On the law of the iterated logarithm for Gaussian processes [1]

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Suggested running head: L.I.L. for Gaussian processes

We present some optimal conditions for the compact law of the iterated logarithm of a sequence of jointly Gaussian processes in different situations. We also discuss the local law of the iterated logarithm for Gaussian processes indexed by arbitrary index sets, in particular for self–similar Gaussian processes. We apply these results to obtain the law of the iterated logarithm for compositions of Gaussian processes.

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1. INTRODUCTION.

We consider different kinds of laws of the iterated logarithm (L.I.L.) for Gaussian processes. In the considered situations the limit set is the unit ball of a reproducing kernel Hilbert space (r.k.h.s.) of a covariance function. So, first we will review the definition of this concept. Let T be a parameter set and let $R(s,t)$ be a covariance function defined on $T\times T$, i.e.

$$\sum_{j=1}^{m}\sum_{k=1}^{m} a_j a_k R(t_j,t_k)\geq 0 \tag{1.1}$$

for each $a_1,\dots,a_m\in \mathbb{R}$ and each $t_1,\dots,t_m\in T$. Then, there exists a mean–zero Gaussian process $\{Z(t):t\in T\}$ such that $E[Z(s)Z(t)]=R(s,t)$ for each $s,t\in T$. Let $\mathcal{L}$ be the linear subspace of L_2, generated by $\{Z(t):t\in T\}$. Then, the reproducing kernel Hilbert space (r.k.h.s.) of the covariance function $R(s,t)$ is the following class of functions on T

$$\{(E[Z(t)\xi])_{t\in T}:\xi\in\mathcal{L}\}. \tag{1.2}$$

This space is endowed of the inner product

$$< f_1,f_2 >:= E[\xi_1\xi_2], \tag{1.2}$$

where $f_i(t)=E[Z(t)\xi_i]$ for each $t\in T$ and each $i=1,2$. The unit ball of this r.k.h.s. is

$$K:=\{(E[Z(t)\xi])_{t\in T}:\xi\in\mathcal{L}\ \text{ and } E[\xi^2]\leq 1\}. \tag{1.4}$$

We refer to Aronszajn [3] for more in r.k.h.s.'s.

In Section 2, we consider the compact L.I.L. for Gaussian processes and random vectors with values in a separable Banach space. Given a sequence $\{X_n(t):t\in T\}$, $n\geq 1$, of jointly Gaussian processes (any linear combination of the random variables $X_n(t)$, $n\in\mathbb{N}$, $t\in T$, is Gaussian), we examine the problem of when there exists a compact set K, such that, with probability one, the sequence $\{(2\log n)^{-1/2}X_n(t):t\in T\}$ is relatively compact in $l_\infty(T)$ and its limit set is K, where $l_\infty(T)$ is the Banach space consisting of the uniformly bounded functions on T with the norm $\|x\|_\infty:=\sup_{t\in T}|x(t)|$. This problem has been considered before by several authors: Nisio [15], Oodaira [16], Lai [9], [10], Mangano [14] and Carmona and Kôno [6], among others. Here, we present some sufficient conditions for the L.I.L. of sequences of Gaussian processes, which are simpler than those from these authors, and have some optimality properties. We also consider the L.I.L. of $\{(2n\log\log n)^{-1/2}\sum_{j=1}^n X_j(t):t\in T\}$, where $\{X_n(t):t\in T\}$ is a stationary sequence of Gaussian processes.

In Section 3, we discuss the local L.I.L. for a Gaussian process. We say that a subset $\{x(u):0\leq u\leq 1\}$ of metric space is relatively compact as $u\to 0+$, if any sequence of positive numbers $\{u_n\}_{n=1}^\infty$, converging to 0, has a further subsequence $\{u_{n_k}\}_{k=1}^\infty$ such that $x(u_{n_k})$ converges as $k\to\infty$. Let T be a parameter set which is a subset of a space having a scalar product defined for $t\in T$ and $0\leq u\leq 1$. Let $\{X(t):t\in T\}$ be a Gaussian process indexed by T. Our main result is to give some sufficient conditions in order that the process

$\{(w(u))^{-1}(2\log\log u^{-1})^{-1/2}X(ut) : t \in T\}$ is a.s. relatively compact (as $u \to 0+$) and its limit set (for all sequences of positive numbers converging to zero) is the unit ball of a r.k.h.s., where $w(u)$ is a weight function. Of course, there is nothing particular about 0, we could have chosen another number, even infinity. A particular case, we will consider, is the local law of the iterated logarithm for self–similar processes.

In Section 4, we apply the results in the previous section to the study of the L.I.L. for compositions of Gaussian processes.

2. ON THE STRASSEN LAW OF THE ITERATED LOGARITHM FOR SEQUENCES OF GAUSSIAN RANDOM PROCESSES.

First, we consider the case of a sequence of jointly Gaussian random variables. The following lemma extends Theorem 2 in Lai [9] (see also Theorem 2 in Nisio [15]).

Lemma 2.1. Let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of centered, jointly Gaussian random variables. Assume that:

(i) $\lim_{n\to\infty} E[\xi_n^2]$ exists.

(ii) For each $0 < \epsilon < 1$,

$$\lim_{r\to 1-} \limsup_{n\to\infty} \sup_{m:\ n+n^{\epsilon}\le m\le n+n^{r}} E[\xi_n\xi_m] \le 0. \tag{2.1}$$

Then,

$$\limsup_{n\to\infty} (2\log n)^{-1/2}\xi_n = \sigma \quad \text{a.s.} \tag{2.2}$$

where $\sigma^2 := \lim_{n\to\infty} E[\xi_n^2]$.

Proof. Since

$$\sum_{n=1}^{\infty} \Pr\{(2\log n)^{-1/2}|\xi_n| \ge \sigma + \eta\} < \infty,$$

for each $\eta > 0$, by the lemma of Borel–Cantelli,

$$\limsup_{n\to\infty} (2\log n)^{-1/2}|\xi_n| \le \sigma \quad \text{a.s.} \tag{2.3}$$

This proves the lemma in the case $\sigma = 0$. If $\sigma \neq 0$, we may assume, without loss of generality, that $E[\xi_n^2] = 1$ for each n. Given $0 \le \eta < 1/3$, take $0 < \epsilon < r < 1 < q < p$ such that

$$\epsilon < p(p+q)^{-1},\ 1-\eta < (p-1)(p+q)^{-1},\ (p+q-1)(p+q)^{-1} < r \tag{2.4}$$

and

$$\limsup_{n\to\infty} \sup_{m:\ n+n^{\epsilon}\le m\le n+n^{r}} E[\xi_n\xi_m] < \eta. \tag{2.5}$$

For example, take

$$p = (3-\eta^2)\eta^{-1},\ q = 1+\eta,\ 0 < \epsilon < (3-\eta^2)(3+\eta)^{-1} \ \text{ and } 3(3+\eta)^{-1} < r.$$

Then, there exists k_0 such that $n + n^\epsilon \le m \le n + n^r$, where $n = [k^q(k^p + j_1)]$ and $m = [k^q(k^p + j_2)]$, for each $1 \le j_1 < j_2 \le (k+1)^p - k^p$ and each $k \ge k_0$. Hence, by (2.5), we have that

$$E[\xi_{[k^q(k^p+j_1)]}\xi_{[k^q(k^p+j_2)]}] \le \eta,$$

for each $k \ge k_0$ and each $1 \le j_1 < j_2 \le [(k+1)^p - k^p]$. Let $g, g_1, g_2, \ldots$ be independent centered normal random variables such that

$$E[g^2] = \eta \quad \text{and} \quad E[g_k^2] = 1 - \eta, \quad \text{for} \quad k \ge 1.$$

We have that

$$E[\xi_{[k^q(k^p+j_1)]}\xi_{[k^q(k^p+j_2)]}] \le E[(g + g_{j_2})(g + g_{j_1})]$$

for each $1 \le j_1 < j_2 \le (k+1)^p - k^p$; and

$$E[\xi^2_{[k^q(k^p+j)]}] = 1 = E[(g + g_j)^2],$$

for each $1 \le j \le (k+1)^p - k^p$. So, by the Slepian lemma (see e.g. Corollary 3.12 in Ledoux and Talagrand [12]),

$$\begin{aligned}
&\Pr\{\max_{1\le j\le[(k+1)^p-k^p]}(2\log([k^q(k^p+j)]))^{-1/2}\xi_{[k^q(k^p+j)]} \le (1-3\eta)\}\\
&\le \Pr\{\max_{1\le j\le[(k+1)^p-k^p]}\xi_{[k^q(k^p+j)]} \le (1-2\eta)(2\log(k^{p+q}))^{1/2}\}\\
&\le \Pr\{\max_{1\le j\le[(k+1)^p-k^p]}(g+g_j) \le (1-2\eta)(2\log(k^{p+q}))^{1/2}\}\\
&\le \Pr\{g \le -\eta(2\log(k^{p+q}))^{1/2}\} + \Pr\{\max_{1\le j\le[(k+1)^p-k^p]} g_j \le (1-\eta)(2\log(k^{p+q}))^{1/2}\}.
\end{aligned}$$

By the usual bound on the tail of a normal distribution

$$\sum_{k=1}^{\infty} \Pr\{g \le -\eta(2\log(k^{p+q}))^{1/2}\} < \infty$$

(by (2.4) $1 < \eta(p+q)$). We also have that

$$\begin{aligned}
&\Pr\{\max_{1\le j\le[(k+1)^p-k^p]} g_j \le (1-\eta)(2\log(k^{p+q}))^{1/2}\}\\
&= \left(\Pr\{g_1 \le (1-\eta)(2\log(k^{p+q}))^{1/2}\}\right)^{[(k+1)^p-k^p]}\\
&\le \exp\left(-[(k+1)^p - k^p]\Pr\{g_1 > (1-\eta)(2\log(k^{p+q}))^{1/2}\}\right).
\end{aligned}$$

Again, by the usual bound on the tail of a normal distribution

$$\sum_{k=1}^{\infty} \exp\left(-[(k+1)^p - k^p]\Pr\{g_1 > (1-\eta)(2\log(k^{p+q}))^{1/2}\}\right) < \infty.$$

Therefore,

$$\sum_{k=1}^{\infty} \Pr\{\max_{1\le j\le [(k+1)^p-k^p]} (2\log([k^q(k^p+j)]))^{-1/2}\xi_{[k^q(k^p+j)]} \le (1-3\eta)\} < \infty$$

and the result follows from this, the lemma of Borel–Cantelli and (2.3). □

The difference between Lemma 2.1 and Theorem 2 in Lai [9] lies in condition (ii). In Theorem of Lai [9], the author impose the stronger condition

$$\limsup_{n,\ m-n\to\infty} E[\xi_n\xi_m] \le 0. \tag{2.6}$$

In the study of self–similar Gaussian processes, we need a weaker condition. For example if $\{B(t) : 0 \le t \le 1\}$ is a Brownian motion and $\xi_n = 2^{n/2}B(2^{-n}) + 2^n B(2^{-2n})$, then $E[\xi_n^2] \to 2$ and $E[\xi_n\xi_{2n}] \to 1$. In this case condition (2.6) is not satisfied, but condition (2.1) is.

Obviously, some condition similar to (2.1) is needed. If $\xi_j = \xi_1$ for each $j \ge 1$, then

$$\lim_{n\to\infty} (2\log n)^{-1/2}|\xi_n| = 0 \quad \text{a.s.}$$

The following example shows that condition (2.1) is sharp. Let $\{g_j\}_{j=1}^{\infty}$ be a sequence of i.i.d.r.v.'s with standard normal distribution. Fix $p > 1$, let $n_k = [k^{p-1}]$. Define $\xi_n = g_j$ if $\sum_{l=1}^{k} n_l < n \le \sum_{l=1}^{k+1} n_l$ and $n = \sum_{l=1}^{k} n_l + j$. Then $\sum_{j=1}^{k} n_j \simeq p^{-1}k^p$. Hence,

$$\limsup_{n\to\infty} (2\log n)^{-1/2}\xi_n = \limsup_{k\to\infty} (2\log \sum_{l=1}^{k+1} n_l)^{-1/2} g_k = p^{-1/2} \quad \text{a.s.}$$

We have that $E[\xi_n^2] = 1$ and

$$\limsup_{n\to\infty} \sup_{m:\, n\le m\le n+n^r} |E[\xi_n\xi_m]| = 0,$$

for $0 < r < p^{-1}(p-1)$, but (2.2) is not satisfied. Of course, (2.1) does not hold for this sequence:

$$\lim_{r\to 1-} \limsup_{n\to\infty} \sup_{m:\, n+n^r\le m\le n+n^\epsilon} E[\xi_n\xi_m] = 1,$$

for $0 < \epsilon < p^{-1}(p-1)$. So, Lemma 2.1 is not true if the group of words "for each $0 < \epsilon < 1$" is substituted by "for some $0 < \epsilon < 1$".

A standard argument (see the proof of Lemma 2 in Finkelstein [8]) gives the compact L.I.L. in the finite dimensional case:

Lemma 2.2. Let $\{\xi_n = (\xi_n^{(1)}, \dots, \xi_n^{(d)})\}_{n=1}^{\infty}$ be a sequence of centered jointly Gaussian random vectors with values in $I\!R^d$. Assume that

(i) For each $1 \le j, k \le d$, $E[\xi_n^{(j)}\xi_n^{(k)}]$ converges as $n \to \infty$.

(ii) For each $\lambda_1, \dots, \lambda_d \in I\!R$ and each $0 < \epsilon < 1$

$$\lim_{r\to 1-} \limsup_{n\to\infty} \sup_{m:\ n+n^\epsilon\le m\le n+n^r} \sum_{j,k=1}^{d} \lambda_j\lambda_k E[\xi_n^{(j)}\xi_m^{(k)}] \le 0.$$

Then, with probability one, $\{(2\log n)^{-1/2}\xi_n\}$ is relatively compact and its limit set is the unit ball K of the reproducing kernel Hilbert space of the covariance function $R(s,t) = \lim_{n\to\infty} \sum_{j,k=1}^{d} s_j t_k E[\xi_n^{(j)}\xi_n^{(k)}]$, where $s = (s_1, \ldots, s_d)$ and $t = (t_1, \ldots, t_d)$.

To get the compact L.I.L. for processes, we need the following two consequences of the Ascoli–Arzela theorem (see e.g. Theorems 4.1 and 4.3 in Arcones and Giné [2]):

Lemma 2.3. Let $\{X_n(t) : t \in T\}$, $n \geq 1$, be a sequence of random processes indexed by T. Let $\rho(s,t)$ be a pseudometric in T. Let K be a compact subset of the space $C_u(T,\rho)$ of uniformly bounded and uniformly continuous functions on (T,ρ). Assume that the sequence of processes $\{X_n(t) : t \in T\}$ satisfies the following conditions:

(i) (T,ρ) is totally bounded.

(ii) $\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{\rho(t_1,t_2)\leq\delta} |X_n(t_1) - X_n(t_2)| = 0$ a.s.

(iii) For each $m \in I\!N$ and each $t_1, \ldots, t_m \in T$, with probability one, the sequence $\{(X_n(t_1), \ldots, X_n(t_m))\}_{n=1}^{\infty}$ is relatively compact in $I\!R^m$ and its limit set is $\{(x(t_1), \ldots, x(t_m)) : x \in K\}$.

Then, with probability one, the sequence $\{X_n(t) : t \in T\}$ (whose terms are eventually a.s. in $l_\infty(T)$) is relatively compact in $l_\infty(T)$ and its limit set is K.

Lemma 2.4. Let $\{X_n(t) : t \in T\}$, $n \geq 1$, be a sequence of stochastic processes indexed by T. Suppose that

(i) There is a set $K \subset l_\infty(T)$ such that for each $t_1, \ldots, t_m \in T$, the sequence $\{(X_n(t_1), \ldots, X_n(t_m))\}_{n=1}^{\infty}$ is a.s. relatively compact in $I\!R^m$ and its limit set is $\{(x(t_1), \ldots, x(t_m)) : x \in K\}$.

(ii) There is a set L such that, with probability one, the sequence $\{X_n(t) : t \in T\}$ is relatively compact in $l_\infty(T)$ and its limit set is L.

Then,

(a) (T,ρ) is totally bounded where $\rho(t,s) = \sup_{x\in K} |x(t) - x(s)|$, $t, s \in T$,

(b) $\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{\rho(t,s)\leq\delta} |X_n(t) - X_n(s)| = 0$ a.s.

(c) The set L coincides with the set K and is compact.

Lemma 2.3 (maybe in a less abstract version) has been used by many authors in similar situations. Lemma 2.4 was probably introduced in the cited reference. Observe that conditions (a) and (b) in Lemma 2.4, and $X_n(t) \xrightarrow{\Pr} 0$ for each $t \in T$, imply that $\sup_{t\in T} |X_n(t)| \xrightarrow{\Pr} 0$. This follows from the fact that condition (b) in Lemma 2.4 implies that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \Pr\{\sup_{\rho(s,t)\leq\delta} |X_n(s) - X_n(t)| \geq \eta\} = 0$$

for each $\eta > 0$.

We also need the following upper bound on the tail probability of a Gaussian process (see e.g. Lemma 3.1 in Ledoux and Talagrand [12]) (a more refined inequality on the tail of a Gaussian process is in Borell [5]):

Lemma 2.5. Let $\{X(t) : t \in T\}$ be a centered Gaussian process. Let M be the median of $\sup_{t\in T} |X(t)|$ and let $\sigma^2 = \sup_{t\in T} E[X^2(t)]$. Then, for each $u > 0$,

$$\Pr\{|\sup_{t\in T} |X(t)| - M| \geq u\} \leq \exp\left(-\frac{u^2}{2\sigma^2}\right).$$

Now, we are ready to prove a compact L.I.L. for Gaussian processes.

Theorem 2.1. Let $\{X_n(t) : t \in T\}$, $n \geq 1$, be a sequence of Gaussian processes and let ρ be a pseudometric on T. Suppose:

(i) $\sup_{t\in T}(2\log n)^{-1/2}|X_n(t)| \xrightarrow{\Pr} 0$.

(ii) For each $s, t \in T$, $E[X_n(s)X_n(t)]$ converges as $n \to \infty$.

(iii) For each $d \geq 1$, each $1 > \epsilon > 0$, each $t_1, \ldots, t_d \in T$ and each $\lambda_1, \ldots, \lambda_d \in I\!R$,

$$\lim_{r\to 1-} \limsup_{n\to\infty} \sup_{m:\ n+n^\epsilon \leq m \leq n+n^r} \sum_{j,k=1}^{d} \lambda_j \lambda_k E[X_n(t_j)X_m(t_k)] \leq 0 \tag{2.7}$$

(iv) (T, ρ) is totally bounded.

(v) For each $\eta > 0$,

$$\lim_{\delta\to 0} \sum_{n=1}^{\infty} \exp\left(-\frac{\eta \log n}{\sup_{\rho(s,t)\leq\delta} \|X_n(t) - X_n(s)\|_2^2}\right) < \infty, \tag{2.8}$$

where $\|X\|_2 := (E[X^2])^{1/2}$.

Then, with probability one, $\{(2\log n)^{-1/2}X_n(t) : t \in T\}$ is relatively compact and its limit set is the unit ball K of the reproducing kernel Hilbert space of the covariance function $R(s,t) = \lim_{n\to\infty} E[X_n(s)X_n(t)]$.

Proof. By Lemmas 2.1 and 2.3 (and hypotheses (ii)–(iv)), it suffices to show that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{\rho(s,t)\leq\delta} (2\log n)^{-1/2}|X_n(t) - X_n(s)| = 0 \ \text{ a.s.}$$

By the lemma of Borel–Cantelli, it suffices to show that, for each η, there is a $\delta > 0$ such that

$$\sum_{n=1}^{\infty} \Pr\{\sup_{\rho(s,t)\leq\delta} (2\log n)^{-1/2}|X_n(t) - X_n(s)| \geq \eta\} < \infty.$$

This follows from Lemma 2.5, using hypotheses (i) and (v). $\square$

A choice of pseudometric, intrisic to the problem, is $\rho(s,t) = \lim_{n\to\infty} \|X_n(t) - X_n(s)\|_2$. Condition (i) in Theorem 2.1 can be restated in terms of majorizing measures (see Talagrand [17]). Condition (v) is satisfied if

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{\rho(s,t)\leq\delta} \|X_n(t) - X_n(s)\|_2 = 0. \tag{2.9}$$

Next, we will discuss the optimality of the conditions in Theorem 2.1. Hypotheses (ii) and (iii) are conditions to get the L.I.L. for the finite dimensional projections of the process. They are quite reasonable conditions.

Proposition 2.1. Let $\{X_n(t) : t \in T\}$, $n \geq 1$, be a sequence of centered, jointly Gaussian processes. Suppose that:

(i) For each $s, t \in T$, $E[X_n(s)X_n(t)]$ converges as $n \to \infty$.

(ii) There is compact set K in $l_\infty(T)$ such that, with probability one, the sequence $\{(2\log n)^{-1/2}X_n(t) : t \in T\}$ is relatively compact and its limit set is K.

Then

(a) (T, ρ) is totally bounded, where $\rho(s,t) := \sup\{|x(s) - x(t)| : x \in K\}$.

(b) $\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\rho(s,t)\leq\delta}(2\log n)^{-1/2}|X_n(t) - X_n(s)| = 0$ a.s.

(c) $\sup_{t\in T}(2\log n)^{-1/2}|X_n(t)| \xrightarrow{\Pr} 0$.

If, in addition, $\{X_n(t) : t \in T\}$, $n \geq 1$, is a sequence of independent Gaussian processes, then

(d)

$$\lim_{\delta\to 0}\sum_{n=1}^{\infty}\exp\left(-\frac{\eta\log n}{\sup_{\rho(s,t)\leq\delta}\|X_n(t) - X_n(s)\|_2^2}\right) < \infty \tag{2.10}$$

for each $\eta > 0$.

Proof. Assertions (a) and (b) follow by Lemma 2.4. By the remark after Lemma 2.4, (c) follows. If we also assume independence of the sequence of processes, by the Kolmogorov zero–one law, for each $\delta > 0$, there is a constant $c(\delta)$ such that

$$\limsup_{n\to\infty}\sup_{\rho(s,t)\leq\delta}(2\log n)^{-1/2}|X_n(t) - X_n(s)| = c(\delta) \quad \text{a.s.}$$

and $\lim_{\delta\to 0} c(\delta) = 0$. So, by the lemma of Borel–Cantelli

$$\lim_{\delta\to 0}\sum_{n=1}^{\infty}\Pr\{\sup_{\rho(s,t)\leq\delta}(2\log n)^{-1/2}|X_n(t) - X_n(s)| \geq \eta\} < \infty$$

for each $\eta > 0$. For a standard normal random variable g, we have that if $\Pr\{|g| \geq x\} \leq 1/4$, then $x^{-1}e^{2^{-1}x^2} \leq \Pr\{|g| \geq x\}$. So, for n large, and $\rho(s,t) \leq \delta$,

$$\begin{aligned}
&\Pr\{\sup_{\rho(s,t)\leq\delta}(2\log n)^{-1/2}|X_n(t) - X_n(s)| \geq \eta\} \\
&\geq \Pr\{(2\log n)^{-1/2}|X_n(t) - X_n(s)| \geq \eta\} \\
&\geq 2^{-1}\eta^{-1}(2\log n)^{-1/2}\|X_n(t) - X_n(s)\|_2 \exp\left(-\frac{\eta^2\log n}{\|X_n(t) - X_n(s)\|_2^2}\right),
\end{aligned}$$

assuming that the processes are Gaussian. Therefore,

$$\sum_{n=1}^{\infty}\sup_{\rho(s,t)\leq\delta}\left((2\log n)^{-1/2}\|X_n(t) - X_n(s)\|_2 \exp\left(-\frac{\eta^2\log n}{\|X_n(t) - X_n(s)\|_2^2}\right)\right) < \infty$$

and (d) follows. $\square$

Observe that in the previous proposition

$$\rho(s,t) = \limsup_{n\to\infty} (2\log n)^{-1/2} |X_n(t) - X_n(s)| \quad \text{a.s.}$$

As a consequence of Theorem 2.1, we easily obtain the following:

Theorem 2.2. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of centered random vectors with values in a separable Banach space B. Let X be a another B–valued Gaussian centered random vector. Assume that the following conditions are satisfied:
(i) $(2\log n)^{-1/2}\|X_n\| \xrightarrow{\Pr} 0$, as $n \to \infty$.
(ii) For each $f, g \in B^*$, $\lim_{n\to\infty} E[f(X_n)g(X_n)] = E[f(X)g(X)]$.
(iii) For each $f \in B^*$ and each $\eta > 0$

$$\lim_{\delta\to 0} \sum_{n=1}^{\infty} \exp\left(- \frac{\eta \log n}{\sup_{\substack{\|f\|,\|g\|\le 1 \\ \|f(X)-g(X)\|_2 \le \delta}} \|f(X_n) - g(X_n)\|_2^2} \right) < \infty.$$

Then, with probability one, $\{(2\log n)^{-1/2}X_n\}_{n=1}^{\infty}$ is relatively compact in B and its limit set is the unit ball K of the reproducing kernel Hilbert space of X.

The observations about the optimality of the conditions in Theorem 2.1 also apply to this case. In particular, we have the following:

Proposition 2.2. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent B–valued centered random vectors and let X be a another B–valued Gaussian centered random vector. Suppose that

$$\lim_{n\to\infty} E[f(X_n)g(X_n)] = E[f(X)g(X)]$$

for each $f, g \in B^*$. Then, the sequence $\{(2\log n)^{-1/2}X_n\}_{n=1}^{\infty}$ satisfies the compact L.I.L. if and only if

$$(2\log n)^{-1/2}\|X_n\| \xrightarrow{\Pr} 0, \quad \text{as} \quad n \to \infty,$$

and

$$\lim_{\delta\to 0} \sum_{n=1}^{\infty} \exp\left(- \frac{\eta \log n}{\sup_{\substack{\|f\|,\|g\|\le 1 \\ \|f(X)-g(X)\|_2 \le \delta}} \|f(X_n) - g(X_n)\|_2^2} \right) < \infty$$

for each $\eta > 0$.

Next, we will consider the laws of the iterated logarithm of the process

$$\{(n2\log\log n)^{-1/2} \sum_{j=1}^{n} X_j(t) : t \in T\}$$

where $\{X_j(t) : t \in T\}_{j=1}^{\infty}$ is a stationary sequence of Gaussian processes.

Theorem 2.3. Let $\{X_n(t) : t \in T\}$, $n \geq 1$, be a sequence of mean–zero jointly Gaussian processes. Assume that the following conditions are satisfied:

(i) $E[X_1(t)X_n(s)] = E[X_{m+1}(t)X_{m+n}(s)]$, for each $n, m \in I\!N$ and each $s, t \in T$.

(ii) $\sum_{n=1}^{\infty} r(n) < \infty$, where $r(n) = \sup_{s,t \in T} |E[X_1(t)X_{n+1}(s)]|$.

(iii) $E[\sup_{t \in T} |Y(t)|] < \infty$, where $\{Y(t) : t \in T\}$ is a mean–zero Gaussian processes with covariance given by

$$E[Y(t)Y(s)] = \sum_{k=-\infty}^{\infty} 2^{-1} \left(|E[X_1(t)X_{k+1}(t)]| + |E[X_1(s)X_{k+1}(s)]| \right.$$

$$\left. -2|E[(X_1(t) - X_1(s))(X_{k+1}(t) - X_{k+1}(s))]| \right).$$

Then, with probability one,

$$\{(n2 \log \log n)^{-1/2} \sum_{j=1}^{[ns]} X_j(t) : 0 \leq s \leq 1,\ t \in T\} \tag{2.11}$$

is relatively compact in $l_\infty([0,1] \times T)$ and its limit set is the unit ball K of the r.k.h.s. of the covariance function

$$R((s_1, t_1), (s_2, t_2)) := \min(s_1, s_2) \sum_{k=-\infty}^{\infty} E[X_1(t_1)X_{k+1}(t_2)].$$

Proof. Let $\{Y_n(t) : t \in T\}$, $n \geq 1$, be a sequence of i.i.d. mean–zero Gaussian processes with covariance given by $E[Y_n(t_1)Y_n(t_2)] = E[Y(t_1)Y(t_2)]$. Define $S_s(t) = \sum_{j=1}^{[s]} X_j(t)$ and $U_s(t) = \sum_{j=1}^{[s]} X_j(t)$, for $s > 0$ and $t \in T$.

First, we prove that, for each $\lambda > 1$, with probability one,

$$\{(\lambda^n 2 \log n)^{-1/2} \sum_{j=1}^{[\lambda^n s]} X_j(t) : 0 \leq s \leq 1,\ t \in T\} \tag{2.12}$$

is relatively compact in $l_\infty([0,1] \times T)$ and its limit set is K. We apply Theorem 2.1. We have that for $0 \leq s_1 < s_2 \leq 1$ and $t_1, t_2 \in T$,

$$E[(S_{\lambda^n s_1}(t_1) - S_{\lambda^n s_2}(t_2))^2] \tag{2.13}$$

$$\leq 2E[(S_{\lambda^n s_1}(t_1) - S_{\lambda^n s_1}(t_2))^2] + 2E[(S_{\lambda^n s_1}(t_2) - S_{\lambda^n s_2}(t_2))^2]$$

$$\leq 2[\lambda^n s_1]E[(Y(t_1) - Y(t_2))^2] + 2([\lambda^n s_2] - [\lambda^n s_1])E[Y_1^2(t_2)]$$

$$= 2E[(U_{\lambda^n s_1}(t_1) - U_{\lambda^n s_2}(t_2))^2].$$

From this inequality, the Gaussian comparison principle (see e.g. Theorem 3.15 in Ledoux and Talagrand [12]) and the Lévy inequality, we get that

$$(\lambda^n 2 \log n)^{-1/2} E[\sup_{0 \leq s \leq 1} \sup_{t \in T} |S_{\lambda^n s}(t)|]$$

$$\leq 4(\lambda^n 2\log n)^{-1/2} E[\sup_{0\leq s\leq 1}\sup_{t\in T}|U_{\lambda^n s}(t)|]$$

$$\leq 8(\lambda^n 2\log n)^{-1/2} E[\sup_{t\in T}|U_{\lambda^n}(t)|] \leq 8(2\log n)^{-1/2} E[\sup_{t\in T}|Y_1(t)|] \to 0.$$

So, condition (i) in Theorem 2.1 follows.

It is easy to see that

$$\lambda^{-n} E[S_{\lambda^n s_1}(t_1) S_{\lambda^n s_2}(t_2)] \to \min(s_1, s_2) \sum_{k=-\infty}^{\infty} E[X_1(t_1) X_{k+1}(t_2)],$$

i.e. condition (ii) in Theorem 2.1 holds.

Let $0 < \epsilon < r < 1$. If $n + n^\epsilon \leq m \leq n + n^r$, $0 \leq s_1, s_2 \leq 1$ and $t_1, t_2 \in T$,

$$\lambda^{-n/2}\lambda^{-m/2}|E[S_{\lambda^n s_1}(t_1) S_{\lambda^m s_2}(t_2)]| \leq b^2 \lambda^{-n^\epsilon/2} \to 0,$$

where $b^2 := \sup_{t\in T} E[Y^2(t)]$ (condition (iii) of Theorem 2.1 holds).

Take $\rho((s_1, t_1), (s_2, t_2)) := |s_1 - s_2| + d(t_1, t_2)$, where $d^2(t_1, t_2) = E[(Y(t_1) - Y(t_2))^2]$. By hypothesis (iii) and the Sudakov inequality (T, d) is totally bounded. So, $([0, 1] \times T, \rho)$ is also totally bounded (condition (iv) of Theorem 2.1 follows).

Hypothesis (v) in Theorem 2.1 follows from (2.13). Therefore, the assertion containing equation (2.12) holds.

From a comparison principle (see e.g. Equation (3.12) in Ledoux and Talagrand [12]) and the Lévy inequality, we get that

$$\sum_{k=1}^{\infty} \Pr\{\sup_{[\lambda^k]\leq n\leq[\lambda^{k+1}]}\sup_{t\in T}|S_n(t) - S_{[\lambda^k]}(t)| \geq 16(\lambda-1)^{1/2} b(2\lambda^k \log k)^{1/2}\} < \infty.$$

So,

$$\limsup_{k\to\infty}\sup_{[\lambda^k]\leq n\leq[\lambda^{k+1}]}\sup_{t\in T}(2\lambda^k\log k)^{-1/2}|S_n(t) - S_{[\lambda^k]}(t)| \leq 16(\lambda-1)^{1/2} b \quad \text{a.s.} \tag{2.14}$$

By (2.14), given $\epsilon > 0$, there exists a k_0 finite (and maybe random) such that

$$\sup_{[\lambda^k]\leq n\leq[\lambda^{k+1}]}\sup_{t\in T}(2\lambda^k\log k)^{-1/2}|S_n(t) - S_{[\lambda^k]}(t)| \leq 16(\lambda-1)^{1/2} b + \epsilon,$$

for $k \geq k_0$. Let $k \geq k_0 + 1$ and let $[\lambda^k] \leq n \leq [\lambda^{k+1}]$. If $0 \leq s \leq [\lambda^{k_0-k-1}]$, then $ns, \lambda^k s \leq \lambda^{k_0}$. So,

$$|S_{ns}(t) - S_{[\lambda^k]s}(t)| \leq 2\sup_{1\leq j\leq\lambda^{k_0}}|S_j(t)|.$$

If $\lambda^{k_0-k-1} \leq s \leq 1$, then there exists an integer k_1 such that $[\lambda^{k_1}] \leq [\lambda^k]s \leq [\lambda^{k_1+1}]$. Then, $[\lambda^{k_1}] \leq ns \leq [\lambda^{k_1+3}]$. So,

$$|S_{ns}(t) - S_{\lambda^k s}(t)| \leq 3(16(\lambda-1)^{1/2} b + \epsilon)(2\lambda^{k+2}\log(k+2))^{1/2}.$$

Hence,

$$\sup_{[\lambda^k]\leq n\leq[\lambda^{k+1}]}\sup_{t\in T}\sup_{0\leq s\leq 1}|S_{ns}(t) - S_{\lambda^k s}(t)|$$

$$\leq 2 \sup_{1\leq j\leq \lambda^{k_0}} \sup_{t\in T} |S_j(t)| + 3(16(\lambda-1)^{1/2}b+\epsilon)(2\lambda^{k+2}\log(k+2))^{1/2}.$$

Therefore,

$$\limsup_{k\to\infty} \sup_{[\lambda^k]\leq n\leq[\lambda^{k+1}]} \sup_{0\leq s\leq 1} \sup_{t\in T} (2\lambda^k \log k)^{-1/2} |S_{ns}(t) - S_{\lambda^k s}(t)| \tag{2.15}$$
$$\leq 48\lambda(\lambda-1)^{1/2}b \quad \text{a.s.}$$

This limit and the assertion containing equation (2.12) imply the thesis of the theorem. □

From previous theorem we get immediately the following two corollaries:

Corollary 2.1. Let $\{\xi_n\}_{n=1}^{\infty}$ be a stationary sequence of jointly Gaussian mean–zero random variables. Assume that $\sum_{n=1}^{\infty} |r(n)| < \infty$, where $r(n) = E[X_1 X_{n+1}]$.

Then, with probability one,

$$\{(n2\log\log n)^{-1/2} \sum_{j=1}^{[ns]} \xi_j : 0 \leq s \leq 1\}$$

is relatively compact in $l_\infty([0,1])$ and its limit set is

$$\left\{ \left(\sigma \int_0^s \alpha(u)\,du\right)_{0\leq s\leq 1} : \int_0^1 \alpha^2(u)\,du \leq 1 \right\},$$

where $\sigma^2 = \sum_{k=-\infty}^{\infty} E[\xi_1 \xi_{k+1}]$.

Corollary 2.2. Let $\{X_n(t)\}_{t\in T}$, $n \geq 1$, be a sequence of independent identically distributed mean–zero Gaussian processes. Then, the following are equivalent:

(a) $E[\sup_{t\in T} |X_1(t)|] < \infty$.

(b) With probability one,

$$\{(n2\log\log n)^{-1/2} \sum_{j=1}^{n} X_j(t) : \ t \in T\}$$

is relatively compact in $l_\infty(T)$ and its limit set is the unit ball K of the r.k.h.s. of the covariance function

$$R(t_1,t_2) := E[X_1(t_1)X_1(t_2)].$$

Deo [7] obtained Corollary 2.1 under the stronger condition

$$\lim_{n\to\infty} n^\alpha r(n) = 0, \quad \text{for some } \ \alpha > 1.$$

Corollary 2.2 is easily deducible from the L.I.L. for empirical processes (see e.g. Theorem 8.6 in Ledoux and Talagrand [12]).

3. ON THE LOCAL L.I.L. FOR GAUSSIAN PROCESSES.

In this section, we consider the local L.I.L. for Gaussian processes.

Theorem 3.1. Let $\{X(t) : t \in T\}$ be a centered Gaussian process and let ρ be a pseudometric on T. Let w be a positive function defined on $(0,1]$. Assume that the following conditions are satisfied:

(i) If $t \in T$ and $0 \le u \le 1$, then $ut \in T$.

(ii) For each $s, t \in T$, the following limit exists

$$\lim_{u \to 0+} E\left[\frac{X(ut)X(us)}{w^2(u)}\right] =: R(s,t). \tag{3.1}$$

(iii) For each $m \ge 1$, each $\epsilon > 0$, each $t_1, \dots, t_m \in T$ and each $\lambda_1, \dots, \lambda_m \in I\!R$

$$\lim_{r \to 1-} \limsup_{u \to 0+} \sup_{v:\, ue^{-(\log u^{-1})^r} \le v \le ue^{-(\log u^{-1})^\epsilon}} \sum_{j,k=1}^{m} \lambda_j \lambda_k E\left[\frac{X(ut_j)X(vt_k)}{w(u)w(v)}\right] \le 0. \tag{3.2}$$

(iv) $\sup_{t \in T} |Z(u,t)| \xrightarrow{\Pr} 0$ as $u \to 0+$, where

$$Z(u,t) := \frac{X(ut)}{w(u)(2\log\log u^{-1})^{1/2}}. \tag{3.3}$$

(v) (T, ρ) is totally bounded.

(vi) For each $\eta > 0$, there is a $\delta > 0$ such that

$$\limsup_{\theta \to 1-} \sum_{n=1}^{\infty} \exp\left(\frac{-\eta w^2(\theta^n)\log n}{\sup_{\substack{s,t \in T\\ \rho(s,t) \le \delta}} \|X(\theta^n t) - X(\theta^n s)\|_2^2}\right) < \infty. \tag{3.4}$$

(vii) $\lim_{u \to 1-} \rho(t, ut) = 0$ for each $t \in T$.

Then, with probability one, $\{Z(u,t) : t \in T\}$ is relatively compact in $l_\infty(T)$, as $u \to 0+$, and its limit set in $l_\infty(T)$, as $u \to 0+$, is the unit ball K of the reproducing kernel Hilbert space of the covariance function $R(s,t)$.

Proof. First, we see that condition (ii) implies that $R(s,t)$ is the covariance function of a self–similar Gaussian process. This fact is similar to Theorem 2 in Lamperti [11]. The difference is that we do not assume that the function w to be increasing. We refer to this reference for the definition and main properties of self–similar (also called semi–stable) processes (see also Mandelbrot and van Ness [13]). We claim that there exists a $\gamma > 0$ such that

$$R(as, at) = a^\gamma R(s,t) \text{ for each } s, t \in T \text{ and each } 0 < a \le 1. \tag{3.5}$$

If $R(t,t) = 0$ for each $t \in T$, (3.5) is trivially true. Otherwise, there exists a $t_0 \in T$ such that $R(t_0, t_0) \ne 0$. We have that

$$\lim_{u \to 0+} \frac{w^2(au)}{w^2(u)} = \lim_{u \to 0+} \frac{w^2(au)}{E[X^2(aut_0)]} \frac{E[X^2(aut_0)]}{w^2(u)} = \frac{R(at_0, at_0)}{R(t_0, t_0)}$$

for each $0 < a < 1$. We also have that

$$\rho^2(ut_0, t_0) = R(ut_0, ut_0) - 2R(ut_0, t_0) + R(t_0, t_0) \to 0$$

as $u \to 1-$. So, $R(t_0, ut_0) \neq 0$, for any u in a left neighborhood of 1. Hence, by e.g. Theorem 1.4.1 in Bingham et al. [4], $w(u)$ is a regularly varying function at 0 and there is a real number γ such that

$$\lim_{u \to 0+} \frac{w^2(au)}{w^2(u)} = a^\gamma, \quad \text{for each } a > 0.$$

Therefore, (3.5) holds. Since (T, ρ) is totally bounded, there are $s, t \in T$ such that $\rho(s, t) \neq 0$. We have that $\rho(as, at) = a^{\gamma/2} \rho(s, t)$ for each $0 < a < 1$. Since (T, ρ) is totally bounded, $\gamma > 0$ (by Theorem 8.5.1 in Bingham et al. [4] $\gamma \neq 0$).

Next, we prove that

$$\lim_{u \to 1-} \sup_{t \in T} \rho(t, ut) = 0. \tag{3.6}$$

Given $\epsilon > 0$, take a δ–covering $t_1, \dots, t_p$ of T, i.e. for each $t \in T$, there is $1 \leq j \leq p$ such that $\rho(t, t_j) \leq \delta$. We have that

$$\rho(t, ut) \leq \rho(t, t_j) + \rho(t_j, ut_j) + \rho(ut_j, ut) \leq \delta(1 + u^{\gamma/2}) + \rho(t_j, ut_j).$$

From this and hypothesis (vii), (3.6) follows.

By Theorem 2.1, with probability one, $\{Z(\theta^n, t) : t \in T\}$ is relatively compact in $l_\infty(T)$ and its limit set is K, for each $0 < \theta < 1$. Here, we use hypotheses (ii)–(v.) So, to end the proof, it suffices to show that there is a constant $A(\theta)$ such that

$$\limsup_{n \to \infty} \sup_{\theta^{n+1} \leq u \leq \theta^n} \sup_{t \in T} |Z(u, t) - Z(\theta^n, t)| \leq A(\theta) \quad \text{a.s.} \tag{3.7}$$

for each $0 < \theta < 1$, and $A(\theta) \to 0$ as $\theta \to 1-$. By hypothesis (vi) and (3.6), given $\eta > 0$, there are $\delta > 0$ and $0 < \theta_0 < 1$ such that

$$\sum_{n=1}^{\infty} \exp\left(\frac{-\eta w^2(\theta^n) \log n}{\sup_{\substack{s,t \in T \\ \rho(s,t) \leq \delta}} \|X(\theta^n t) - X(\theta^n s)\|_2^2} \right) < \infty \tag{3.8}$$

for each $\theta_0 < \theta < 1$ and

$$\sup_{\theta_0 \leq u \leq 1} \sup_{t \in T} \rho(t, ut) \leq \delta. \tag{3.9}$$

We have that

$$|Z(u, t) - Z(\theta^n, t)| \leq \frac{|X(\theta^n t) - X(ut)|}{w(u)(2 \log \log u^{-1})^{1/2}} + \left| \frac{w(\theta^n)(2 \log \log \theta^{-n})^{1/2}}{w(u)(2 \log \log u^{-1})^{1/2}} - 1 \right| |Z(\theta^n, t)| \tag{3.10}$$

and

$$\frac{|X(\theta^n t) - X(ut)|}{w(u)(2 \log \log u^{-1})^{1/2}} \leq \frac{w(\theta^n)(2 \log \log \theta^{-n})^{1/2}}{w(u)(2 \log \log u^{-1})^{1/2}} |Z(\theta^n, t) - Z(\theta^n, u\theta^{-n} t)|.$$

By Theorem 1.5.2 in Bingham et al. [4]

$$\lim_{n \to \infty} \sup_{\theta^{n+1} \leq u \leq \theta^n} \left| \frac{w(\theta^n)}{w(u)} \right| = \theta^{-\gamma/2}. \tag{3.11}$$

By Lemma 2.5 and hypothesis (iv) and (3.8),

$$\limsup_{n\to\infty} \sup_{\substack{s,t\in T\\ \rho(s,t)\le\delta}} |Z(\theta^n, s) - Z(\theta^n, t)| \le (2\eta)^{1/2} \quad \text{a.s.}$$

for $\theta_0 < \theta < 1$. From this and (3.9)

$$\limsup_{n\to\infty} \sup_{\theta^{n+1}\le u\le\theta^n} \sup_{t\in T} |Z(\theta^n, t) - Z(\theta^n, u\theta^{-n}t)| \le (2\eta)^{1/2} \quad \text{a.s.}$$

for $\theta_0 < \theta < 1$. Last fact and (3.11) imply that

$$\limsup_{n\to\infty} \sup_{\theta^{n+1}\le u\le\theta^n} \sup_{t\in T} \frac{|X(\theta^n t) - X(ut)|}{w(u)(2\log\log u^{-1})^{1/2}} \le \theta^{-\gamma/2}(2\eta)^{1/2} \quad \text{a.s.} \tag{3.12}$$

for $\theta_0 < \theta < 1$. Again, by Theorem 1.5.2 in Bingham et al. [4]

$$\lim_{n\to\infty} \sup_{\theta^{n+1}\le u\le\theta^n} \left| \frac{w(\theta^n)}{w(u)} - 1 \right| = \theta^{-\gamma/2} - 1 \tag{3.13}$$

for each $0 < \theta < 1$. By Theorem 2.1

$$\limsup_{n\to\infty} \sup_{t\in T} |Z(\theta^n, t)| = \sup_{t\in T} (R(t,t))^{1/2} \quad \text{a.s.} \tag{3.14}$$

From (3.13) and (3.14), it follows that

$$\begin{aligned} \limsup_{n\to\infty} \sup_{\theta^{n+1}\le u\le\theta^n} \sup_{t\in T} \left| \frac{w(\theta^n)(2\log\log\theta^{-n})^{1/2}}{w(u)(2\log\log u^{-1})^{1/2}} - 1 \right| |Z(\theta^n, t)| \\ \le (\theta^{-\gamma/2} - 1) \sup_{t\in T} (R(t,t))^{1/2} \quad \text{a.s.} \end{aligned} \tag{3.15}$$

for each $0 < \theta < 1$. Finally observe that (3.10), (3.12) and (3.15) imply (3.7). $\square$

The comments on Section 2 on optimality of hypotheses apply to previous theorem. Conditions (i) and (ii) more than conditions are part of the set–up. A condition like (iii) is needed to obtain the L.I.L. for the finite dimensional distributions (see Example 3.1 below). This condition (iii) is weak enough to allow us to obtain the L.I.L. for self–similar Gaussian processes under best possible conditions (see Corollary 3.1). Conditions (i) and (ii), and the compact law of the iterated logarithm with limit set K, imply conditions (iv), (v) and

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{\substack{s,t\in T\\ \rho(s,t)\le\delta}} \frac{|X(ut) - X(us)|}{w(u)(2\log\log u^{-1})^{1/2}} = 0 \quad \text{a.s.} \tag{3.16}$$

with $\rho^2(s,t) = R(s,s) + R(t,t) - 2R(s,t)$. Condition (vi) seems to the right condition to obtain (3.16), since an analogous condition is also sufficient in a similar L.I.L. for empirical processes (see Arcones [1]). Observe that condition (vi) is satisfied if

$$\lim_{\delta\to 0} \lim_{u\to 0+} \sup_{\substack{s,t\in T\\ \rho(s,t)\le\delta}} \frac{\|X(ut) - X(us)\|_2}{w(u)} = 0. \tag{3.17}$$

Conditions (vii) is a very weak regularity condition. We also must observe that (iii) is implied for the stronger condition:

(iii)' For each $m \geq 1$, each $t_1, \ldots, t_m \in T$ and each $\lambda_1, \ldots, \lambda_m \in I\!R$

$$\lim_{\substack{r\to 1+\\ c\to 0+}} \limsup_{u\to 0+} \sup_{v:\, u^r \leq v \leq cu} \sum_{j,k=1}^{m} \lambda_j \lambda_k E\left[\frac{X(ut_j)X(vt_k)}{w(u)w(v)}\right] \leq 0, \tag{3.18}$$

which is easier to check and holds in all the examples considered.

In the case that the Gaussian process is self–similar, the hypotheses in the previous theorem simplify:

Corollary 3.1. Let $\{X(t) : t \in T\}$ be a mean–zero Gaussian process and let $\gamma > 0$. Suppose that the following conditions are satisfied:

(i) If $t \in T$ and $0 \leq u \leq 1$, then $ut \in T$.

(ii) $E[X(ut)X(us)] = u^{2\gamma} E[X(t)X(s)]$ for each $0 \leq u \leq 1$ and each $s, t \in T$.

(iii) $\sup_{t\in T} |X(t)| < \infty$ a.s.

(iv) $\lim_{u\to 1-} E[X(ut)X(t)] = E[X^2(t)]$ for each $t \in T$.

(v) For each $m \geq 1$, each $t_1, \ldots, t_m \in T$ and each $\lambda_1, \ldots, \lambda_m \in I\!R$

$$\limsup_{u\to 0+} \sum_{j,k=1}^{m} \lambda_j \lambda_k u^{-\gamma} E[X(t_j)X(ut_k)] \leq 0.$$

Then, with probability one,

$$\left\{ \frac{X(ut)}{u^{\gamma}(2\log\log u^{-1})^{1/2}} : t \in T \right\}$$

is relatively compact in $l_\infty(T)$, as $u \to 0+$, and its limit set, as $u \to 0+$, is the unit ball K of the reproducing kernel Hilbert space of the Gaussian process $\{X(t) : t \in T\}$.

Proof. Without loss of generality we may assume that $E[X^2(t_0)] > 0$ for some $t_0 \in T$. We apply Theorem 1. Observe that by self–similarity $\sup_{t\in T} \frac{|X(ut)|}{u^{\gamma}(2\log\log u^{-1})^{1/2}}$ has the distribution as $\sup_{t\in T} \frac{|X(t)|}{(2\log\log u^{-1})^{1/2}}$. So, condition (ii) in Theorem 3.1 follows. We also have that by the Sudakov inequality (see e.g. Theorem 3.18 in Ledoux and Talagrand [12]), (T, ρ) is totally bounded, where $\rho^2(t, s) = E[(X(t) - X(s))^2]$. The rest of the conditions in that theorem follow trivially. $\square$

Observe that condition (v) in Corollary 3.1 is satisfied if $\lim_{u\to 0+} u^{-\gamma} E[X(t)X(us)] = 0$, for each $s, t \in T$.

Example 3.1. Let $T = [0, 1]$, let g be a standard normal random variable and let $\gamma > 0$. Consider the Gaussian process $\{X(t) : t \in T\}$ defined by $X(t) = t^{\gamma} g$. This Gaussian process satisfies conditions (i)–(iv) in Corollary 3.1. However, it does not satisfy neither condition (v) nor the compact law of the iterated logarithm.

Example 3.2. Let T be a collection of measurable subsets of $\mathbb{R}^d$. Let λ be the Lebesgue measure on $\mathbb{R}^d$. Let $\{X(A) : A \in T\}$ be a centered Gaussian process such that

$$E[X(A)X(B)] = \lambda(A \cap B).$$

Suppose that

(i) If $0 \le u \le 1$ and $A \in T$, then $uA \in T$.

(ii) $E[\sup_{A\in T} |X(A)|] < \infty$.

It follows from Corollary 3.1, that, with probability one,

$$\left\{ \frac{X(uA)}{u^{1/2}(2\log\log u^{-1})^{1/2}} : A \in T \right\}$$

is relatively compact, as $u \to 0$, and its limit set, as $u \to 0$, is the unit of the r.k.h.s. of the covariance function $\lambda(A \cap B)$, i.e., the limit set is

$$\left\{ \left(\int_A \alpha(s_1, \dots, s_d)\, ds_d \cdots ds_1 \right)_{A\in T} : \int_{\mathbb{R}^d} \alpha^2(s_1, \dots, s_d)\, ds_d \cdots ds_1 \le 1 \right\}.$$

Observe that $\lim_{u\to 1-} \lambda((uA) \cap A) = \lambda(A)$, for each measurable set A, with finite Lebesgue measure, because a standard argument based on approximation by open sets. By the Sudakov minorization (see e.g. Theorem 3.18 in Ledoux and Talagrand [12]), $\sup_{A\in T} \lambda(A) < \infty$. So, condition (iv) in Corollary 3.1 follows. We also have that $u^{-1/2}\lambda(A \cap (uB)) \le u^{1/2}\lambda(B)$ for each $A, B \in T$. Hence, condition (v) in Corollary 3.1 is satisfied.

In particular, if $T = \{[0, t_1] \times \cdots \times [0, t_d] : 0 \le t_1, \dots, t_d \le 1\}$, the process $\{X(t) : t \in T\}$ is a Brownian sheet, i.e.

$$E[X(t)X(s)] = \prod_{j=1}^{d} (t_j \wedge s_j)$$

for each $t, s \in [0, 1]^d$, where $t = (t_1, \dots, t_d)$ and $s = (s_1, \dots, s_d)$. So, we have that, with probability one,

$$\{u^{-1/2}(2\log\log u^{-1})^{-1/2}X(ut) : t \in [0, 1]^d\}$$

is relatively compact, as $u \to 0$, and its limit set, as $u \to 0$, is

$$\left\{ \left(\int_0^{t_1} \cdots \int_0^{t_d} \alpha(s_1, \dots, s_d)\, ds_d \dots ds_1 \right)_{t\in[0,1]^d} : \int_0^1 \cdots \int_0^1 \alpha^2(s_1, \dots, s_d)\, ds_d \cdots ds_1 \le 1 \right\}.$$

4. A LOCAL LAW OF THE ITERATED LOGARITHM FOR COMPOSITIONS OF GAUSSIAN PROCESSES.

In this section, we consider the law of the iterated logarithm for compositions of Gaussian processes. First, we present a variation of Theorem 3.1, which is more suited for the applications in this section.

Theorem 4.1. Let $\{X(t) : t \in T\}$ be a centered Gaussian process, let ρ be a pseudometric on T and let $w, \tau : (0, 1] \to (0, \infty)$ be two functions. Assume that the following conditions are satisfied:

(i) If $t \in T$ and $0 \le u \le 1$, then $ut \in T$.
(ii) For each $s, t \in T$ the following limit exists

$$\lim_{u \to 0+} E\left[\frac{X(\tau(u)s)X(\tau(u)t)}{w^2(u)}\right] =: R(s,t).$$

(iii) For each $m \ge 1$, each $\epsilon > 0$, each $t_1, \ldots, t_m \in T$ and each $\lambda_1, \ldots, \lambda_m \in I\!R$

$$\lim_{r \to 1-} \limsup_{u \to 0+} \sup_{v:\, ue^{-(\log u^{-1})^r} \le v \le ue^{-(\log u^{-1})^\epsilon}} \sum_{j,k=1}^{m} \lambda_j \lambda_k E\left[\frac{X(ut_j)X(vt_k)}{w(u)w(v)}\right] \le 0.$$

(iv)

$$\sup_{t \in T} \frac{|X(\tau(u)t)|}{w(u)(2\log\log u^{-1})^{1/2}} \xrightarrow{\Pr} 0 \quad \text{as} \quad u \to 0+\,.$$

(v) (T, ρ) is totally bounded.
(vi) For each $\eta > 0$, there is a $\delta > 0$ such that

$$\limsup_{\theta \to 1-} \sum_{n=1}^{\infty} \exp\left(\frac{-\eta w^2(\theta^n)\log n}{\sup_{\substack{s,t \in T \\ \rho(s,t) \le \delta}} \|X(\tau(\theta^n)t) - X(\tau(\theta^n)s)\|_2^2}\right) < \infty.$$

(vii) $\tau(u)$ is an nondecreasing function, which tends to 0, as $u \to 0+$, and

$$\lim_{\theta \to 1-} \limsup_{n \to \infty} \left|\frac{\tau(\theta^{n+1})}{\tau(\theta^n)} - 1\right| = 0.$$

(viii)

$$\lim_{\theta \to 1-} \limsup_{n \to \infty} \sup_{\theta^{n+1} \le u \le \theta^n} \left|\frac{w(u)}{w(\theta^n)} - 1\right| = 0.$$

(ix) $\lim_{u \to 1-} \sup_{t \in T} \rho(t, ut) = 0$.
Then, with probability one,

$$\left\{\frac{X(\tau(u)t)}{w(u)(2\log\log u^{-1})^{1/2}} : t \in T\right\} \tag{4.1}$$

is relatively compact in $l_\infty(T)$, as $u \to 0+$, and its limit set is the unit ball K of the reproducing kernel Hilbert space of the covariance function $R(s,t)$.

The proof of last theorem is very similar to that of Theorem 3.1 and it is omited. If the function $\tau(\cdot)$ is not continuous, last theorem is not just a change of scale in Theorem 3.1.

Next, we recall the definition of r.k.h.s. of a covariance function in the multivariate case. Let T_j be a parameter set for $1 \le j \le d$. Let $R_{j,k}(t_j, t_k)$, $1 \le j \le k \le d$, be joint covariance functions, i.e. $R_{j,k}(t_j, t_k) = R_{k,j}(t_k, t_j)$ for each $t_j \in T_j$ and each $t_k \in T_k$, and

$$\sum_{j,k=1}^{d} \sum_{l,m=1}^{p} a_{j,l} a_{k,m} R_{j,k}(t_{j,l}, t_{k,m}) \ge 0 \tag{4.2}$$

where $a_{j,l} \in I\!R$ and $t_{j,l} \in T_j$, for each $1 \le j \le d$ and each $1 \le l \le p$. Then, there are Gaussian processes $\{Z_j(t_j) : t_j \in T_j\}$, $1 \le j \le d$, defined in the same probability space, such that $R_{j,k}(t_j, t_k) = E[Z_j(t_j)Z_k(t_k)]$, for each $t_j \in T_j$ and each $t_k \in T_k$. Let $\mathcal{L}$ be the linear subspace of L_2, generated by $\{Z_j(t_j) : t_j \in T_j, 1 \le j \le d\}$. Then, the r.k.h.s. of the joint covariance functions $R_{j,k}(t_j, t_k)$ is the class of functions on $T_1 \times \cdots \times T_d$

$$\{(E[Z_1(t_1)\xi], \dots, E[Z_d(t_d)\xi])_{t_1 \in T_1, \dots, t_d \in T_d} : \xi \in \mathcal{L}\} \tag{4.3}$$

This space is endowed of the inner product

$$< f_1, f_2 >:= E[\xi_1 \xi_2],$$

where $f_i(t_1, \dots, t_d) = (E[Z_1(t_1)\xi_i], \dots, E[Z_d(t_d)\xi_i])$ each $t_1 \in T_1, \dots, t_d \in T_d$ and each $i = 1, 2$. The unit ball of this r.k.h.s. is

$$K := \{(E[Z_1(t_1)\xi], \dots, E[Z_d(t_d)\xi])_{t_1 \in T_1, \dots, t_d \in T_d} : E[\xi^2] \le 1\}. \tag{4.4}$$

Theorem 4.2. Let $\{X_j(t) : t \in T_j\}$ be a centered Gaussian process, let ρ_j be a pseudometric on T_j and let $w_j, \tau_j : (0,1] \to (0,\infty)$ be two functions, for $j = 1, \dots, d$. Assume that, for each $1 \le j \le d$, the conditions in Theorem 4.1 are satified for $\{X_j(t) : t \in T_j\}$, ρ_j, w_j and τ_j. Assume also that the following conditions are satisfied:

(i) For each $t_j \in T_j$ and each $t_k \in T_k$, where $1 \le j, k \le d$, the following limit exists

$$\lim_{u \to 0+} E\left[\frac{X_j(\tau_j(u)t_j)X_k(\tau_k(u)t_k)}{w_j(u)w_k(u)}\right] =: R_{j,k}(t_j, t_k).$$

(ii) For each $p \ge 1$, each $\epsilon > 0$, each $t_{j,1}, \dots, t_{j,p} \in T_j$ and each $\lambda_{j,1}, \dots, \lambda_{j,p} \in I\!R$,

$$\lim_{r \to 1-} \limsup_{u \to 0+} \sup_{v :\, ue^{-(\log u^{-1})^r} \le v \le ue^{-(\log u^{-1})^\epsilon}} \sum_{j,k=1}^{d} \sum_{l,m=1}^{p} \lambda_{j,l}\lambda_{k,m} E\left[\frac{X_j(\tau_j(u)t_{j,l})X_k(\tau_k(v)t_{k,m})}{w_j(u)w_k(v)}\right] \le 0.$$

Then, with probability one,

$$\left\{\left(\frac{X_1(\tau_1(u)t_1)}{w_1(u)(2\log\log u^{-1})^{1/2}}, \dots, \frac{X_d(\tau_d(u)t_d)}{w_d(u)(2\log\log u^{-1})^{1/2}}\right) : t_1 \in T_1, \dots, t_d \in T_d\right\} \tag{4.5}$$

is relatively compact in $l_\infty(T_1 \times \cdots \times T_d)$ and its limit set is the unit ball K of the reproducing kernel Hilbert space of the covariance function $R_{j,k}(t_j, t_k)$.

Proof. By Lemma 2.1

$$\limsup_{n \to \infty} \sum_{j=1}^{d} \sum_{l=1}^{p} \lambda_{j,l} \frac{X_j(\tau_j(\theta^n)t_{j,l})}{w_j(\theta^n)(2\log n)^{1/2}} = \left(\sum_{j,k=1}^{d} \sum_{l,m=1}^{p} \lambda_{j,l}\lambda_{k,m} R_{j,k}(t_{j,l}, t_{k,m})\right)^{1/2} \quad \text{a.s.}$$

for each $0 < \theta < 1$, each $\lambda_{j,l} \in \mathbb{R}$ and each $t_{j,l} \in T_j$. So, this implies the compact law of the iterated logarithm for the finite dimensional distributions of the process

$$\left\{ \left(\frac{X_1(\tau_1(\theta^n)t_1)}{w_1(\theta^n)(2\log\log\theta^{-n})^{1/2}}, \dots, \frac{X_d(\tau_d(\theta^n)t_d)}{w_d(\theta^n)(2\log\log\theta^{-n})^{1/2}} \right) : t_1 \in T_1, \dots, t_d \in T_d \right\}. \tag{4.6}$$

The same arguments as in Theorem 3.1 imply the uniform L.I.L. for the process in (4.6). Again, by the arguments in Theorem 3.1, the blocking, i.e. (3.7), holds for each $1 \le j \le d$. So, the result follows. $\square$

From Theorem 4.2, it is easy to get the following law of the iterated logarithm for compositions of Gaussian processes.

Corollary 4.1. Let $\{X_j(t) : t \in T_j\}$ be a centered Gaussian process, let ρ_j be a pseudometric on T_j and let $w_j, \tau_j : (0,1] \to (0,\infty)$ be functions, for $1 \le j \le d$. Assume that the conditions in Theorem 4.2 are satisfied. Assume also that:

(i) $T_1 = \cdots = T_{d-1} = [-M, M]$, where M is so large that there is a $\eta > 0$ such that

$$\limsup_{u\to 0+} \sup_{t_j \in T_j} \left| \frac{X_j(\tau_j(u)t_j)}{w_j(u)(2\log\log u^{-1})^{1/2}} \right| \le M - \eta \quad \text{a.s.}$$

for $j = 2, \dots, d$.

(ii) $\tau_j(u) = w_{j+1}(u)(2\log\log u^{-1})^{1/2}$, for $j = 1, \dots, d-1$.

Then, with probability one,

$$\left\{ \frac{X_1 \circ \cdots \circ X_d(\tau_d(u)t_d)}{w_1(u)(2\log\log u^{-1})^{1/2}} : t_d \in T_d \right\}$$

is relatively compact, as $u \to 0+$, in $l_\infty(T_d)$ and its limit set, as $u \to 0+$, is

$$\left\{ (f_1 \circ \cdots \circ f_d(t_d))_{t_d \in T_d} \ : (f_1, \dots, f_d) \in K \right\}.$$

Of course, we could have taken absolute values, before taking a composition, i.e. get a L.I.L for $X_1(|\cdots(|X_d(\tau_d(u)t_d)|)\cdots|)$. Another variation is when a Gaussian process is composed with itself:

Corollary 4.2. Let $\{X(t) : t \ge 0\}$ be a mean–zero Gaussian process, let $b > 0$ and let $\gamma > 0$, $\gamma \ne 1$. Assume that the conditions are satisfied:

(i) $E[X(ut)X(us)] = u^{2\gamma}E[X(t)X(s)]$ for each $u, s, t \ge 0$.

(ii) $\sup_{0\le t\le 1} |X(t)| < \infty$ a.s.

(iii) $\lim_{u\to 1-} E[X(ut)X(t)] = E[X^2(t)]$ for each $t \ge 0$.

(iv) For each $s, t \ge 0$

$$\lim_{u\to 0+} u^{-\gamma}E[X(s)X(ut)] = 0.$$

Then, with probability one,

$$\{\frac{X(|X(ut)|)}{u^{\gamma^2}(2\log\log u^{-1})^{(\gamma+1)/2}} : \ 0 \le t \le b\} \tag{4.7}$$

is relatively compact, as $u \to 0+$, in $l_\infty([0,b])$ and its limit set is

$$\{(\alpha(|\beta(t)|))_{0\le t\le b} : (\alpha,\beta) \in K\}$$

where K is the unit ball of the r.k.h.s. of the process $\{(X(s),Y(t)) : 0 \le s \le M, 0 \le t \le 1\}$, $\{Y(t) : t \in I\!\!R\}$ is an independent copy of the process $\{X(t) : t \in I\!\!R\}$ and $M^2 > E[X^2(b)]$.

Moreover, with probability one,

$$\left\{\frac{X(|X(ub)|)}{u^{\gamma^2}(2\log\log u^{-1})^{(\gamma+1)/2}} :\right\} \tag{4.8}$$

is relatively compact, as $u \to 0+$ and its limit set is $[-\sigma,\sigma]$, where

$$\sigma := \sup_{0\le r\le m(b)} m(r)(1-r^2(m(b))^{-2})^{1/2} \quad \text{a.s.} \tag{4.9}$$

and $m(t) = (E[X^2(t)])^{1/2}$.

Proof. Assume that $0 < \gamma < 1$ (the case $\gamma > 1$ is similar). Let $\tau(u) = u^\gamma(2\log\log u^{-1})^{1/2}$. By Theorem 4.2 and the conditions checked in Corollary 3.1, in order to prove the first part of the claim, it suffices to show that

$$E\left[\frac{X(\tau(u)s)X(ut)}{(\tau(u))^\gamma u^\gamma}\right] \to 0, \quad \text{as} \quad u \to 0+, \tag{4.10}$$

for each $s,t \ge 0$, and that

$$\sup_{u^r\le v\le cu}\left|E\left[\frac{X(\tau(u)s)X(vt)}{(\tau(u))^\gamma v^\gamma}\right]\right| \to 0 \quad \text{as } u \to 0+, \tag{4.11}$$

and

$$\sup_{u^r\le v\le cu}\left|E\left[\frac{X(\tau(v)s)X(u)t)}{(\tau(v))^\gamma u^\gamma}\right]\right| \to 0, \quad \text{as } u \to 0+, \tag{4.12}$$

for each $s,t \ge 0$, where $1 < r < \gamma^{-1}$ and $0 < c < 1$. By hypotheses (i) and (iv)

$$E\left[\frac{X(\tau(u)s)X(ut)}{(\tau(u))^\gamma u^\gamma}\right] = E\left[\frac{X(s)X((u/\tau(u))t)}{(u/\tau(u))^\gamma}\right] \to 0 \quad \text{as } u \to 0+.$$

(4.11) and (4.12) follow by the same argument.

To show (4.8), it suffices to show that

$$\{\alpha(|\beta(b)|) : (\alpha,\beta) \in K\} = [-\sigma,\sigma]. \tag{4.13}$$

Let $\mathcal{L}_1$ be the linear subspace of L_2 generated by $\{X(s) : 0 \le s \le M\}$. Let $\mathcal{L}_2$ be the linear subspace of L_2 generated by $\{X(t) : 0 \le t \le b\}$. Then, it is easy to see that

$$K = \{(E[X(s)\xi_1], E[Y(t)\xi_2])_{0\le s\le M,\, 0\le t\le b} : \xi_1 \in \mathcal{L}_1, \xi_2 \in \mathcal{L}_2, E[\xi_1^2 + \xi_2^2] \le 1\}.$$

It follows from this that $\{\alpha(|\beta(b)|) : (\alpha, \beta) \in K\}$ is a symmetric closed interval. Hence, it suffices to show that

$$\sup\{\alpha(|\beta(b)|) : (\alpha, \beta) \in K\} = \sigma. \tag{4.14}$$

Given $\xi_1 \in \mathcal{L}_1$ and $\xi_2 \in \mathcal{L}_2$ such that $E[\xi_1^2+\xi_2^2] \le 1$, let $\alpha(s) = E[X(s)\xi_1]$, let $\beta(t) = E[X(t)\xi_2]$ and let $r = |\beta(b)|$. By the Cauchy–Schwarz inequality

$$r \le m(b)\|\xi_2\|_2 \le m(b), \tag{4.15}$$

$$\|\xi_1\|_2^2 \le 1 - \|\xi_2\|_2^2 \le 1 - (m(b))^{-2}r^2$$

and

$$|\alpha(r)| \le m(r)\|\xi_1\|_2 \le m(r)(1 - m^{-2}(b)r^2)^{1/2} \le \sigma.$$

So, in (4.14), the left hand side is smaller o equal than the right hand side. Given $0 \le r \le m(b)$, if

$$\xi_1 = (m(r))^{-1}(1 - (m(b))^{-2}r^2)^{1/2}X(r) \ \text{ and } \xi_2 = rm^{-2}(b)Y(b),$$

then $\alpha(\beta(b)) = m(r)(1 - (m^{-2}(b))^{-2}r^2)^{1/2}$. So, (4.14) follows. □

Example 4.1. A mean–zero Gaussian process $\{X(t) : t \ge 0\}$ is called a fractional Brownian motion of order γ, $1/2 \le \gamma \le 1$, if its covariance is given by

$$E[X(t)X(s)] = 2^{-1}(t^{2\gamma} + s^{2\gamma} - |t - s|^{2\gamma}),\ \ s, t \ge 0. \tag{4.16}$$

This process was introduced in Mandelbrot and van Ness [13]. It is very easy to see that previous corollary applies to this process, if $1/2 \le \gamma < 1$: given $b > 0$, with probability one,

$$\frac{X(|X(ub)|)}{u^{\gamma^2}(2\log\log u^{-1})^{(\gamma+1)/2}}$$

is relatively compact, as $u \to 0+$ and its limit set is $[-\sigma, \sigma]$ where $\sigma = b^{\gamma^2}\gamma^{\gamma/2}(\gamma + 1)^{-(\gamma+1)/2}$.

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References

[1] Arcones, M. A. (1996). Necessary and sufficient conditions for the law of the iterated logarithm of triangular arrays of empirical processes. Preprint.

[2] Arcones, M. A. and Giné, E. (1996). On the law of the iterated logarithm for canonical U–statistics and processes. *Stoch. Proc. Appl.*

[3] Aronszajn, N. (1950). Theory of reproducing kernels. *Trans. Amer. Math. Soc.* **68** 337–404.

[4] Bingham, N. H.; Goldie, C.M. and Teugels, J. L. (1987). *Regular Variation.* Cambridge University Press, Cambridge, United Kingdom.

[5] Borell, C. (1975). The Brunn–Minkowski inequality in Gauss space. *Invent. Math.* **30** 207–216.

[6] Carmona, R. and Kôno, N. (1976). Convergence en loi et lois du logarithme itéré pour les vecteurs gaussiens. *Z. Wahrsch. verw. Geb.* **36** 241–267.

[7] Deo, C. M. (1974). A note on stationary Gaussian sequences. *Ann. Probab.* **2** 954–957.

[8] Finkelstein, H. (1971). The law of the iterated logarithm for empirical distributions. *Ann. Math. Statist.* **42** 607–615.

[9] Lai, T. L. (1973). Gaussian processes, moving averages and quick detection problems. *Ann. Probab.* **1** 825–837.

[10] Lai, T. L. (1974). Reproducing kernel Hilbert spaces and the law of the iterated logarithm for Gaussian processes. *Z. Wahrsch. verw. Gebiete* **29** 7–19.

[11] Lamperti, J. (1962). Semi–stable stochastic processes. *Trans. Amer. Math. Soc.* **104** 62–78.

[12] Ledoux, M. and Talagrand, M. (1991). *Probability in Banach Spaces.* Springer–Verlag, New York.

[13] Mandelbrot, B. B. and van Ness, J. W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM, Review* **10** 422-437.

[14] Mangano, G. C. J. (1976). Sequential compactness of certain sequences of Gaussian random variables with values in $C[0,1]$. *Ann. Probab.* **4** 902–913.

[15] Nisio, M. (1967). On the extreme values of Gaussian processes. *Osaka J. Math.* **4** 313–326.

[16] Oodaira, H. (1972). On Strassen's version of the law of the iterated logarithm for Gaussian processes. *Z. Wahrsch. verw. Gebiete* **21** 289–299.

[17] Talagrand, M. (1987). Regularity of Gaussian processes. *Acta Mathem.* **159** 99–149.