1}{\overline{n}|}} - \left(E\left[L_{x:\overset{1}{\overline{n}|}}\right] + \frac{P}{d}\right)^2.$$

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Example 10

Austin is 40 years old and purchases a 20–year pure endowment policy with face value of 50000. This policy will be paid by a level benefit annual premium at the beginning of the next 20 years while Austin is alive. Assume that $v = 0.94$ *and death is modeled using the De Moivre model with terminal age 110.*

(i) Find the net single premium for this policy.

(ii) Find the benefit annual premium for this policy using the equivalence principle.

(iii) Calculate the standard deviation of the loss for this policy.

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Solution: (i) We have that

$$A_{40:\overset{1}{\overline{20}|}} = v^{20} \cdot {}_{20}p_{40} = (0.94)^{20} \frac{70-20}{70} = 0.2072187437.$$

The net single premium for this policy is
$(50000)(0.2072187437) = 10360.93719.$

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Solution: (ii) We have that

$$A_{40:\overline{20}|} = \frac{a_{\overline{20}|6/94}}{70} + (0.94)^{20}\frac{70-20}{70} = 0.3660997278,$$

$$\ddot{a}_{40:\overline{20}|} = \frac{1 - A_{40:\overline{20}|}}{d} = \frac{1-0.3660997278}{1-0.94} = 10.56500454.$$

The benefit annual premium for this policy using the equivalence principle is

$$P_{\substack{1\\20:\overline{40}|}} = \frac{10360.93719}{10.56500454} = 980.6845942.$$

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Solution: (iii) We have that

$$
\begin{aligned}
{}^2A^{1}_{40:\overline{20}|} &= \frac{a_{\overline{20}|6/94(2+6/94)}}{70} = 0.09931698364,\\
{}^2A_{40:\overset{1}{\overline{20}|}} &= (0.94)^{40}\frac{70-20}{70} = 0.06011545082,\\
\frac{P^{1}_{40:\overline{20}|}}{d} &= \frac{0.01961369187}{0.06} = 0.3268948645,\\
\mathrm{Var}(L_{40:\overset{1}{\overline{20}|}}) &= (0.3268948645)^2(0.09931698364)\\
&+ (1.3268948645)^2(0.06011545082) - (0.3268948645)^2\\
=&0.009595052877.
\end{aligned}
$$

The standard deviation of the loss for this policy is $(50000)\sqrt{0.009595052877} = 4897.717039.$

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If a n–year pure endowment is funded only t years where $1 \leq t \leq n$, then the present value of the loss for a n–year pure endowment is

$$v^n I(K_x > n) - P\ddot{a}_{\overline{\min(K_x,t)}|} = Z_{x:\overset{1}{\overline{n}|}} - P\ddot{Y}_{x:\overline{t}|}.$$

The actuarial present value of the loss for a t–year funded n–year term insurance is

$$A_{x:\overset{1}{\overline{n}|}} - P\ddot{a}_{x:\overline{t}|}.$$

The benefit premium which satisfies the equivalence principle is

$${}_tP_{x:\overset{1}{\overline{n}|}} = P({}_tA_{x:\overset{1}{\overline{n}|}}) = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{t}|}}.$$

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n–year endowment.

The loss for a fully discrete n–year endowment is

$$L_{x:\overline{n}|} = v^{\min(n,K_x)} - P\ddot{a}_{\overline{\min(K_x,n)}|} = Z_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{n}|} = Z_{x:\overline{n}|} - P\frac{1 - Z_{x:\overline{n}|}}{d}$$

$$= \left(1 + \frac{P}{d}\right) Z_{x:\overline{n}|} - \frac{P}{d}.$$

The actuarial present value of the loss for a fully discrete n–year term insurance is

$$E[L_{x:\overline{n}|}] = A_{x:\overline{n}|} - P\ddot{a}_{x:\overline{n}|}.$$

We have that

$$\mathrm{Var}(L_{x:\overline{n}|}) = \left(1 + \frac{P}{d}\right)^2 \mathrm{Var}(Z_{x:\overline{n}|}) = \left(1 + \frac{P}{d}\right)^2 \left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right).$$

The benefit premium which satisfies the equivalence principle is

$$P_{x:\overline{n}|} = P(A_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

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Example 11

A 20–year endowment insurance policy to (x) *with face value of 10000 payable at the end of the year of death is funded by a level benefit annual premium at the beginning of the next 20 years while* (x) *is alive. Assume that* $\delta = 0.05$ *and death is modeled using the constant force of mortality* $\mu = 0.02$.

(i) Find the benefit annual premium for this policy.

(ii) Calculate the variance for the loss at–issue random variable.

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Solution: (i) We have that

$$A_{x:\overline{20}|} = (1 - e^{-(20)(0.05+0.02)})\frac{1 - e^{-0.02}}{1 - e^{-0.02} + e^{0.05} - 1} + e^{-(20)(0.05+0.02)}$$

$$=0.4565008761,$$

$$\ddot{a}_{x:\overline{20}|} = (1 - e^{-(20)(\delta+\mu)})\frac{1}{1 - vq_x}$$

$$=(1 - e^{-(20)(0.05+0.02)})\frac{1}{1 - e^{-(0.05+0.02)}}$$

$$=11.14399653,$$

$$P = (10000) \cdot P_{x:\overline{20}|} = (10000)\frac{A_{x:\overline{20}|}}{\ddot{a}_{x:\overline{20}|}} = (10000)\frac{0.4565008761}{11.14399653}$$

$$=409.6383868.$$

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Solution: (ii)

$$
\begin{aligned}
&{}^2A_{x:\overline{20}|}\\
=&(1-e^{-(20)((2)0.05+0.02)})\frac{1-e^{-0.02}}{1-e^{-0.02}+e^{(2)0.05}-1}+e^{-(20)(0.05+0.02)}\\
=&0.2347898702,
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Var}(L_{x:\overline{20}|})=\left(1+\frac{P}{d}\right)^2\left({}^2A_{x:\overline{20}|}-(A_{x:\overline{20}|})^2\right)\\
=&\left(10000+\frac{409.6383868}{1-e^{-0.05}}\right)^2(0.2347898702-(0.4565008761)^2)\\
=&8936221.384.
\end{aligned}
$$

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The present value of the loss for a t–year funded n–year endowment is

$$v^{\min(n,K_x)} - P\ddot{a}_{\overline{\min(K_x,t)}|} = Z_{x:\overline{n}|} - P\ddot{Y}_{x:\overline{t}|}.$$

The actuarial present value of the loss for a t–year funded n–year term insurance is

$$A_{x:\overline{n}|} - P\ddot{a}_{x:\overline{t}|}.$$

The benefit premium which satisfies the equivalence principle is

$${}_tP_{x:\overline{n}|} = P({}_tA_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{t}|}}.$$

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Theorem 8

$$P_{x:\overline{n}|} = P^{1}_{x:\overline{n}|} + P_{x:\overset{1}{\overline{n}|}}.$$

Proof: We have that

$$P^{1}_{x:\overline{n}|} + P_{x:\overset{1}{\overline{n}|}} = \frac{A^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} + \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = P_{x:\overline{n}|}.$$

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Theorem 9

$$ {}_nP_x = P^1_{x:\overline{n}|} + P_{x:\overset{1}{\overline{n}|}} A_{x+n}. $$

Proof: We have that

$$ P^1_{x:\overline{n}|} + P_{x:\overset{1}{\overline{n}|}} A_{x+n} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} + \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}} A_{x+n} = \frac{A^1_{x:\overline{n}|} + {}_n|A_x}{\ddot{a}_{x:\overline{n}|}} = \frac{A_x}{\ddot{a}_{x:\overline{n}|}} = {}_nP_x. $$

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n–year deferred insurance.

The present value of the loss for a n–year deferred insurance is

$${}_{n|}L_x = v^{K_x} I(K_x > n) - P\ddot{a}_{\overline{K_x|}} = {}_{n|}Z_x - P\ddot{Y}_x.$$

The actuarial present value of the loss for a n–year term insurance is

$${}_{n|}A_x - P\ddot{a}_x.$$

The benefit premium which satisfies the equivalence principle is

$$P({}_{n|}A_x) = \frac{{}_{n|}A_x}{\ddot{a}_x}.$$

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Theorem 10

We have that

$$\mathrm{Var}({}_n|L_x) = \frac{P^2}{d^2} \cdot {}^2A^1_{x:\overline{n}|} + \left(1 + \frac{P}{d}\right)^2 \cdot {}^2{}_n|A_x - \left(E\left[{}_n|L_x\right] + \frac{P}{d}\right)^2.$$

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Proof: Using that $Z_x = Z^1_{x:\overline{n}|} + {}_{n|}Z_x$,

$${}_{n|}L_x = {}_{n|}Z_x - P\ddot{Y}_x = {}_{n|}Z_x - P\frac{1-Z_x}{d} = -\frac{P}{d} + \frac{P}{d}Z^1_{x:\overline{n}|} + \left(1+\frac{P}{d}\right){}_{n|}Z_x.$$

Since

$$Z^1_{x:\overline{n}|}\cdot {}_{n|}Z_x = v^{K_x}I(n<K_x)v^{K_x}I(K_x\le n) = 0,$$

$$E\left[\left({}_{n|}L_x + \frac{P}{d}\right)^2\right] = \frac{P^2}{d^2}\cdot {}^2A^1_{x:\overline{n}|} + \left(1+\frac{P}{d}\right)^2\cdot {}^2{}_{n|}A_x$$

Hence,

$$\begin{aligned}&\mathrm{Var}({}_{n|}L_x) = \mathrm{Var}\left({}_{n|}L_x + \frac{P}{d}\right)\\ =&\frac{P^2}{d^2}\cdot {}^2A^1_{x:\overline{n}|} + \left(1+\frac{P}{d}\right)^2\cdot {}^2{}_{n|}A_x - \left(E\left[{}_{n|}L_x\right] + \frac{P}{d}\right)^2.\end{aligned}$$

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The present value of the loss for a t–year funded n–year deferred insurance is

$$v^{K_x} I(K_x > n) - P\ddot{a}_{\overline{\min(K_x,t)}|} = {}_{n|}Z_x - P\ddot{Y}_{x:\overline{t}|}.$$

The actuarial present value of the loss for a t–year funded n–year term insurance is

$${}_{n|}A_x - P\ddot{a}_{x:\overline{t}|}.$$

The benefit premium which satisfies the equivalence principle is

$${}_tP({}_{n|}A_x) = \frac{{}_{n|}A_x}{\ddot{a}_{x:\overline{t}|}}.$$

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Plan	Premium
Whole life insurance	$P_x = \frac{A_x}{\ddot{a}_x}$
t–year funded whole life insurance	${}_tP_x = \frac{A_x}{\ddot{a}_{x:\overline{t}\vert}}$
n–year term insurance	$P^1_{x:\overline{n}\vert} = \frac{A^1_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{n}\vert}}$
t–year funded n–year term insurance	${}_tP^1_{x:\overline{n}\vert} = \frac{A^1_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{t}\vert}}$
n–year pure endowment insurance	$P_{x:\overset{1}{\overline{n}\vert}} = \frac{A_{x:\overset{1}{\overline{n}\vert}}}{\ddot{a}_{x:\overline{n}\vert}}$
t–year funded n–year pure endowment insurance	${}_tP_{x:\overset{1}{\overline{n}\vert}} = \frac{A_{x:\overset{1}{\overline{n}\vert}}}{\ddot{a}_{x:\overline{n}\vert}}$
n–year endowment	$P_{x:\overline{n}\vert} = \frac{A_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{n}\vert}}$
t–year funded n–year endowment insurance	${}_tP_{x:\overline{n}\vert} = \frac{A_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{t}\vert}}$
n–year deferred insurance	$P({}_n\vert A_x) = \frac{{}_n\vert A_x}{\ddot{a}_x}$
t–year funded n–year deferred insurance	${}_tP({}_n\vert A_x) = \frac{{}_n\vert A_x}{\ddot{a}_{x:\overline{t}\vert}}$

Table: Annual benefit premiums in the fully discrete case

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