Manual for SOA Exam MLC.

Chapter 6. Benefit premiums.

Section 6.8. Non-level premiums and/or benefits.

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Non-level premiums and/or benefits.

Let b_k be the benefit paid by an insurance company at the end of year k, $k = 1, 2, \dots$. The contingent cashflow of benefits is

benefits	0	b_1	b_2	b_3	$\cdots$
Time after issue	0	1	2	3	$\cdots$

Hence, the APV of the contingent benefit is

$$\sum_{k=1}^{\infty} b_k v^k \mathbb{P}\{K_x = k\} = \sum_{k=1}^{\infty} b_k v^k \cdot {}_{k-1|}q_x.$$

Let π_{k-1} be the benefit premium received by an insurance company at the beginning of year k, $k = 1, 2, \dots$. The contingent cashflow of benefit premiums is

benefit premiums	π_0	π_1	π_2	π_3	$\cdots$
Time after issue	0	1	2	3	$\cdots$

Hence, the APV of the contingent benefit premiums is

$$\sum_{k=0}^{\infty} \pi_k v^k \mathbb{P}\{K_x > k\} = \sum_{k=0}^{\infty} \pi_k v^k \cdot {}_k p_x.$$

Under the equivalence principle,

$$\sum_{k=1}^{\infty} b_k v^k \cdot {}_{k-1|}q_x = \sum_{k=0}^{\infty} \pi_k v^k \cdot {}_k p_x.$$

Example 1

For a special fully discrete 15-payment whole life insurance on (20):
(i) The death benefit is 1000 for the first 10 years and is 6000 thereafter.
(ii) The benefit premium paid during the each of the first 5 years is half of the benefit premium paid during the subsequent years.
(iii) Mortality follows the life table for the USA population in 2004.
(iv) $i = 0.06$.
Calculate the initial annual benefit premium.

Solution: Let π be the initial premium. We have that

$$2\pi\ddot{a}_{20:\overline{15}|} - \pi\ddot{a}_{20:\overline{5}|} = (1000)A_{20} + (5000)\cdot {}_{10}E_{20}A_{30}.$$

The APV of benefits is

$$(1000)A_{20} + (5000)\cdot {}_{10}E_{20}\cdot A_{30}$$
$$=52.45587 + (5)(0.553116815)(82.29543) = 280.0508007.$$

As to the APV of premiums,

$$\ddot{a}_{20:\overline{15}|} = \ddot{a}_{20} - {}_{15}E_{20}\cdot\ddot{a}_{35} = 16.739946 - (1.06)^{-15}\frac{97250}{98709}(15.817689)$$
$$=10.23733295,$$
$$\ddot{a}_{20:\overline{5}|} = \ddot{a}_{20} - {}_{5}E_{20}\cdot\ddot{a}_{25} = 16.739946 - (0.743753117)(16.514250)$$
$$=4.457421088,$$
$$2\pi\ddot{a}_{20:\overline{15}|} - \pi\ddot{a}_{20:\overline{5}|} = ((2)(10.23733295) - 4.457421088)\pi$$
$$=16.01724481\pi.$$

Hence, $\pi = \frac{280.0508007}{16.01724481} = 17.48433042.$

Example 2

Consider a whole life insurance policy to (40) *with face value of 250000 payable at the end of the year of death. This policy will be paid by benefit annual premiums paid at the beginning of each year while* (40) *is alive. Suppose that the premiums increase by 6% each year. Assume that* $i = 6\%$ *and death is modeled using the De Moivre model with terminal age 100. Find the amount of the first benefit annual premium for this policy.*

Solution: We have that

$$A_{40} = \frac{a_{\overline{60}|0.06}}{60} = 0.2693571284$$

and the net single premium is
$(250000)(0.2693571284) = 67339.28211$. Let π be the amount of the first benefit premium. Then, $\pi_k = \pi(1.06)^k$, $k = 0, 1, 2, \dots$ The APV of the benefit annual premiums is

$$\sum_{k=0}^{59} v^k \pi (1.06)^k \, {}_kp_{40} = \sum_{k=0}^{59} (1.06)^{-k} \pi (1.06)^k \frac{60-k}{60}$$
$$= \sum_{k=1}^{60} \pi \frac{k}{60} = \pi \frac{(60)(61)}{(2)(60)} = 31.5\pi.$$

Hence, $\pi = \frac{67339.28211}{31.5} = 2137.754988$.

Often, insurance products guarantee the return of net annual premiums.

Theorem 1

Suppose that an insurance contract is funded with an annual benefit premium of π *at the beginning of each year while the individual is alive. This contract returns the total paid premiums without interest at the end of the year of death. The APV of the net benefit premiums retained by the insurer is*

$$\pi\left(\ddot{a}_x - (IA)_x\right).$$

Proof: The insuree has made payments of π at times $0, 1, \ldots, K_x - 1$. The total amount of these payments is πK_x. Hence, the insuree gets a death benefit of πK_x at time K_x. This death benefit is that a unit increasing life insurance. Hence, the APV of the returned benefit premiums is $\pi\,(IA)_x$. The APV of the net benefit premiums retained by the insurer is $\pi\left(\ddot{a}_x - (IA)_x\right)$.

Consider a fully discrete unit whole life insurance to (x) that returns the benefit premiums without interest. Let π the annual benefit premium for this insurance. If (x) dies in the k–th year, the death benefit is $1 + k\pi$. Hence, the APV of benefits is $A_x + \pi(IA)_x$. To find π, we solve $\pi\ddot{a}_x = A_x + \pi(IA)_x$. Equivalently, the APV of the net benefit premiums retained by the insurer is $\pi\left(\ddot{a}_x - (IA)_x\right)$, which equals A_x. Hence, $\pi\left(\ddot{a}_x - (IA)_x\right) = A_x$.

Example 3

Consider a whole life insurance policy to (40) *with pays 250000 plus the return of the annual premiums without interest at the end of the year of death. This policy will be paid by benefit annual premiums paid at the beginning of each year while* (40) *is alive. Assume that* $i = 6\%$ *and death is modeled using De Moivre's model with terminal age 100. Find the amount of the benefit annual premium for this policy using the equivalence principle.*

Solution: Let π be the amount of the benefit annual premium. Using the equivalence principle, $\pi \ddot{a}_{40} = 250000 A_{40} + \pi (IA)_{40}$ and $\pi = \frac{250000 A_{40}}{\ddot{a}_{40} - (IA)_{40}}$. We have that

$$A_{40} = \frac{a_{\overline{60}|0.06}}{60} = 0.2693571284,$$

$$\ddot{a}_{40} = \frac{1 - A_{40}}{d} = \frac{1 - 0.2693571284}{(0.06)/1.06} = 12.90802406,$$

$$(IA)_{40} = \frac{(Ia)_{\overline{60}|0.06}}{60} = 4.253403641,$$

$$\pi = \frac{250000 A_{40}}{\ddot{a}_{40} - (IA)_{40}} = \frac{(250000)(0.2693571284)}{12.90802406 - 4.253403641} = 7780.732007.$$

Theorem 2

Suppose that an insurance contract is funded with an annual benefit premium of π at the beginning of each year for n years while the individual is alive. If the individual dies within n years, it returns the total accumulated premiums without interest. The APV of the net benefit premiums retained by the insurer is

$$\pi\left(\ddot{a}_{x:\overline{n}|} - (IA)^{1}_{x:\overline{n}|}\right).$$

Proof: If $K_x \le n$, the insuree has made payments of P at times $0, 1, \dots, K_x - 1$. The total amount of these payments is πK_x. Hence, if $K_x \le n$, the insuree gets a death benefit of πK_x at time K_x. This death benefit is that an n–year term unit increasing life insurance. Hence, the APV of the returned benefit premiums is $\pi\,(IA)^{1}_{x:\overline{n}|}$. The APV of the net benefit premiums retained by the insurer is $\pi\left(\ddot{a}_{x:\overline{n}|} - (IA)^{1}_{x:\overline{n}|}\right)$.

Example 4

Consider a special 20*–payment whole life insurance policy to* (40) *with face value \$250,000. This policy will be paid by benefit annual premiums paid at the beginning of each of the next 20 years while* (40) *is alive. If the insuree dies within 20 years, this policy will return the annual benefit premiums without interest at the end of the year of death. Assume that* $i = 6\%$ *and death is modeled using De Moivre's model with terminal age 100. Calculate the amount of the benefit annual premium for this policy using the equivalence principle.*

Solution: Let π be the amount of the benefit annual premium. Using the equivalence principle, $\pi \ddot{a}_{40:\overline{20}|} = 250000 A_{40} + \pi (IA)^1_{40:\overline{20}|}$ and $\pi = \frac{250000 A_{40}}{\ddot{a}_{40:\overline{20}|} - (IA)^1_{40:\overline{20}|}}$. We have that

$$A_{40} = \frac{a_{\overline{60}|0.06}}{60} = 0.2693571284,$$

$$\ddot{a}_{40} = \frac{1 - A_{40}}{d} = \frac{1 - 0.2693571284}{(0.06)/1.06} = 12.90802406,$$

$$A_{40:\overline{20}|} = \frac{a_{\overline{20}|0.06}}{60} + v^{20} \frac{60 - 20}{60} = 0.7945002282,$$

$$\ddot{a}_{40:\overline{20}|} = \frac{1 - A_{40:\overline{20}|}}{d} = \frac{1 - 0.7945002282}{(0.06)/1.06} = 3.630495968,$$

$$(IA)^1_{40:\overline{20}|} = \frac{(Ia)_{\overline{20}|0.06}}{60} = 2.145942157,$$

$$\pi = \frac{250000 A_{40}}{\ddot{a}_{40:\overline{20}|} - (IA)^1_{40:\overline{20}|}} = \frac{(250000)(0.2693571284)}{3.630495968 - 2.145942157} = 45359.94694.$$

Example 5

A special 10–year deferred whole life insurance of \$50,000 on (40) *returns the benefit premiums paid without interest if death happens during the deferral period. This is insurance is funded by annual benefit premiums made at the beginning of the year during the deferral period.* $i = 0.075$. *Mortality follows de Moivre model with terminal age 110. Calculate the annual benefit premiums.*

Solution: Let π be the annual benefit premium. We have that

$$\pi \ddot{a}_{40:\overline{10}|} = \pi\,(IA)^{1}_{40:\overline{10}|} + (50000)_{10|}A_{40} = \pi\,(IA)^{1}_{40:\overline{10}|} + (50000)_{10}E_{40}A_{50},$$

$$\pi = \frac{(50000)_{10}E_{40}A_{50}}{\ddot{a}_{40:\overline{10}|} - (IA)^{1}_{40:\overline{10}|}}.$$

We have that

$$_{10}E_{40} = v^{10}\frac{70-10}{70} = 0.41588051, \quad A_{50} = \frac{a_{\overline{60}|}}{60} = 0.2193230125,$$

$$A_{40:\overline{10}|} = \frac{a_{\overline{10}|}}{70} + {}_{10}E_{40} = 0.09805829937 + 0.41588051 = 0.5139388094,$$

$$\ddot{a}_{40:\overline{10}|} = \frac{1-0.5139388094}{75/1075} = 6.966877065,$$

$$(IA)^{1}_{40:\overline{10}|} = \frac{(Ia)_{\overline{10}|}}{70} = \frac{\frac{\ddot{a}_{\overline{10}|}-(10)v^{10}}{i}}{70} = \frac{\frac{7.378887028-4.851939283}{0.075}}{70} = 0.48132338$$

Hence,

$$\pi = \frac{(50000)(0.41588051)(0.2193230125)}{6.966877065-0.48132338} = 703.1949061.$$

Consider a fully discrete unit whole life insurance to (x) that returns the benefit premiums with interest if death happens within n years. Let π the annual benefit premium for this insurance. If $K_x \le n$, then the insurer returns $\pi \ddot{s}_{\overline{K_x}|i}$ at time K_x. The present value at time zero of this payment is $\pi \ddot{a}_{\overline{K_x}|i}$. We have that $\ddot{a}_{\overline{K_x}|i} I(K_x \le n) + \ddot{a}_{\overline{n}|i} I(K_x > n) = \ddot{a}_{\overline{\min(K_x,n)}|i}$. So,

$$E[\ddot{a}_{\overline{K_x}|i} I(K_x \le n)] = \ddot{a}_{x:\overline{n}|} - \ddot{a}_{\overline{n}|i} \cdot {}_n p_x.$$

The APV of the return of the benefit premiums is $\pi(\ddot{a}_{x:\overline{n}|} - \ddot{a}_{\overline{n}|i} \cdot {}_n p_x)$. To find π, we solve

$$\pi \ddot{a}_x = A_x + \pi(\ddot{a}_{x:\overline{n}|} - \ddot{a}_{\overline{n}|} \cdot {}_n p_x).$$

This equation is equivalent to

$$A_x = \pi(\ddot{a}_x - \ddot{a}_{x:\overline{n}|} + \ddot{a}_{\overline{n}|} \cdot {}_n p_x) = \pi({}_n|\ddot{a}_x + \ddot{a}_{\overline{n}|} \cdot {}_n p_x).$$

An interpretation of this formula is that the deceased within n years do not make any contributions to the insurer's fund. Its benefit premiums are returned with interest. The n-year survivors make contribution to the insurer's fund while they are alive. The APV of these contributions is $\ddot{a}_{\overline{n}|} \cdot {}_{n}p_x + {}_{n}|\ddot{a}_x$.

Example 6

Consider a special whole life insurance policy to (40) *with pays 250000 at the end of the year of death as well as the return of the annual premiums with interest payable at the end of the year if death happens in the first 15 years. This policy will be paid by benefit annual premiums paid at the beginning of each year while* (40) *is alive. Assume that* $i = 6\%$ *and death is modeled using De Moivre's model with terminal age 100. Find the amount of the benefit annual premium for this policy using the equivalence principle.*

Solution: Let π be the amount of the benefit annual premium. Using the equivalence principle,
$\pi\ddot{a}_{40} = (250000)A_{40} + \pi(\ddot{a}_{40:\overline{15}|} - \ddot{a}_{\overline{15}|} \cdot {}_{15}p_{40})$. and
$\pi = \frac{250000A_{40}}{\ddot{a}_{40}-\ddot{a}_{40:\overline{15}|}+\ddot{a}_{\overline{15}|}\cdot{}_{15}p_{40}}$. We have that

$$A_{40} = \frac{a_{\overline{60}|0.06}}{60} = 0.2693571284,$$
$$\ddot{a}_{40} = \frac{1 - A_{40}}{d} = \frac{1 - 0.2693571284}{(0.06)/1.06} = 12.90802406,$$
$$A_{40:\overline{15}|} = \frac{a_{\overline{15}|0.06}}{60} = 0.1618708165,$$
$$\ddot{a}_{40:\overline{15}|} = \frac{1 - 0.1618708165}{0.06/1.06} = 14.80694891,$$
$$\ddot{a}_{\overline{15}|} \cdot {}_{15}p_{40} = (10.29498393)\frac{60 - 15}{60} = 7.721237945,$$
$$\pi = \frac{(250000)(0.2693571284)}{12.90802406 - 14.80694891 + 7.721237945} = 11565.72671.$$

Theorem 3

Suppose that an insurance contract is funded with an annual benefit premium of P at the beginning of each year for n years while the individual is alive. If the individual dies within n years, it returns the total accumulated premiums with interest. The APV of the net benefit premiums retained by the insurer is

$$\pi \ddot{a}_{\overline{n}|} \cdot {}_n p_x = \pi \ddot{s}_{\overline{n}|} \cdot {}_n E_x.$$

Proof 1: The insurer only retains the benefit premiums if the insuree survives n years. The APV of the retained benefit premiums is $\pi \ddot{a}_{\overline{n}|} \cdot {}_n p_x = \pi \ddot{s}_{\overline{n}|} \cdot {}_n E_x$.

Proof 2: If $K_x \le n$, the insuree has made payments of π at times $0, 1, \dots, K_x - 1$. The present value at time zero of these payments is $\pi \ddot{a}_{\overline{K_x}|}$. Hence, the APV of the returned benefit premiums is

$$\pi \sum_{k=1}^{n} \ddot{a}_{\overline{k}|} \mathbb{P}\{K_x = k\}.$$

The APV of all benefit premiums (including the returned ones) is

$$\pi \ddot{a}_{x:\overline{n}|} = \pi \sum_{k=1}^{n} \ddot{a}_{\overline{k}|} \mathbb{P}\{K_x = k\} + \pi \ddot{a}_{\overline{n}|} \mathbb{P}\{K_x > n\}.$$

Hence, the APV of the net benefit premiums retained by the insurer is

$$\pi \ddot{a}_{x:\overline{n}|} - \pi \sum_{k=1}^{n} \ddot{a}_{\overline{k}|} = \pi \ddot{a}_{\overline{n}|} \mathbb{P}\{K_x = k\} = \pi \ddot{a}_{\overline{n}|} \mathbb{P}\{K_x > n\} = \pi \ddot{a}_{\overline{n}|} \cdot {}_n p_x$$

$= \pi \ddot{s}_{\overline{n}|} \cdot {}_n E_x$.

Consider an n–year deferred life annuity which pays B at the end of the year of death plus a return with interest of the payments made if death happens in the deferral period. Let π be the amount of the benefit annual premium. Using the equivalence principle,

$$\pi \ddot{s}_{\overline{n}|} \cdot {}_nE_x = B \cdot {}_{n|}\ddot{a}_x.$$

Hence,

$$\pi = \frac{B \cdot {}_{n|}\ddot{a}_x}{\ddot{s}_{\overline{n}|} \cdot {}_nE_x} = \frac{B \cdot {}_nE_x \ddot{a}_{n+x}}{\ddot{s}_{\overline{n}|} \cdot {}_nE_x} = \frac{B\ddot{a}_{x+n}}{\ddot{s}_{\overline{n}|}}.$$

Example 7

A special 20–year deferred life annuity on (50) with face value 50000 is funded by annual benefit premiums at the beginning of the first 20 years while (50) is alive. If death happens during the deferral period, the insurer will return of the annual premiums with interest at the end of the year of death. $i = 6\%$. Mortality is given by the life table for the USA population in 2004. Calculate the amount of the benefit annual premium for this policy using the equivalence principle.

Solution: Let π be the benefit annual premium. Using the equivalence principle, $\pi \ddot{s}_{\overline{20}|} \cdot {}_{20}E_{50} = (50000)_{20|}\ddot{a}_{50}$. Hence,

$$\pi = \frac{(50000)_{20|}\ddot{a}_{50}}{\ddot{s}_{\overline{20}|} \cdot {}_{20}E_{50}} = \frac{(50000) \cdot {}_{20}E_{50} \cdot \ddot{a}_{70}}{\ddot{s}_{\overline{20}|} \cdot {}_{20}E_{50}} = \frac{(50000)\ddot{a}_{70}}{\ddot{s}_{\overline{20}|}}.$$

We have that $\ddot{a}_{70} = 9.762512$, $\ddot{s}_{\overline{20}|} = 38.99272668$ and

$$\pi = \frac{(50000)(9.762512)}{38.99272668} = 12518.37564.$$

