The large deviation principle of stochastic processes [1]

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Abstract

We discuss the large deviation principle of stochastic processes as random elements of $l_\infty(T)$. We show that the large deviation principle in $l_\infty(T)$ is equivalent to the large deviation principle of the finite dimensional distributions plus an exponential asymptotic equicontinuity condition with respect to a pseudometric which makes T a totally bounded pseudometric space. This result allows to obtain necessary and sufficient conditions for the large deviation principle of different types of stochastic processes. We discuss the large deviation principle of Gaussian and Poisson processes. As application, we determine the integrability of the iterated fractional Brownian motion.

April 13, 2004

[1] *AMS 2000 subject classifications.* Primary 60F10.
Key words and phrases. Large deviations, stochastic processes, Gaussian processes, iterated Brownian motion, Poisson process.

1 The large deviation principle for stochastic processes

In many different situations, it is of interest to estimate the rate of convergence of certain probabilities. Often, these probabilities converge exponentially fast. Several authors have considered large deviations and obtained different types of applications mainly to mathematical physics. General references on large deviations are: Bahadur (1971), Varadhan (1984), Deuschel and Stroock (1989), and Dembo and Zeitouni (1998).

We study functional large deviations of stochastic processes following the approach to deal with measurability problems for the weak convergence of stochastic processes in Hoffmann–Jørgensen (1991). We assume very little measurability restrictions and we use outer and inner probabilities. We refer to van der Vaart and Wellner (1996) and Dudley (1999) for measurability considerations. We consider stochastic processes as random elements. By a random element, we mean a (non necessarily measurable) function from a probability space to an arbitrary set. We use the following definition of (LDP) large deviation principle for random elements:

Definition 1.1 *Given a sequence of random elements $\{X_n\}_{n=1}^{\infty}$ with values in a topological space $(S, \mathcal{T})$, a sequence of positive numbers $\{\epsilon_n\}_{n=1}^{\infty}$ such that $\epsilon_n \to 0$, and a function $I : S \to [0, \infty]$, it is said that $\{X_n\}$ satisfies the (LDP) large deviation principle with speed ϵ_n^{-1} and with rate function I if:*

(i) For each $0 \le c < \infty$, $\{z \in S : I(z) \le c\}$ is a compact set.

(ii) For each set $A \subset S$,

$$\begin{aligned} & -I(A^o) \le \liminf_{n\to\infty} \epsilon_n \log(\Pr{}_*\{X_n \in A\}) \\ \le & \limsup_{n\to\infty} \epsilon_n \log(\Pr{}^*\{X_n \in A\}) \le -I(\bar{A}), \end{aligned}$$

where for a set B, $I(B) = \inf\{I(x) : x \in B\}$.

In the previous definition and in the future, we denote $\inf(\emptyset) = \infty$. Condition (i) in Definition 1.1 implies that I is a lower semicontinuous function. The assumptions in Definition 1.1 imply that there exists a $z \in S$ such that $I(z) = 0$. Typically, z is unique and $X_n \overset{\Pr}{\to} z$. A function $I : S \to [0, \infty]$ is called a good rate function if condition (i) in Definition 1.1 holds.

We study the LDP for a sequence of stochastic processes $\{U_n(t) : t \in T\}$ with values in a Banach space B, which are bounded with probability one, where T is an index set. We consider $\{U_n(t) : t \in T\}$ as a random element with values in the Banach space $l_\infty(T, B)$, the set of bounded functions in T with values in B with the norm $|z|_\infty := \sup_{t\in T} |z(t)|_B$, where $|\cdot|_B$ denotes the norm in B. We do not assume that $\{U_n(t) : t \in T\}$ is a random variable with values in $l_\infty(T, B)$ endowed with the Borel σ–field. We only assume that for each $t \in T$, $U_n(t)$ is a r.v.

Definition 1.2 *Given a sequence of stochastic processes* $\{U_n(t) : t \in T\}$ *with values in a Banach space* B*, such that for* n *large enough* $\Pr_*\{\sup_{t\in T} |U_n(t)|_B < \infty\} = 1$*, a sequence of positive numbers* $\{\epsilon_n\}_{n=1}^\infty$ *such that* $\epsilon_n \to 0$*, and a function* $I : l_\infty(T, B) \to [0, \infty]$*, we say that* $\{U_n(t) : t \in T\}$ *satisfies the LDP in* $l_\infty(T, B)$ *with speed* ϵ_n^{-1} *and with rate function* I *if:*

(i) For each $0 \le c < \infty$*,* $\{z \in l_\infty(T, B) : I(z) \le c\}$ *is a compact set of* $l_\infty(T, B)$*.*

(ii) For each set $A \subset l_\infty(T, B)$*,*

$$\begin{aligned} & -I(A^o) \le \liminf_{n\to\infty} \epsilon_n \log(\Pr_*\{\{U_n(t) : t \in T\} \in A\}) \\ \le \quad & \limsup_{n\to\infty} \epsilon_n \log(\Pr^*\{\{U_n(t) : t \in T\} \in A\}) \le -I(\bar{A}). \end{aligned}$$

We denote $l_\infty(T) = l_\infty(T, \mathbb{R})$. It is easy to see that a sequence of stochastic processes $\{U_n(t) : t \in T\}$ with values in a Banach space B satisfies the LDP in $l_\infty(T, B)$ with speed ϵ_n^{-1} if and only if the sequence of stochastic processes $\{V_n(t, f) : t \in T, f \in B_1^*\}$ satisfies the LDP in $l_\infty(T)$ with speed ϵ_n^{-1}, where B_1^* is the unit ball of the dual of B and $V_n(t, f) = f(U_n(t))$, for $t \in T$ and $f \in B_1^*$. So, the study of the LDP for stochastic processes with values in a Banach space reduces to the study of the LDP for real valued stochastic processes. We will usually consider stochastic processes with values in $\mathbb{R}$. But, sometimes, we will need to consider multidimensional valued stochastic processes. We also have that a sequence r.v.'s $\{X_n\}_{n=1}^\infty$ with values in a Banach space B satisfies the LDP with speed ϵ_n^{-1} if and only if the stochastic process $\{f(X_n) : f \in B_1^*\}$ satisfies the LDP with speed ϵ_n^{-1}. So, our results give necessary and sufficient conditions for the LDP of Banach space valued r.v.'s (see Corollary 3.6).

It is well known that functional formulations of limit theorems have many different applications (see for example van der Vaart and Wellner, 1996; and Dudley, 1999). We will use the functional LDP to obtain the tail behavior of the iterated fractional Brownian motion. For stochastic processes whose paths are not bounded in T, but they are bounded in subsets of T, it is possible to obtain a LDP in another spaces (see Theorem 3.9).

In Section 2, we present an extension of the contraction principle. The contraction principle says that we may apply a continuous function to a sequence of random elements satisfying the LDP and still have the LDP for the transformed sequence. We extend this technique to not necessarily continuous functions. We will need this extended contraction principle because the composition of stochastic processes is not a continuous functional in $l_\infty(\mathbb{R}) \times l_\infty([0, M])$, where $M > 0$.

In Section 3, we show that a sequence of bounded stochastic processes satisfies the LDP if and only if the finite dimensional distributions satisfy the LDP and an exponential asymptotic equicontinuity condition holds with respect to certain pseudometric which makes T totally bounded. Some applications of this characterization are given. We see that the LDP in Definition

1.2 with $B = I\!R$ implies that (T, ρ) is a totally bounded pseudometric space and an exponential asymptotic equicontinuity condition holds with respect to this pseudometric, where

$$\rho(s,t) = \sum_{k=1}^{\infty} k^{-2} \min(\rho_k(s,t), 1) \tag{1.1}$$

and

$$\rho_k(s,t) = \sup\{|u_2 - u_1| : I_{s,t}(u_1, u_2) \le k\} \tag{1.2}$$

and $I_{s,t}$ is the rate function for the LDP of $(U_n(s), U_n(t))$. An easy method (see Theorem 6.3) to prove that a sequence of stochastic processes $\{U_n(t) : t \in T\}$ in $l_\infty(T)$ does not satisfy the LDP is to check that (T, ρ_k) is not totally bounded. Several applications of this characterization of the LDP are given. We obtain necessary and sufficient conditions for the LDP of a sequence of r.v.'s with values in a separable Banach space. We obtain minimal sufficient conditions for the LDP of stochastic processes with either increasing or convex paths. We present minimal conditions for obtaining that the composition of two stochastic processes satisfies the LDP.

Several authors have studied the tightness in the large deviation principle in a similar way weak convergence is studied. Puhalskii (1991) showed that is a sequence of r.v.'s with values in a metric space are exponential tight, then there exists a subsequence satisfying the large deviation principle. He also showed that for stochastic processes with values in $D[0, M]$ an exponential asymptotic equicontinuity condition implies tightness.

In Section 4, we give the form of the rate function for the LDP in $l_\infty(T)$ for many of the considered cases. We will that under certain conditions the rate function is given by

$$I(z) = \inf\left\{\int \Psi(\gamma(x))\, d\mu(x) : \int f(x,t)\gamma(x)\, d\mu(x) = z(t) \text{ for each } t \in T\right\},$$

where Ψ is a convex function, μ is a (positive) measure in a measurable space $(S, \mathcal{S})$ and $\{f(x,t) : t \in T\}$ is a class of measurable functions. In the cases considered in this paper either $\Phi(x) = p^{-1}|x|^p$, for some $p > 0$ or $\Phi(x) = e^x - 1$. We also see that under certain conditions, the rate of function in the LDP of some certain stochastic processes has the form

$$I(z) = \begin{cases} \int_0^M \Psi(z'(t))\, dt, & \text{if } z(0) = 0 \text{ and } z \text{ is absolutely continuous} \\ \infty, & \text{else.} \end{cases} \tag{1.3}$$

We obtain the rate function of the the composition of two stochastic proceses, when the rate of each of the considered stochastic processes has the form in (1.3) with different functions Ψ.

In Section 5, we give necessary and sufficient conditions for the LDP of a sequence of Gaussian processes. Several applications to the iterated Brownian motion are presented. A Brownian motion $\{B(t) : t \in I\!R\}$ is a centered Gaussian process with covariance

$$E[B(s)B(t)] = \min(|s|, |t|)I(st \geq 0), s, t \in I\!R.$$

If $\{B_1(t) : t \geq 0\}$ and $\{B_2(t) : t \geq 0\}$ are two independent Brownian motion, then $\{B(t) : t \in I\!R\}$ is a Brownian motion, where $B(t) = B_1(t)$ for $t \geq 0$, and $B(t) = B_2(-t)$ for $t \leq 0$. The process $\{B(B(t)) : t \in I\!R\}$ is called an iterated Brownian motion. Some authors call an iterated Brownian process to the process $\{B_1(B_2(t)) : t \in I\!R\}$, where $\{B_1(t) : t \in I\!R\}$ and $\{B_2(t) : t \in I\!R\}$ are two independent Brownian motions. Funaki (1979) used a modification of the iterated Brownian motion to give a probability solution to the partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{1}{8}\frac{\partial^4 u}{\partial x^4} \text{ with } u(0, x) = u_0(x).$$

Deheuvels and Mason (1992), Burdzy (1993, 1994), Arcones (1995); Hu and Shi (1995); Shi (1995; Hu, Pierre–Loti–Viaud, and Shi, (1995); Csáki; Csörgö, Földes, and Révész (1995); Khoshnevisan and Lewis (1996a, 1996b) and Csáki, Földes and Révész (1997) have studied different properties of the iterated Brownian motion. We will prove that for each $0 < M < \infty$,

$$\lim_{\lambda\to\infty} \lambda^{-2^k/(2^k-1)} \log(\Pr\{|B^{(k)}(M)| \geq \lambda\}) = \frac{-(2^{k+1}-2)}{2^{k2^k/(2^k-1)}M^{1/(2^k-1)}}$$

and

$$\lim_{\lambda\to\infty} \lambda^{-2^k/(2^k-1)} \log(\Pr\{\sup_{0\leq t\leq M} |B^{(k)}(t)| \geq \lambda\}) = \frac{-(2^{k+1}-2)}{2^{k2^k/(2^k-1)}M^{1/(2^k-1)}},$$

where $B \circ \overset{(k)}{\cdots} \circ B(t)$. Our results also apply to compositions of independent Brownian motions. We also consider the iterated fractional Brownian motion. Compact laws of the iterated logarithm for the iterated fractional Brownian motion are obtained.

In Section 6, we give necessary and sufficient conditions for the LDP's of a (nonhomogeneous) Poisson process, under different normalizations. Under some normalization, the LDP does not hold in $l_\infty[0, M]$ and we have to consider the LDP in a space of measures.

Given a metric space (S, d), $B(z, \delta)$ denotes the open ball with center z and radius δ. Given a subset A of S and $\delta > 0$, $A(\delta) = \{x \in S : d(x, A) < \delta\}$. In $I\!R^d$, we denote $|z| = (\sum_{i=1}^d z_i^2)^{1/2}$ and $|z|_\infty = \max_{1\leq i\leq d} |z_i|$, where $z = (z_1, \ldots, z_d)$.

2 An Extension of the contraction principle

An useful technique to study large deviations is the contraction principle (Donsker and Varadhan, 1976, Theorem 2.1). We will need the following extension of this theorem:

Theorem 2.1 *Let $\{X_n\}_{n=1}^\infty$ be a sequence of random elements with values in a metric space (S_1, d_1). Let $\{\epsilon_n\}$ be a sequence of positive numbers which converges to zero. Let $\{f_n\}$ be a sequence of Borel functions from (S_1, d_1) into (S_2, d_2), where (S_2, d_2) is a metric space. Let f be a Borel function from S_1 into S_2. Suppose that:*

(i) $\{X_n\}_{n=1}^\infty$ satisfies the LDP with rate ϵ_n^{-1} and rate function I_1.

(ii) If $\{x_n\}$ is a sequence in S_1 such that $x_n \to x$, for some x with $I_1(x) < \infty$, then $f_n(x_n) \to f(x)$.

Then,

(a) For each open set U in S_2,

$$\liminf_{n\to\infty} \epsilon_n \log(\Pr{}_*\{f_n(X_n) \in U\}) \geq -I_2(U),$$

where $I_2(y) = \inf\{I_1(x) : f(x) = y\}$.

(b) For each closed set F of S_2,

$$\limsup_{n\to\infty} \epsilon_n \log(\Pr{}^*\{f_n(X_n) \in F\}) \leq -I_2(F).$$

(c) I_2 is a good rate function.

Proof. To prove part (a), it suffices to show that given $\epsilon > 0$, and $x_0 \in S_1$ with $I_1(x_0) < \infty$, then

$$\liminf_{n\to\infty} \epsilon_n \log(\Pr{}_*\{f_n(X_n) \in B(y_0, \epsilon)\}) \geq -I_1(x_0),$$

where $y_0 = f(x_0)$. By condition (ii), there are $\delta > 0$ and an integer n_0 such that for $n \geq n_0$, then $f_n(B(x_0, \delta)) \subset B(y_0, \epsilon)$. Hence,

$$\begin{aligned} & \liminf\nolimits_{n\to\infty} \epsilon_n \log(\Pr{}_*\{f_n(X_n) \in B(y_0, \epsilon)\}) \\ \geq\ & \liminf\nolimits_{n\to\infty} \epsilon_n \log(\Pr{}_*\{X_n \in B(x_0, \delta)\}) \geq -I_1(x_0). \end{aligned}$$

To prove part (b), it suffices to show that given a closed set $F \in S_2$,

$$\limsup_{n\to\infty} \epsilon_n \log(\Pr{}^*\{f_n(X_n) \in F\}) \leq -I_1(f^{-1}(F)).$$

For each $\delta > 0$ and each positive integer k, we have that

$$\begin{aligned} & \limsup\nolimits_{n\to\infty} \epsilon_n \log(\Pr{}^*\{f_n(X_n) \in F\}) \\ \leq\ & \limsup\nolimits_{n\to\infty} \epsilon_n \log(\Pr{}^*\{X_n \in G_k(\delta)\}) \leq -I_1(G_k(\delta)), \end{aligned}$$

where $G_k(\delta) = \overline{\cup_{j=k}^{\infty}\{x \in S_1 : d(f_j(x), F) \leq \delta\}}$. We need to prove that

$$I_1(f^{-1}(F)) = \lim_{\delta\to 0}\lim_{k\to\infty} I_1(G_k(\delta)). \tag{2.1}$$

Given x with $I_1(x) < \infty$ and $f(x) \in F$, $f_k(x) \to f(x)$. So, for k large enough, $x \in G_k(\delta)$. Hence,

$$I_1(f^{-1}(F)) \geq \lim_{\delta\to 0}\lim_{k\to\infty} I_1(G_k(\delta)).$$

We may assume that $\lim_{\delta\to 0}\lim_{k\to\infty} I_1(G_k(\delta)) < \infty$. Take $\delta_j \to 0$, $k_j \nearrow \infty$ and $x_j \in G_{k_j}(\delta_j)$ such that

$$\lim_{j\to\infty} I_1(x_j) = \lim_{\delta\to 0}\lim_{k\to\infty} I_1(G_k(\delta)).$$

Take $l_j \geq k_j$ and z_j with $d(f_{l_j}(z_j), F) \leq \delta_j$ and $d(z_j, x_j) \leq j^{-1}$. Since I_1 is a good rate, there exists a subsequence $\{x_{j_k}\}$ and $x \in S_1$ such that $x_{j_k} \to x$. So, $z_{j_k} \to x$ and $f_{l_{j_k}}(z_{j_k}) \to f(x) \in F$. But, by the lower semicontinuity of the function I_1, $I_1(x) \leq \liminf_{j\to\infty} I_1(x_j)$. Therefore, (2.1) follows.

To prove part (c), it suffices to show that the restriction of f to $\{x : I_1(x) < \infty\}$ is continuous. Note that for each $c \geq 0$, $\{y : I_2(y) \leq c\} = f(\{x : I_1(x) \leq c\})$. Given $\epsilon > 0$ and $x_0 \in \{x : I_1(x) < \infty\}$, there are $\delta > 0$ and a positive integer n_0 such that for each $n \geq n_0$, $f_n(B(x_0, \delta)) \subset B(f(x_0), \epsilon)$. Given $x \in B(x_0, \delta) \cap \{x : I_1(x) < \infty\}$, for $n \geq n_0$,
$d_2(f_n(x), f(x_0)) < \epsilon$. From this and hypothesis (ii), for each $x \in B(x_0, \delta) \cap \{x : I_1(x) < \infty\}$, $d_2(f(x), f(x_0)) < \epsilon$. $\square$

The following corollary follows immediately from the previous theorem.

Corollary 2.2 *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random elements with values in a metric space (S_1, d_1) such that $\{X_n\}_{n=1}^{\infty}$ satisfies the LDP with rate ϵ_n^{-1} and rate function I_1. Let $f : (S_1, d_1) \to (S_2, d_2)$ be a function which is continuous at each x with $I_1(x) < \infty$, where (S_2, d_2) is a metric space. Then, $\{f(X_n)\}_{n=1}^{\infty}$ satisfies the LDP with rate ϵ_n^{-1} and rate function $I_2(y) = \inf\{I_1(x) : f(x) = y\}$.*

3 Asymptotic equicontinuity for the large deviation principle

In this section, we prove that the LDP in $l_\infty(T)$ is equivalent to the LDP for the finite dimensional distributions plus an exponential asymptotic equicontinuity condition with respect with certain pseudometric, which makes T totally bounded. This condition can be interpreted as a tightness condition.

Assuming the LDP for the finite dimensional distributions, we claim that for each $k \geq 1$,

$$\rho_k(s,t) = \sup\{|u_2 - u_1| : I_{s,t}(u_1, u_2) \leq k\} \tag{3.1}$$

defines a pseudometric on T. The LDP for the finite dimensional distributions implies that for each $s, t \in T$ and each $k \geq 1$,

$$\{(u_1, u_2) : I_{s,t}(u_1, u_2) \leq k\}$$

is a compact set. So, for each $s, t \in T$, $\rho_k(s,t) < \infty$. By the contraction principle, given $r, s, t \in T$,

$$\begin{aligned} & \rho_k(r,t) = \sup\{|u_2 - u_1| : I_{r,t}(u_1, u_2) \leq k\} \\ = \ & \sup\{|u_3 - u_1| : I_{r,s,t}(u_1, u_2, u_3) \leq k\} \\ \leq \ & \sup\{|u_2 - u_1| : I_{r,s,t}(u_1, u_2, u_3) \leq k\} \\ & + \sup\{|u_3 - u_2| : I_{r,s,t}(u_1, u_2, u_3) \leq k\} \\ = \ & \rho_k(r,s) + \rho_k(s,t). \end{aligned}$$

Therefore, ρ_k is a pseudometric. The pseudometrics ρ_k play a role in the exponential asymptotic tightness of a sequence of stochastic processes.

First, we prove the following lemma:

Lemma 3.1 *Let $\{U_n(t) : t \in T\}$ be a sequence of stochastic processes, where T is an index set. Let $\{\epsilon_n\}$ be a sequence of positive numbers that converges to zero. Suppose that:*

(i) For each $t_1, \dots, t_m \in T$, $(U_n(t_1), \dots, U_n(t_m))$ satisfies the LDP with speed ϵ_n^{-1} and good rate function $I_{t_1,\dots,t_m}$.

(ii) For each $k \geq 1$, (T, ρ_k) is a totally bounded pseudometric space.

Then, for each $0 \leq c < \infty$, $\{z \in l_\infty(T) : I(z) \leq c\}$ is a set of uniformly bounded and uniformly equicontinuous functions in (T, ρ) and it is closed in $l_\infty(T)$, where

$$I(z) = \sup\{I_{t_1,\dots,t_m}(z(t_1), \dots, z(t_m)) : t_1, \dots, t_m \in T, m \geq 1\}. \tag{3.2}$$

and

$$\rho(s,t) = \sum_{k=1}^{\infty} k^{-2} \min(\rho_k(s,t), 1). \tag{3.3}$$

Consequently, for each $0 \leq c < \infty$, $\{z \in l_\infty(T) : I(z) \leq c\}$ is a compact set of $l_\infty(T)$.

Proof. Since each $I_{t_1,\dots,t_m}$ is lower semicontinuous, so is $I(\cdot)$. This implies that for each $0 \leq c < \infty$ the set $\{z \in l_\infty(T) : I(z) \leq c\}$ is closed. Let $\rho_k^*(s,t) = \sup\{|z(t) - z(s)| : I(z) \leq k\}$ and let $\rho^*(s,t) = \sum_{k=1}^{\infty} k^{-2} \min(\rho_k^*(s,t), 1)$. It is easy to see that the set of functions $\{z \in$

$l_\infty(T) : I(z) \le k\}$ is a set of uniformly equicontinuous functions in (T, ρ^*). Since $I_{s,t}(z(s), z(t)) \le I(z)$, $\rho_k^*(s,t) \le \rho_k(s,t)$. Hence, the set of functions $\{z \in l_\infty(T) : I(z) \le k\}$ is a set of uniformly equicontinuous functions in (T, ρ). Since for each $t \in T$, I_t is a good rate, for each $0 \le c < \infty$,

$$\sup\{|z(t)| : I(z) \le c\} \le \sup\{|u| : I_t(u) \le c\} < \infty.$$

So, $\{z \in l_\infty(T) : I(z) \le c\}$ is a set of uniformly bounded functions. The Arzelà–Ascoli theorem (see for example Theorem IV.6.7 in Dunford and Schwartz, 1988) implies that $\{z \in l_\infty(T) : I(z) \le c\}$ is a compact set of $l_\infty(T)$. We may apply this theorem even when (T, ρ) is a totally bounded pseudometric space and not a compact space because identifying the points which are a zero distance (see Problem 2C in Willard, 1970), we may assume that (T, ρ) is a metric space and imbedding T in its completion, we may assume that (T, ρ) is complete. A metric space can be isometrically embedded as a dense subset of the complete metric space consisting by the Cauchy sequences in this space (see for example Theorem 24.4 in Willard, 1970). The considered functions can be extended as functions in the completion by the principle of extension by continuity (see Theorem I.6.17 in Dunford and Schwartz, 1988). □

We denote a finite partition function π of T to a function $\pi : T \to T$ such for each $t \in T$, $\pi(\pi(t)) = \pi(t)$, and the cardinality of $\{\pi(t) : t \in T\}$ is finite. Let $\pi(T) = \{t_1, \ldots, t_m\}$ and $A_j = \{t \in T : \pi(t) = t_j\}$ for $1 \le j \le m$, then $\{A_1, \ldots, A_m\}$ is a partition of T. Finite partition functions can be used to characterize compactness of $l_\infty(T)$. A set K of $l_\infty(T)$ is compact if and only if it is closed, bounded and for each $\tau > 0$, there exists a finite partition function $\pi : T \to T$ such that $\sup_{x \in K} |x(t) - x(\pi(t))| \le \tau$ (see Theorem IV.5.6 in Dunford and Schwartz, 1988). We also have that if K is a compact set of $l_\infty(T)$, then K is a set of uniformly bounded and equicontinuous functions in the pseudometric space (T, d), where $d(s,t) = \sup_{x \in K} |x(s) - x(t)|$.

Theorem 3.2 *Let $\{U_n(t) : t \in T\}$ be a sequence of stochastic processes, where T is an index set. Let $\{\epsilon_n\}$ be a sequence of positive numbers that converges to zero. Let $I : l_\infty(T) \to [0, \infty]$ and let $I_{t_1,\ldots,t_m} : \mathbb{R}^m \to [0, \infty]$ be a function for each $t_1, \ldots, t_m \in T$. Let d be a pseudometric in T.*

Consider the conditions:

(a.1) (T, d) is totally bounded.

(a.2) For each $t_1, \ldots, t_m \in T$, $(U_n(t_1), \ldots, U_n(t_m))$ satisfies the LDP with speed ϵ_n^{-1} and good rate function $I_{t_1,\ldots,t_m}$.

(a.3) For each $\tau > 0$,

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \epsilon_n \log \Pr{}^*\{\sup_{d(s,t) \le \eta} |U_n(t) - U_n(s)| \ge \tau\} = -\infty.$$

(b.1) For each $0 \le c < \infty$, $\{z \in l_\infty(T) : I(z) \le c\}$ is a compact set of $l_\infty(T)$.

(b.2) For each $A \subset l_\infty(T)$,

$$-\inf_{z \in A^o} I(z) \le \liminf_{n\to\infty} \epsilon_n \log \Pr{}_*\{U_n \in A\}$$

$$\le \limsup_{n\to\infty} \epsilon_n \log \Pr{}^*\{U_n \in A\} \le -\inf_{z\in \bar{A}} I(z).$$

If the set of conditions (a) is satisfied, then the set of conditions (b) holds with $I(\cdot)$ given by (3.2).

If the set of conditions (b) is satisfied, then the set of conditions (a) holds with

$$\begin{aligned} &I_{t_1,\dots,t_m}(u_1,\dots,u_m) \\ &= \inf\{I(z) : z \in l_\infty(T), (z(t_1),\dots,z(t_m)) = (u_1,\dots,u_m)\} \end{aligned} \tag{3.4}$$

and the pseudometric ρ in (3.3).

Proof. Assume that the set of conditions (a) holds. First, we show that for each $k \ge 1$,

$$\lim_{\eta\to 0} \sup_{d(s,t)\le\eta} \rho_k(s,t) = 0. \tag{3.5}$$

Given $\tau > 0$, take $\eta > 0$, such that

$$\limsup_{n\to\infty} \epsilon_n \log \Pr{}^*\{\sup_{d(s,t)\le\eta} |U_n(t) - U_n(s)| \ge \tau\} \le -k-1.$$

This implies that

$$\sup_{d(s,t)\le\eta} \limsup_{n\to\infty} \epsilon_n \log \Pr\{|U_n(t) - U_n(s)| \ge \tau\} \le -k-1,$$

which gives that $\sup_{d(s,t)\le\eta} \rho_k(s,t) \le \tau$. Therefore, (3.5) holds. This implies that for each $k \ge 1$, (T,ρ_k) is totally bounded. Hence, (b.1) follows from Lemma 3.1.

Define

$$\begin{aligned} &I^{(1)}_{t_1,\dots,t_m}(u_1,\dots,u_m) \\ =\ &\inf\{I(z) : z \in l_\infty(T),\ (z(t_1),\dots,z(t_m)) = (u_1,\dots,u_m)\}, \end{aligned}$$

where $I(\cdot)$ is defined in (3.2). We claim that for each $t_1 \dots, t_m \in T$ and each $u_1,\dots,u_m \in I\!R$,

$$I_{t_1,\dots,t_m}(u_1,\dots,u_m) = I^{(1)}_{t_1,\dots,t_m}(u_1,\dots,u_m). \tag{3.6}$$

It is easy to see that $I_{t_1,\dots,t_m}(u_1,\dots,u_m) \le I^{(1)}_{t_1,\dots,t_m}(u_1,\dots,u_m)$. To prove the inverse inequality, we may assume that $I_{t_1,\dots,t_m}(u_1,\dots,u_m) < \infty$. Let $\tau > 0$ and let $k_0 > 2\tau + I_{t_1,\dots,t_m}(u_1,\dots,u_m)$. We can find $t_{m+1}, t_{m+2} \dots$ such that $\{t_n\}_{n=1}^{\infty}$ is a dense set in (T, ρ_{k_0}). By the contraction principle, for each $r_1, \dots, r_m, s_1, \dots, s_p \in T$ and each $u_1, \dots, u_m \in I\!R$,

$$\begin{aligned}(3.7)\qquad & I_{r_1,\dots,r_m}(u_1,\dots,u_m) \\ &= \inf\{I_{r_1,\dots,r_m,s_1,\dots,s_p}(u_1,\dots,u_m,v_1,\dots,v_p) : v_1,\dots,v_p \in I\!R\}.\end{aligned}$$

So, we can find $u_{m+1}, u_{m+2}, \dots$ such that for each $n \geq m$,

$$I_{t_1,\dots,t_n}(u_1,\dots,u_n) < \tau + I_{t_1,\dots,t_m}(u_1,\dots,u_m).$$

Define $z(t_j) = u_j$. By the definition of the pseudometric ρ_{k_0}, we have that z is an equicontinuous function in $(\{t_n\}_{n=1}^{\infty}, \rho_{k_0})$. So, there exists a unique extension of z to an equicontinuous function in (T, ρ_{k_0}) (see Theorem I.6.17 in Dunford and Schwartz, 1988). By an abuse of notation, we call this extension z. To finish the proof of (3.6), it suffices to show that

$$(3.8)\qquad I(z) \leq 2\tau + I_{t_1,\dots,t_m}(u_1,\dots,u_m).$$

Hence, we need to prove that for each $s_1, \dots, s_l \in T$,

$$(3.9)\qquad I_{s_1,\dots,s_l}(z(s_1),\dots,z(s_l)) \leq \tau + I_{t_1,\dots,t_m}(u_1,\dots,u_m).$$

For each $1 \leq i \leq l$, take $t_{n_j^{(i)}}$ such that $\rho_{k_0}(t_{n_j^{(i)}}, s_i) \to 0$, as $j \to \infty$. By (3.7), there are $v_i^{(j)}$ such that

$$\begin{aligned}(3.10)\qquad & I_{t_{n_j^{(1)}},\dots,t_{n_j^{(m)}},s_1,\dots,s_l}(z(t_{n_j^{(1)}}),\dots,z(t_{n_j^{(m)}}),v_1^{(j)},\dots,v_l^{(j)}) \\ &\leq I_{t_{n_j^{(1)}},\dots,t_{n_j^{(m)}}}(z(t_{n_j^{(1)}}),\dots,z(t_{n_j^{(m)}})) + \tau < I_{t_1,\dots,t_m}(u_1,\dots,u_m) + 2\tau.\end{aligned}$$

Hence,

$$I_{t_{n_j^{(i)}},s_i}(z(t_{n_j^{(i)}}), v_i^{(j)}) \leq k_0.$$

So, $z(t_{n_j^{(i)}}) - v_i^{(j)} \to 0$, as $j \to \infty$. Hence, $v_i^{(j)} \to z(s_i)$, as $j \to \infty$. From (3.7) and (3.10),

$$I_{s_1,\dots,s_l}(v_1^{(j)},\dots,v_l^{(j)}) < 2\tau + I_{t_1,\dots,t_m}(u_1,\dots,u_m).$$

This inequality and the lower semicontinuity of $I_{s_1,\dots,s_l}$ implies (3.9).

In order to prove that for each set $A \subset l_\infty(T)$,

$$(3.11)\qquad \limsup_{n\to\infty} \epsilon_n \log(\Pr{}^*\{U_n \in A\}) \leq -\inf_{z\in\bar{A}} I(z),$$

we may suppose that $\inf_{z\in\bar{A}} I(z) > 0$. Let $0 < a < \inf_{z\in\bar{A}} I(z)$ and let $K = \{z \in l_\infty(T) : I(z) \leq a\}$. Then, $K \cap \bar{A} = \emptyset$ and K is a compact set.

Thus, there exists a $\delta > 0$ such that $K(4\delta) \cap \bar{A} = \emptyset$. By Lemma 3.1, (3.5) and (a.3), there exists a $\eta > 0$ such that

$$\sup_{z \in K} \sup_{d(s,t) \le \eta} |z(s) - z(t)| \le \delta \tag{3.12}$$

and

$$\limsup_{n\to\infty} \epsilon_n \log \left(\Pr{}^*\{ \sup_{d(s,t)\le\eta} |U_n(t) - U_n(s)| \ge \delta\} \right) \le -2a. \tag{3.13}$$

By condition (a.1) there exists a finite partition function $\pi : T \to T$ such that $\sup_{t\in T} d(t, \pi(t)) \le \eta$. Let $\pi(T) = \{t_1, \dots, t_m\}$ and let $C_\delta = \{(z(t_1), \dots, z(t_m)) : z \in K(\delta)\}$. It is easy to see that if $(z(t_1), \dots, z(t_m)) \in C_\delta$ and $\sup_{t\in T} |z(t) - z(\pi(t))| \le \delta$, then $z \in K(4\delta)$. Hence,

$$\begin{aligned} & \Pr{}^*\{U_n \in \bar{A}\} \le \Pr{}^*\{U_n \notin K(4\delta)\} \\ \le\ & \Pr{}^*\{(U_n(t_1), \dots, U_n(t_m)) \notin C_\delta\} + \Pr{}^*\{\sup_{t\in T} |U_n(t) - U_n(\pi(t))| \ge \delta\}. \end{aligned}$$

Since C_δ is an open set, by the LDP for the finite dimensional distributions,

$$\begin{aligned} & \limsup_{n\to\infty} \epsilon_n \log \left(\Pr{}^*\{(U_n(t_1), \dots, U_n(t_m)) \notin C_\delta\}\right) \\ \le\ & -\inf\{I_{t_1,\dots,t_m}(u_1, \dots, u_m) : (u_1, \dots, u_m) \notin C_\delta\} \end{aligned}$$

By (3.6)

$$\begin{aligned} & \inf\{I_{t_1,\dots,t_m}(u_1, \dots, u_m) : (u_1, \dots, u_m) \notin C_\delta\} \\ \ge\ & \inf\{I(z) : z \notin K(\delta)\} \ge a. \end{aligned}$$

So,

$$\limsup_{n\to\infty} \epsilon_n \log \left(\Pr{}^*\{U_n \in A\}\right) \le -a.$$

Letting $a \to \inf_{z\in\bar{A}} I(z)$, (3.11) follows.

Next, we prove that for each set $A \subset l_\infty(T)$,

$$-\inf_{z\in A^o} I(z) \le \liminf_{n\to\infty} \epsilon_n \log(\Pr{}_*\{U_n \in A\}).$$

It suffices to prove that if $z_0 \in A^o$ and $I(z_0) < \infty$, then

$$-I(z_0) \le \liminf_{n\to\infty} \epsilon_n \log(\Pr{}_*\{U_n \in A\}) \tag{3.14}$$

Take $a > I(z_0) > b$. Let $K = \{z \in l_\infty(T) : I(z) \le a\}$. There exists a $\delta > 0$ such that $B(z_0, 3\delta) \subset A^o$ and $\inf\{I(z) : z \in B(z_0, 3\delta)\} > b$. By Lemma 3.1, (3.5) and (a.3) there exists a $\eta > 0$ such that (3.12) and (3.13) hold. Take a finite partition function π such that $\sup_{t\in T} d(t, \pi(t)) \le \eta$. Let

$\{t_1,\dots,t_m\} = \pi(T)$. Then, if $\max_{1\le j\le m} |z(t_j) - z_0(t_j)| < \delta$ and $\sup_{d(s,t)\le\eta} |z(s) - z(t)| < \delta$, then $\sup_{t\in T} |z(t) - z_0(t)| < 3\delta$. So,

$$\begin{aligned} & \Pr\{\max_{1\le j\le m} |U_n(t_j) - z_0(t_j)| < \delta\} \\ \le\ & \Pr{}_*\{\sup_{t\in T} |U_n(t) - z_0(t)| < 3\delta\} + \Pr{}^*\{\sup_{d(s,t)\le\eta} |U_n(s) - U_n(t)| \ge \delta\}. \end{aligned}$$

We claim that

$$(3.15)\ \{z \in l_\infty(T) : \sup_{1\le j\le m} |z(t_j) - z_0(t_j)| < \delta\} \subset \{z \in l_\infty(T) : I(z) \ge b\}.$$

If $\sup_{1\le j\le m} |z(t_j) - z_0(t_j)| < \delta$ and $z \notin K$, then $I(z) > a > b$. If $\sup_{1\le j\le m} |z(t_j) - z_0(t_j)| < \delta$ and $z \in K$, then $z \in B(z_0, 3\delta)$ and $I(z) > b$. Therefore, (3.15) holds. So, by condition (a.2),

$$-b \le \liminf_{n\to\infty} \epsilon_n \log \Pr{}_*\{\Pr\{\max_{1\le j\le m} |U_n(t_j) - z_0(t_j)| < \delta\})\}.$$

Therefore, (3.14) follows.

Assume that the set of conditions (b) holds. It is easy to see that if K is a compact set of $l_\infty(T)$ and $e(s,t) = \sup_{z\in K} |z(s) - z(t)|$, then (T, e) is a totally bounded pseudometric space and K is a collection of uniformly bounded and uniformly e–equicontinuous functions. Hence, for each $k \ge 1$, (T, ρ_k^*) is a totally bounded pseudometric space, where ρ_k^* is as in the proof of Lemma 3.1. (3.6) implies that $\rho_k^* = \rho_k$ ((3.6) follows from the contraction principle). Therefore, (T, ρ) is also totally bounded, that is (a.1) holds.

Given $t_1,\dots,t_m \in T$, the function $g : l_\infty(T) \to I\!R^m$ defined by $g(z) = (z(t_1),\dots,z(t_m))$ is a continuous function. So, the contraction principle implies (a.2).

To prove (a.3), it suffices to show that given $0 < \tau, c < \infty$, there exists $\eta > 0$ such that

$$(3.16)\qquad \limsup_{n\to\infty} \epsilon_n \log \Pr{}^*\{\sup_{d(s,t)\le\eta} |U_n(t) - U_n(s)| \ge \tau\} \le -c.$$

$\{z \in l_\infty(T) : I(z) \le c\}$ is a set of uniformly bounded and uniformly ρ–continuous functions. So, there exists a $\eta > 0$ such that

$$\{z \in l_\infty(T) : I(z) \le c\} \subset \{z \in l_\infty(T) : \sup_{\rho(s,t)\le\eta} |z(s) - z(t)| \le 2^{-1}\tau\}.$$

So, $F := \{z \in l_\infty(T) : \sup_{\rho(s,t)\le\eta} |z(s) - z(t)| \ge \tau\}$ is a closed in $l_\infty(T)$ and $\inf_{z\in F} I(z) \ge c$. Hence, (3.16) holds. □

Of course, if conditions (a) in Theorem 3.1 hold for some pseudometric d and e is a uniformly equivalent to d, then conditions (a) hold for e. d and e are uniformly equivalent if

$$\lim_{\delta\to 0} \sup_{d(s,t)\le\delta} e(s,t) = \lim_{\delta\to 0} \sup_{e(s,t)\le\delta} d(s,t) = 0.$$

Theorem 3.2 is true with ρ^* instead of ρ.

Alternatively, conditions (a.1) and (a.3) in Theorem 3.1 can be put using finite partition functions. Conditions (a.1) and (a.3) are equivalent to: for each $c, \eta > 0$, there exists a finite partition function π of T such that

$$\limsup_{n\to\infty} \epsilon_n \log \Pr{}^*\{\sup_{t\in T} |U_n(t) - U_n(\pi(t))| \geq \eta\} \leq -c. \tag{3.17}$$

Under the conditions in Theorem 3.2, if $z \in l_\infty(T)$ and $I(z) < \infty$, then z is a uniformly continuous in (T, d).

The next corollary characterizes when the asymptotic equicontinuity condition is satisfied with respect to the Euclidean distance when T is a bounded set of $\mathbb{R}^d$.

Corollary 3.3 *Let T is a compact set of $\mathbb{R}^d$, let $\{U_n(t) : t \in T\}$ be a sequence of stochastic processes and let $\{\epsilon_n\}$ be a sequence of positive numbers that converges to zero. Then, the following sets of conditions ((a) and (b)) are equivalent:*

(a.1) $\{U_n(t) : t \in T\}$ satisfies the LDP in $l_\infty(T)$ with speed ϵ_n^{-1}.

(a.2) For each $t_0 \in T$, $\lim_{t\to t_0} \rho(t, t_0) = 0$.

(b.1) For each $t_1, \ldots, t_m \in T$, $(U_n(t_1), \ldots, U_n(t_m))$ satisfies the LDP with speed ϵ_n^{-1} and good rate function $I_{t_1,\ldots,t_m}$.

(b.2) For each $\tau > 0$,

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \epsilon_n \log \Pr{}^*\{\sup_{\substack{s,t\in T\\ |s-t|\leq\eta}} |U_n(t) - U_n(s)| \geq \tau\} = -\infty.$$

Proof. Assume the conditions (a). (b.1) follows from (a.1) and the contraction principle. We have that the identity function $(T, |\cdot|) \to (T, \rho)$ is continuous, where ρ is as in (3.3). Since $(T, |\cdot|)$ is a compact set, this function is also uniformly continuous. So, (b.2) holds.

Assume the set of conditions (b). By Theorem 3.2 (a.1) holds. Given $\tau > 0$ and $k > 0$, there exists a $\eta > 0$ such that

$$\limsup_{n\to\infty} \epsilon_n \log \Pr{}^*\{\sup_{\substack{s,t\in T\\ |s-t|\leq\eta}} |U_n(t) - U_n(s)| \geq \tau\} \leq -k - 1.$$

Hence, if $|s - t| \leq \eta$,

$$\limsup_{n\to\infty} \epsilon_n \log \Pr\{|U_n(t) - U_n(s)| \geq \tau\} \leq -k - 1.$$

By the LDP for $(U_n(s), U_n(t))$,

$$\liminf_{n\to\infty} \epsilon_n \log \Pr\{|U_n(t) - U_n(s)| > \tau\} \geq -\inf\{I_{s,t}(u_1, u_2) : |u_1 - u_2| > \tau\}.$$

Therefore, for $|s-t| \le \eta$,

$$k < \inf\{I_{s,t}(u_1, u_2) : |u_1 - u_2| > \tau\}.$$

This implies that

$$\sup_{\substack{s,t \in T \\ |s-t| \le \eta}} \sup\{|u_2 - u_1| : I_{s,t}(u_1, u_2)| \le k\} \le \tau. \quad \square$$

Observe that in the previous corollary, the condition (a.2) can be substituted by the condition:

(a.2)' For each $0 < k < \infty$,

$$\lim_{\eta \to 0} \sup_{\substack{s,t \in T \\ |s-t| \le \eta}} \sup\{|u_2 - u_1| : I_{s,t}(u_1, u_2)| \le k\} = 0.$$

It may happen that a sequence of stochastic processes $\{U_n(t) : 0 \le t \le M\}$ satisfies the LDP, but condition (b.2) in the previous theorem is not satisfied. For the stochastic processes in Theorem 6.2, if $(\mu[0, Mn])^{-1}\mu[0, Mn) \to 0$, the LDP holds, but neither (b.2) nor (a.2) in the previous theorem hold.

The next corollary allows to combine the LDP for several index sets.

Corollary 3.4 *Let $\{U_n(t) : t \in T\}$ be a sequence of stochastic processes with values in $\mathbb{R}^d$. Let $T^{(1)}$ and let $T^{(2)}$ be two subsets of T such that $T = T^{(1)} \cup T^{(2)}$. Suppose that:*

(i) For each $t_1, \dots, t_m \in T$, $(U_n(t_1), \dots, U_n(t_m))$ satisfies the LDP with speed ϵ_n^{-1} and rate function $I_{t_1,\dots,t_n}$.

(ii) For each $j = 1, 2$, $\{U_n(t) : t \in T_j\}$ satisfies the LDP $l_\infty(T_j)$ with speed ϵ_n^{-1}.

Then, $\{U_n(t) : t \in T\}$ satisfies the LDP in $l_\infty(T)$ with speed ϵ_n^{-1} and rate function

$$I(z) = \sup\{I_{t_1,\dots,t_n}(z(t_1), \dots, z(t_n)) : t_1, \dots, t_m \in T, m \ge 1\}.$$

Proof. Given $c, \tau > 0$, there exist a partition functions $\pi^{(i)}$, $i = 1, 2$, such that

$$\limsup_{n \to \infty} \epsilon_n \log(\Pr{}^*\{\sup_{t \in T^{(i)}} |U_n(t) - U_n(\pi^{(i)}(t))| \ge \eta\}) \le -c$$

Let $\pi(t) = \pi^{(1)}(t)$, if $t \in T^{(1)}$, and $\pi(t) = \pi^{(2)}(t)$, if $t \in T^{(2)} - T^{(1)}$. Then,

$$\limsup_{n \to \infty} \epsilon_n \log(\Pr{}^*\{\sup_{t \in T} |U_n(t) - U_n(\pi(t))| \ge 2\eta\}) \le -c,$$

which implies the claim. $\square$

The next corollary allows to obtain the LDP for stochastic processes in $\mathbb{R}^d$.

Corollary 3.5 *Let $\{U_n^{(i)}(t) : t \in T\}$, $1 \le i \le d$, be sequences of stochastic processes defined in the same probability space. Suppose that:*

(i) For each $t_1, \dots, t_m \in T$,

$$(U_n^{(1)}(t_1), \dots, U_n^{(1)}(t_m), \dots, V_n^{(d)}(t_1), \dots, V_n^{(d)}(t_m))$$

satisfies the LDP in $I\!R^{dm}$ with speed ϵ_n^{-1}.

(ii) For each $1 \le i \le d$, $\{U_n^{(i)}(t) : t \in T\}$ satisfies the LDP in $l_\infty(T)$ with speed ϵ_n^{-1}.

Then, $\{(U_n^{(1)}(t), \dots, U_n^{(d)}(t)) : t \in T\}$ satisfies the LDP in $l_\infty(T, I\!R^d)$ with speed ϵ_n^{-1}.

Proof. Let $T^* = \{1, \dots, m\} \times T$. Let $W_n(i,t) = U_n^{(i)}(t)$ for $t \in T$. Corollary implies that for each $1 \le i \le d$, $\{W_n(i,t) : t \in T\}$ satisfies the LDP with speed ϵ_n^{-1}. By Corollary 3.4, $\{W_n(t^*) : t^* \in T^*\}$ satisfies the LDP in $l_\infty(T^*)$ with speed ϵ_n^{-1}. $l_\infty(T^*)$ is isometric to $l_\infty(T, I\!R^d)$. $\square$

The previous theorem implies Slutsky theorem for the LPD in $l_\infty(T)$. Under the conditions in the previous theorem, by the contraction principle for any continuous function g in $I\!R^d$, $\{g(U_n^{(1)}(t), \dots, U_n^{(d)}(t)) : t \in T\}$ satisfies the LDP in $l_\infty(T)$ with speed ϵ_n^{-1}.

The next corollary gives necessary and sufficient conditions for the LDP for Banach space valued r.v.'s.

Corollary 3.6 *Let $\{X_n\}_{n=1}^\infty$ be a sequence of r.v.'s with values in Banach space B. Then, $\{X_n\}_{n=1}^\infty$ satisfies the LDP in B with speed ϵ_n^{-1} if and only if for each $f_1, \dots, f_m \in B^*$, $(f_1(X_n), \dots, f_m(X_n))$ satisfies the LDP with speed ϵ_n^{-1} and rate function $I_{f_1,\dots,f_m}$, and for each $\tau > 0$,*

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \epsilon_n \log \Pr{}^*\{\sup_{f_1,f_2 \in B_1^*,\ \rho(f_1,f_2) \le \eta} |f_1(X_n) - f_2(X_n)| \ge \tau\} = -\infty,$$

where ρ is as in (3.3).

Proof. Consider $\phi : B \to l_\infty(B_1^*)$, defined by $\phi(z) = \{f(z) : f \in B_1^*\}$. It is easy to see that $\phi : B \to \phi(B)$ is a continuous one–to–one function with continuous inverse. Thus, by the contraction principle X_n satisfies the LDP in B if and only $\{f(X_n) : f \in B_1^*\}$ satisfy the LDP in $l_\infty(T)$. Theorem 3.2 implies the claim. $\square$

Next, we consider the LDP for stochastic processes whose sample paths are a convex function on the parameter. It is well known that if a sequence of convex functions converges, then the convergence is uniformly on a compact set (see Theorem 10.8, in Rockafellar, 1970). A similar result holds for the weak convergence of stochastic processes (see Arcones, 1998). We will use some of the techniques in this paper. In particular Lemma 13 in Arcones (1998) says that for each convex function $f : [-1, 1]^d \to I\!R$,

$$\sup_{x\in[-1,1]^d} |f(x)| \le 3^d \sup_{x\in\{-1,0,1\}^d} |f(x)|. \tag{3.18}$$

We also will need that if Let T_0 be a set of R^d, let $\epsilon > 0$, let $T_0^\epsilon = \{x + y : x \in T_0,\ |y| \le \epsilon\}$ and let $f : T_0^\epsilon \to I\!R$ be a convex function, then for each $x, y \in T_0$

$$|f(x) - f(y)| \le |x - y|\epsilon^{-1} 2 \sup_{t\in T_0^\epsilon} |f(t)|, \tag{3.19}$$

(see Lemma 14 in Arcones, 1998).

Corollary 3.7 *Let T_0 be an open convex set of $I\!R^d$. Let T be a compact set of T_0. Let $\{\epsilon_n\}$ be a sequence of positive numbers converging to zero. Let $\{U_n(t) : t \in T_0\}$ be a sequence of stochastic processes. Suppose that:*

(i) $U_n(t)$ is a convex function in t.

(ii) For each $t_1, \dots, t_m \in T_0$, $(U_n(t_1), \dots, U_n(t_m))$ satisfies the LDP with speed ϵ_n^{-1}.

Then, $\{U_n(t) : t \in T\}$ satisfies the LDP in $l_\infty(T)$ with speed ϵ_n^{-1}.

Proof. We have to prove that given $c, \eta > 0$, there exists a finite partition function π of T_0 such that

$$\limsup_{n\to\infty} \epsilon_n \log(\Pr{}^*\{\sup_{t\in T_0} |U_n(t) - U_n(\pi(t))| \ge \eta\}) \le -c.$$

Take $\epsilon > 0$ such that $T_0^\epsilon \subset T$. Since T_0^ϵ is a compact set, it can be covered by a finite number of hypercubes. So, by (3.18) , there are $t_1, \dots, t_m \in T$ such that for each convex function h defined on T,

$$\sup_{t\in T_0^\epsilon} |h(t)| \le 3^d \max_{1\le l\le m} |h(t_l)|.$$

Hence, there exists a finite constant M such that

$$\lim_{n\to\infty} \epsilon_n \log(\Pr\{\max_{1\le l\le m} |U_n(t_l)| \ge M\}) \le -c.$$

Take a finite partition function π of T_0 such that $\sup_{t\in T_0} |t - \pi(t)| \le 2^{-1}M^{-1}3^{-d}\eta\epsilon$. By (3.19),

$$\sup_{t\in T_0} |U_n(t) - U_n(\pi(t))| \le M^{-1}3^{-d}\eta \sup_{t\in T_0^\epsilon} |U_n(t)|.$$

Hence,

$$\begin{aligned} & \limsup_{n\to\infty} \epsilon_n \log(\Pr{}^*\{\sup_{t\in T_0} |U_n(t) - U_n(\pi(t))| \ge \eta\}) \\ \le\ & \limsup_{n\to\infty} \epsilon_n \log(\Pr{}^*\{\sup_{t\in T_0^\epsilon} |U_n(t)| \ge M3^d\}) \\ \le\ & \limsup_{n\to\infty} \epsilon_n \log(\Pr\{\max_{1\le l\le m} |U_n(t_l)| \ge M\} \le -c. \end{aligned}$$

Hence, the claim follows. □

Next, we consider the case of nondecreasing processes.

Corollary 3.8 *Let $\{U_n(t) : 0 \le t \le M\}$ be a sequence of stochastic processes. Let $\{\epsilon_n\}$ be a sequence of positive numbers converging to zero. Suppose that:*

(i) With probability one, $U_n(t)$ is a nondecreasing function in t.

(ii) For each $0 \le t_1 < \cdots < t_m \le M$, $(U_n(t_1), \ldots, U_n(t_m))$ satisfies the LDP with speed ϵ_n^{-1}.

(iii) For each $0 < k < \infty$ and each $0 \le t_0 \le M$,

$$\lim_{t\to t_0} \sup\{|u_2 - u_1| : I_{t,t_0}(u_1, u_2)| \le k\} = 0,$$

where I_{t,t_0} is the rate function of the LDP of $(U_n(t), U_n(t_0))$.

Then, $\{U_n(t) : 0 \le t \le M\}$ satisfies the LDP in $l_\infty([0, M])$ with speed ϵ_n^{-1}.

Proof. By an argument in the proof of Corollary 3.3, condition (ii) implies that for each $0 < k < \infty$,

$$\lim_{\eta\to 0} \sup_{\substack{0\le s,t\le M\\ |s-t|\le\eta}} \sup\{|u_2 - u_1| : I_{s,t}(u_1, u_2)| \le k\} = 0.$$

First, we prove that for each $\tau > 0$,

$$\lim_{\eta\to 0} \inf\{I_{s,t}(u_1, u_2) : |u_2 - u_1| \ge \tau, 0 \le s, t \le M, |s - t| \le \eta\} = -\infty. \tag{3.20}$$

Given $0 < k < \infty$ and $\tau > 0$, there exists a $\eta > 0$ such that

$$\sup_{\substack{0\le s,t\le M\\ |s-t|\le\eta}} \sup\{|u_2 - u_1| : I_{s,t}(u_1, u_2)| \le k\} < \tau.$$

Hence,

$$\inf\{I_{s,t}(u_1, u_2) : |u_2 - u_1| \ge \tau, 0 \le s, t \le M, |s - t| \le \eta\} \ge k.$$

and (3.20) holds.

Given a positive integer m, we have that

$$\begin{aligned} &\Pr\{\max_{1\le i\le m} \sup_{t_{i-1}\le t\le t_i} |U_n(t) - U_n(t_{i-1})| \ge \tau\} \\ \le\ &\max_{1\le i\le m} \Pr\{U_n(t_i) - U_n(t_{i-1}) \ge \tau\} \end{aligned}$$

where $t_i = m^{-1}Mi$. So,

$$\begin{aligned} &\limsup_{n\to\infty} \epsilon_n \log(\Pr\{\max_{1\le i\le m} \sup_{t_{i-1}\le t\le t_i} |U_n(t) - U_n(t_{i-1})| \ge \tau\}) \\ &\le -\inf\{I_{t_{i-1},t_i}(u, v) : |v - u| \ge \tau, 1 \le i \le m\}. \end{aligned}$$

which tends to ∞, as $m \to \infty$. □

In the previous theorem, conditions (i) and (ii), without condition (iii), are not sufficient to obtain the thesis of theorem. In the example considered in Theorem 6.2 when $\mu[0,x]$ is slowly varying at infinity, conditions (i) and (ii) in the previous theorem hold, but the stochastic process does not satisfy the LDP. In other words, the LDP of the finite dimensional distributions of a nondecreasing stochastic process does not imply the uniform LDP.

We will need the following proposition, whose proof is omitted because it is trivial.

Theorem 3.9 *Let $\{U_n(t) : t \in I\!R\}$ and be a sequence of stochastic processes. Suppose that for each $0 < M < \infty$, $\{U_n(t) : |t| \le M\}$ satisfies the LDP in $l_\infty[-M, M]$ with speed ϵ_n^{-1} and rate function I_M.*

Then, $\{U_n(t) : t \in I\!R\}$ satisfies the LDP in $(\mathcal{F}(I\!R), d_{\mathrm{comp}})$ with speed ϵ_n^{-1} and rate function I, where

$$\mathcal{F}(I\!R) = \{\alpha : I\!R \to I\!R : \sup_{|t|\le M} |\alpha(t)| < \infty, \text{ for each } M < \infty\},$$

$$d_{\mathrm{comp}}(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{|\alpha - \beta|_{l_\infty[-k,k]} \wedge 1}{2^k},$$

$$I(z) = \lim_{M\to\infty} I_M(z|_{[-M,M]})$$

and $z|_{[-M,M]}$ is z is restricted to $[-M, M]$.

Next, we consider the compositions of stochastic processes.

Theorem 3.10 *Let $\{U_n(t) : t \in I\!R\}$ and let $\{V_n(t) : 0 \le t \le M_2\}$ be two sequences of stochastic processes. Suppose that:*

(i) For each $M_1 < \infty$,

$$\{U_n(t) : |t| \le M_1\} \times \{V_n(t) : 0 \le t \le M_2\}$$

satisfies the LDP in $l_\infty([-M_1, M_1]) \times l_\infty([0, M_2])$ with speed ϵ_n^{-1} and rate function $I_{M_1}^{(U,V)}$.

(ii) For each $t \in I\!R$ and each positive integer k

$$\lim_{t\to t_0} \sup\{|u_2 - u_1| : I_{t,t_0}^{(U)}(u_1, u_2) \le k\} = 0,$$

where $I_{t,t_0}^{(U)}$ is the rate function of the LDP of $(U_n(t), U_n(t_0))$.

Then, $\{U_n(V_n(t)) : 0 \le t \le M_2\}$ satisfies the LDP in $l_\infty([0, M_2])$ with speed ϵ_n^{-1}. Moreover, the rate function is

$$I(z) = \inf\{I^{(U,V)}(\alpha, \beta) : \alpha \circ \beta = z\},$$

where $I^{(U,V)}(\alpha, \beta) = \lim_{M_1\to\infty} I_{M_1}^{(U,V)}(\alpha|_{[-M_1,M_1]}, \beta)$.

PROOF. By the Theorem 3.9, we have that (U_n, V_n) satisfies the LDP in $(\mathcal{F}(I\!R), d_{\text{comp}}) \times l_\infty[-M_2, M_2]$ with speed ϵ_n^{-1} and rate function $I^{(U,V)}$. Let $\phi : (\mathcal{F}(I\!R), d_{\text{comp}}) \times l_\infty[-M_2, M_2] \to l_\infty[-M_2, M_2]$ be defined by $\phi(\alpha, \beta) = \alpha \circ \beta$. By condition (ii), if $I^{(U,V)}(\alpha, \beta) < \infty$, then α is continuous. We claim that ϕ is continuous at each (α, β) with $I^{(U,V)}(\alpha, \beta) < \infty$. Observe that if $(\alpha_n, \beta_n) \to (\alpha, \beta)$ in $\mathcal{F}(I\!R) \times l_\infty[-M_2, M_2]$, then
$M_1 := \sup_{n\geq 1} \sup_{|t|\leq M_2} |\beta_n(t)| < \infty$. Since $\alpha_n \to \alpha$ in $l_\infty[-M_2, M_2]$, $\beta_n \to \beta$ in $l_\infty[-M_1, M_1]$ and α is uniformly continuous in $[-M_1, M_1]$,

$$\begin{aligned}
&\sup_{|t|\leq M_2} |\alpha_n(\beta_n(t)) - \alpha(\beta(t))| \\
&\leq \sup_{|t|\leq M_2} |\alpha_n(\beta_n(t)) - \alpha(\beta_n(t))| + \sup_{|t|\leq M_2} |\alpha(\beta_n(t)) - \alpha(\beta(t))| \\
&\leq \sup_{|t|\leq M_1} |\alpha_n(t) - \alpha(t)| + \sup_{|t|\leq M_2} |\alpha(\beta_n(t)) - \alpha(\beta(t))| \to 0.
\end{aligned}$$

Hence, by Corollary 2.2, $\{U_n \circ V_n\}$ satisfies the LDP in $l_\infty[-M_2, M_2]$ with speed ϵ_n^{-1}. $\square$

There are variations of the previous theorem which hold in an obvious way. For example, we may consider the processes $\{U_n(t) : t \geq 0\}$ and $\{V_n(t) : 0 \leq t \leq M_2\}$. Under the conditions in the previous theorem, we obtain the LDP for $\{U_n(|V_n(t)|) : 0 \leq t \leq M_2\}$.

4 The rate function of the LDP of stochastic processes

In many situations, the large deviations for the finite dimensional distributions can be obtained from the following theorem:

Theorem 4.1 *(Ellis, 1984, Theorem II.2). Let $(U_n(1), \ldots, U_n(m))$ be a sequence of r.v.'s with values in $I\!R^m$. Let ϵ_n be a sequence of positive numbers converging to zero. Suppose that:*

(i) For each $\lambda_1, \ldots, \lambda_m$, the following limit exists (the limit could be infinity)

$$\lim_{n\to\infty} \epsilon_n \log \left(E[\exp(\epsilon_n^{-1} \sum_{j=1}^{m} \lambda_j U_n(j))] \right) =: l(\lambda)$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)$.

(ii) Zero is in the interior of $\mathcal{D}$ $(l) := \{\lambda \in I\!R^m : l(\lambda) < \infty\}$.

(iii) l is a lower semicontinuous convex function on $I\!R^m$.

(iv) $l(\lambda)$ is differentiable in the interior of $\mathcal{D}(l)$.

(v) If λ_n is a sequence in the interior of $\mathcal{D}(l)$ converging to a boundary point of $\mathcal{D}(l)$, then $\|\text{grad } l(\lambda_n)\| \to \infty$.

Then, $(U_n(1),\dots,U_n(m))$ satisfies the LDP in $I\!R^m$ with speed ϵ_n^{-1} and rate function

$$I(u_1,\dots,u_m)=\sup\{\sum_{j=1}^m \lambda_j u_j - l(\lambda_1,\dots,\lambda_m):\lambda_1,\dots,\lambda_m\in I\!R\}.$$

In many cases, the function l in Theorem 4.1 can be written as

$$l(\lambda_1,\dots,\lambda_m)=\int_S \Phi(\sum_{j=1}^m \lambda_j f_j(x))\,d\mu(x), \tag{4.1}$$

where $(S,\mathcal{S})$ is a measurable space, $f_1,\dots,f_m$ are measurable functions, μ is a measure on S and $\Phi: I\!R\to(-\infty,\infty]$ is a convex function. We will take either $\Phi(x)=e^x-1$ or $\Phi(x)=p^{-1}|x|^p$ for some $p>1$. In this section, we study the rate function in $l_\infty(T)$, when (4.1) holds for the finite dimensional distributions.

Lemma 4.2 *Let Φ be a convex function. Let $(S,\mathcal{S})$ be a measurable space. Let μ be a measure on S. Let $f_1,\dots,f_m$ be measurable functions in S such that for each $\lambda_1,\dots,\lambda_m\in I\!R$, $\int\Phi(\sum_{j=1}^m\lambda_j f_j(x))\,d\mu(x)<\infty$. Let*

$$\begin{aligned}&I^{(1)}(u_1,\dots,u_m)\\ =\ &\sup\left\{\textstyle\sum_{j=1}^m \lambda_j u_j-\int\Phi(\sum_{j=1}^m\lambda_j f_j(x))\,d\mu(x):\lambda_1,\dots,\lambda_m\in I\!R\right\}\end{aligned}$$

and let

$$\begin{aligned}I^{(2)}(u_1,\dots,u_m)=\inf\Big\{&\textstyle\int\Psi(\gamma(x))\,d\mu(x):\\ &\textstyle\int f_j(x)\gamma(x)\,d\mu(x)=u_j \text{ for each } 1\le j\le m\Big\},\end{aligned}$$

where Ψ is the conjugate convex function of Φ defined by

$$\Psi(y)=\sup_x(xy-\Phi(x)). \tag{4.2}$$

Then, for each $u_1,\dots,u_m\in I\!R$, $I^{(1)}(u_1,\dots,u_m)=I^{(2)}(u_1,\dots,u_m)$.

Proof. If $\int f_j(x)\gamma(x)\,d\mu(x)=u_j$, for each $1\le j\le m$, by (4.2),

$$\sum_{j=1}^m\lambda_j u_j-\int\Phi(\sum_{j=1}^m\lambda_j f_j(x))\,d\mu(x)$$

$$=\int\left(\sum_{j=1}^m\lambda_j f_j(x)\gamma(x)-\Phi(\sum_{j=1}^m\lambda_j f_j(x))\right)d\mu(x)\le\int\Psi(\gamma(x))\,d\mu(x).$$

Thus, $I^{(1)}(u_1,\dots,u_m)\le I^{(2)}(u_1,\dots,u_m)$. Now, we may assume that $I^{(1)}(u_1,\dots,u_m)<\infty$. Since Φ is convex, it has a left and a right derivative (see for example Chapter I in Rao and Ren, 1991). Let φ be the right derivative of Φ. For the $\lambda_1, \dots, \lambda_m$ attaining the sup in $I^{(1)}(u_1, \dots, u_m)$, for each $1 \le j \le m$,

$$u_j = \int \varphi \left(\sum_{j=1}^{m} \lambda_j f_j(x) \right) f_j(x)\, d\mu(x).$$

Let $\gamma(x) = \varphi \left(\sum_{j=1}^{m} \lambda_j f_j(x) \right)$. Since

$$x\varphi(x) = \Phi(x) + \Psi(\varphi(x))$$

(see for example Theorem I.3.3 in Rao and Ren, 1991),

$$\begin{aligned} & \textstyle\sum_{j=1}^{m} \lambda_j u_j - \int \Phi(\sum_{j=1}^{m} \lambda_j f_j(x))\, d\mu(x) \\ = & \textstyle\int \left(\sum_{j=1}^{m} \lambda_j f_j(x) \varphi(\sum_{j=1}^{m} \lambda_j f_j(x)) - \Phi(\sum_{j=1}^{m} \lambda_j f_j(x)) \right) d\mu(x) \\ = & \textstyle\int \Psi(\gamma(x))\, d\mu(x). \end{aligned}$$

Thus, $I^{(2)}(u_1, \dots, u_m) \le I^{(1)}(u_1, \dots, u_m)$. $\square$

Assuming that $\{U_n(t) : t \in T\}$ satisfies the LDP and the conditions in the previous theorem hold, by Theorem 3.2 we have that for each $k \ge 1$, (T, ρ_k) is totally bounded, where ρ_k is as in (3.1). It is easy to see that this condition is equivalent to for each $k \ge 1$, (T, d_k) is totally bounded, where

$$\begin{aligned} & d_k(s,t) \\ & = \sup\{ | \int (f(x,s) - f(x,t)) \gamma(x)\, d\mu(x) | : \int \Psi(\gamma(x))\, d\mu(x) \le k \}. \end{aligned} \tag{4.3}$$

In some cases, previous pseudometric is an Orlicz norm. We recall some notation in Orlicz spaces from Rao and Ren (1991). A function $\Phi_1 : \mathbb{R} \to [0, \infty]$ is said to be a Young function if it is convex, $\Phi_1(0) = 0$, $\Phi_1(x) = \Phi_1(-x)$, and $\lim_{x \to \infty} \Phi_1(x) = \infty$. Given a measurable space $(S, \mathcal{S})$ and a measure μ on S, the Orlicz space $\mathcal{L}^{\Phi_1}(\mu)$ associated with the Young function Φ_1 is the class of measurable functions f on $(S, \mathcal{S})$ such that for some $\lambda > 0$ $\int \Phi_1(\lambda f)\, d\mu < \infty$. Define the Orlicz norm by

$$\|f\|_{\Phi_1} = \sup\{ | \int fg\, d\mu | : \int \Psi_1(|g|)\, d\mu \le 1 \}, \tag{4.4}$$

and the gauge norm of the Orlicz space $\mathcal{L}^{\Phi_1}(\mu)$ by

$$N_{\Phi_1}(f) = \inf\{ t > 0 : \int \Phi_1(f/t)\, d\mu \le \Phi(1) \},$$

where Ψ_1 be the conjugate function of Φ_1 in the sense of (4.2). Assuming that $\Phi(1) < 1$, we have that

$$N_{\Phi_1}(f) \le \|f\|_{\Phi_1} \le 2N_{\Phi_1}(f)$$

(see Proposition III.3.4 in Rao and Ren, 1991). It is well known that the linear space $\mathcal{L}^{\Phi_1}(\mu)$ with the norm N_{Φ_1} is a Banach space. If the convex function Ψ is a Young function, we have that the distance d_k in (4.3) is an Orlicz norm. Given a Young function Φ_1, $\mathcal{M}^{\Phi_1}$ denotes the Banach space consisting by the class of functions f such that for each $\lambda > 0$, $\int \Phi_1(\lambda|f|)\, d\mu < \infty$, with the norm N_{Φ_1}. We will use that $(\mathcal{M}^{\Phi_1})^* = \mathcal{L}^{\Psi_1}$ (see Theorem IV.1.7 in Rao and Ren, 1991). We will say that a sequence of functions γ_n in $\mathcal{L}^{\Psi_1}$ converges weakly to γ_0 in $\sigma(\mathcal{L}^{\Psi_1}, \mathcal{M}^{\Phi_1})$ if

$$\int \gamma_n f\, d\mu \to \int \gamma_0 f\, d\mu$$

for each $f \in \mathcal{M}^{\Phi_1}$. A function Φ is called an N–function (a nice Young function) if Φ is a continuous Young function such that $\Phi(x) > 0$ for $x \neq 0$, $\lim_{x\to 0} x^{-1}\Phi(x) = 0$ and $\lim_{x\to\infty} x^{-1}\Phi_1(x) = \infty$. We will use that if Φ is an N–function, then a bounded set in $\mathcal{L}^{\Psi_1}$ is $\sigma(\mathcal{L}^{\Psi_1}, \mathcal{M}^{\Phi_1})$–sequentially relatively compact (see Corollary IV.5.5 in Rao and Ren, 1991).

We will need the following lemma:

Lemma 4.3 *Let $\Psi : I\!\!R \to [0, \infty]$ be a convex function. Let $(S, \mathcal{S})$ be a measurable space. Let μ be a measure on S. Let γ be a function on S. Then,*

$$\begin{aligned}&\int \Psi(\gamma(x))\, d\mu(x) = \sup\{\textstyle\sum_{j=1}^m \mu(B_j)\Psi\left(\frac{1}{\mu(B_j)}\int_{B_j}\gamma(x)\, d\mu(x)\right)\\ &: B_1, \dots, B_m \text{ are disjoint sets and } 0 < \mu(B_j) < \infty\}.\end{aligned}$$

The proof of the previous lemma is omitted since it is trivial.

Theorem 4.4 *Let $\Phi : I\!\!R \to [0, \infty)$ be a convex function such that $\Phi(0) = 0$, $\Phi'(0) = a$ exists, $\max(\Phi(x) - ax, \Phi(-x) + ax) > 0$ for each $x \neq 0$, and $\lim_{x\to\infty} x^{-1}\max(\Phi(x), \Phi(-x)) = \infty$. Let Ψ be the conjugate function of Φ in the sense of (4.2). Let $(S, \mathcal{S})$ be a measurable space. Let μ be a measure on S. Let $\{f(x,t) : t \in T\}$ be a class of measurable functions. Suppose that:*

(i) For each $t \in T$ and each $\lambda > 0$,

$$\int \Phi_1(\lambda f(x,t))\, d\mu(x) < \infty,$$

where $\Phi_1(x) = \max(\Phi(x) - ax, \Phi(-x) + ax)$.

(ii) (T, d) is totally bounded, where $d(s,t) = \sum_{k=1}^\infty k^{-2}\min(d_k(s,t), 1)$ and

$$d_k(s,t) = \sup\{|\int (f(x,s) - f(x,t))\gamma(x)\, d\mu(x)| : \int \Psi(\gamma(x))\, d\mu(x) \le k\}.$$

(iii) If $a \neq 0$, suppose also that $\mu(S) < \infty$.

Then,

$$I(z) = \sup\{I_{t_1,\ldots,t_m}(z(t_1),\ldots,z(t_m)) : t_1,\ldots,t_m \in T, m \geq 1\},$$

where

$$I_{t_1,\ldots,t_m}(u_1,\ldots,u_m) = \inf\{\int \Psi(\gamma(x))\,d\mu(x) : \int f(x,t_j)\gamma(x)\,d\mu(x) = u_j, \text{ for each } 1 \leq j \leq m\} \tag{4.5}$$

and

$$I(z) = \inf\left\{\int \Psi(\gamma(x))\,d\mu(x) : \int f(x,t)\gamma(x)\,d\mu(x) = z(t) \text{ for each } t \in T\right\}. \tag{4.6}$$

Proof. We have that $\Phi_1(x)$ is an N–function with conjugate $\Psi_1(x) = \min(\Psi(a+x),\Psi(a-x))$. Let

$$I^{(1)}(z) = \sup\{I_{t_1,\ldots,t_m}(z(t_1),\ldots,z(t_m)) : t_1,\ldots,t_m \in T, m \geq 1\}.$$

Obviously, $I^{(1)}(z) \leq I(z)$. We need to prove that if $I^{(1)}(z) < \infty$, then $I^{(1)}(z) \geq I(z)$. Take $\{s_n\}$ such that $\{s_n\}$ is dense in (T,d) and $I^{(1)}(z) = \lim_{n\to\infty} I_{s_1,\ldots,s_n}(z(s_1),\ldots,z(s_n))$. Take γ_n such that $\int \gamma_n(x) f(x,s_j)\,d\mu(x) = z(s_j)$ for each $1 \leq j \leq n$ and

$$\int \Psi(\gamma_n(x))\,d\mu(x) \leq I_{s_1,\ldots,s_n}(z(s_1),\ldots,z(s_n)) + n^{-1}.$$

Let $k_0 > I^{(1)}(z)$. We have that for n large enough $\int \Psi_1(\gamma_n(x)+a)\,d\mu(x) \leq k_0$. So, by Corollary IV.5.5 in Rao and Ren (1991), $\{\gamma_n + a\}$ is weakly compact in $(\mathcal{M}^{\Phi_1})^* = \mathcal{L}^{\Psi_1}$. Hence, there exists a subsequence n_k and $\gamma_0 + a \in \mathcal{L}^{\Psi_1}$ such that such $\gamma_{n_k} + a$ converges weakly to $\gamma_0 + a$ in $\sigma(\mathcal{L}^{\Psi_1},\mathcal{M}^{\Phi_1})$. This implies that $\int \gamma_0(x) f(x,s_j)\,d\mu(x) = z(s_j)$ for each $j \geq 1$. By Lemma 4.3, $\int \Psi(\gamma_0(x))\,d\mu(x) \leq I^{(1)}(z)$. Since z and $\int f(x,t)\gamma(x)\,d\mu(x)$ are d–uniformly continuous functions, $\int \gamma_0(x) f(x,t)\,d\mu(x) = z(t)$ for each $t \in T$. □

Unless $\lim_{x\to\infty} |x|^{-1} \max(\Phi(x),\Phi(-x)) = \infty$, the rate function does not have the form in the previous lemma (see Lynch and Sethuraman, 1987).

The results in this section translate to a Banach space in an usual way. Let B be a separable Banach space. Let $\{U_n\}$ be a sequence of r.v.'s with values in B. Suppose that for each $f \in B^*$,

$$\lim_{n\to\infty} \epsilon_n \log(E[\exp(\epsilon_n^{-1} f(U_n))]) = \int \Phi(f(x))\,d\mu(x),$$

where μ is a measure on B and Φ is a convex function. Under the conditions in Theorem 4.4, the rate function for the LDP of $\{U_n\}$ with speed ϵ_n^{-1} is

$$I(z) = \inf\{\int \Psi(\gamma(x))\,d\mu(x) : \int x\gamma(x)\,d\mu(x) = z,\ \gamma : B \to \mathbb{R} \text{ is a measurable function}\}.$$

Next, we consider the simplest case to which the previous lemmas apply.

Theorem 4.5 *Let $\{U_n(t) : 0 \leq t \leq M\}$ be a sequence of stochastic processes. Let $\{\epsilon_n\}$ be a sequence of positive numbers that converges to zero. Let Φ be a nonnegative convex function. Suppose that:*

(i) For each $0 \leq t_1 \leq \cdots \leq t_m \leq M$ and each $\lambda_1, \ldots, \lambda_m \in I\!R$,

$$\lim_{n\to\infty} \epsilon_n \log \left(E[\exp(\epsilon_n^{-1} \sum_{j=1}^m \lambda_j U_n(t_j))] \right) = \sum_{j=1}^m \Phi(\sum_{i=j}^m \lambda_j)(t_j - t_{j-1}).$$

(ii) For each $\eta > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \epsilon_n \log \Pr\{ \sup_{\substack{|s-t|\leq\delta \\ 0\leq s,t\leq M}} |U_n(s) - U_n(t)| \geq \eta\} = -\infty.$$

(iii) $\Phi(0) = 0$, $\Phi'(0) = a$ exists, $\max(\Phi(x) - ax, \Phi(-x) + ax) > 0$ for each $x \neq 0$, and $\lim_{x\to\infty} x^{-1} \max(\Phi(x), \Phi(-x)) = \infty$.

Then, $\{U_n(t) : 0 \leq t \leq M\}$ satisfies the LDP in $l_\infty[0, M]$ with speed ϵ_n^{-1} and rate function

$$I(z) = \begin{cases} \int_0^M \Psi(z'(t))\, dt, & \text{if } z(0) = 0 \text{ and } z \text{ is absolutely continuous} \\ \infty, & \text{else.} \end{cases}$$

Proof. It follows from theorems 4.1 and 4.4, and Lemma 4.2 with $f(x, t) = I(0 \leq x \leq t)$ and μ equal to the Lebesgue measure. Observe that have condition (iii) in Theorem 4.4, we need that for each $k \geq 1$, $([0, M], d_k)$ is totally bounded, where

$$d_k(s, t) = \sup\{\left|\int_s^t \gamma(x)\, dx\right| : \int_0^M \Psi(\gamma(x))\, dx \leq k\}.$$

It suffices to show that $\lim_{\eta\to 0} \sup_{\substack{0\leq s,t\leq M \\ |s-t|\leq\eta}} d_k(s, t) = 0$. But, given $\lambda > 0$, $0 \leq s, t \leq M$ and γ with $\int_0^M \Psi(\gamma(x))\, dx \leq k$,

$$\int_s^t \gamma(x)\, dx \leq \int_s^t \lambda^{-1}(\Psi(\gamma(x)) + \Phi(\lambda))\, dx \leq \lambda^{-1}k + \lambda^{-1}|s - t|\Phi(\lambda),$$

and

$$-\int_s^t \gamma(x)\, dx \leq \int_s^t \lambda^{-1}(\Psi(\gamma(x)) + \Phi(-\lambda))\, dx \leq \lambda^{-1}k + \lambda^{-1}|s - t|\Phi(-\lambda).$$

Hence,

$$\sup_{\substack{0\leq s,t\leq M \\ |s-t|\leq\eta}} d_k(s, t) \leq \inf_{\lambda>0} (\lambda^{-1}k + \lambda^{-1}\eta \max(\Phi(\lambda), \Phi(\lambda))),$$

which implies condition (iii) in Theorem 4.4. □

An analogous of the previous theorem hold for processes defined in $[-M_1, M_2]$ where $M_1, M_2 > 0$.

The previous theorem can be used to give compositions of processes.

Theorem 4.6 *Let $\{U_n(t) : t \in I\!R\}$ and let $\{V_n(t) : 0 \le t \le M_0\}$ be two sequences of stochastic processes, where $M_0 > 0$. Let $\{\epsilon_n\}$ be a sequence of positive numbers that converges to zero. Suppose that:*

(i) For each $0 \le t_1 \le \cdots \le t_m$, each $0 \le s_1 \le \cdots \le s_m$, each $0 \le r_1 \le \cdots \le r_m \le M_0$, each $\lambda_1, \dots, \lambda_m \in I\!R$, each $\tau_1, \dots, \tau_m \in I\!R$ and each $\nu_1, \dots, \nu_m \in I\!R$,

$$\begin{array}{rl} & \lim_{n\to\infty} \epsilon_n \log \Big(E \Big[\exp \Big(\epsilon_n^{-1} \Big(\sum_{j=1}^m \lambda_j U_n(t_j) \\ & \quad + \sum_{j=1}^m \tau_j U_n(-s_j) + \sum_{j=1}^m \nu_j V_n(r_j) \Big) \Big) \Big] \Big) \\ = & \sum_{j=1}^m \Phi_1(\sum_{i=j}^m \lambda_j)(t_j - t_{j-1}) + \sum_{j=1}^m \Phi_1(\sum_{i=j}^m \tau_j)(s_j - s_{j-1}) \\ & + \sum_{j=1}^m \Phi_2(\sum_{i=j}^m \nu_j)(r_j - r_{j-1}) \end{array}$$

where Φ_1 and Φ_2 are two nonnegative convex functions.

(ii) For each $\eta > 0$ and each $0 < M < \infty$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \epsilon_n \log \Pr\{ \sup_{\substack{|s-t|\le\delta \\ |s|,|t|\le M}} |U_n(s) - U_n(t)| \ge \eta\} = -\infty.$$

(iii) For each $\eta > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \epsilon_n \log \Pr\{ \sup_{\substack{|s-t|\le\delta \\ 0\le s,t\le M_0}} |V_n(s) - V_n(t)| \ge \eta\} = -\infty.$$

(iv) For $i = 1, 2$, $\Phi_i(0) = 0$, $\Phi_i'(0) = a_i$ exists, $\max(\Phi_i(x) - a_i x, \Phi_i(-x) + a_i x) > 0$ for $x > 0$ and $\lim_{x\to\infty} x^{-1} \max(\Phi_i(x), \Phi_i(-x)) = \infty$.

(v) $\max(\Phi_2(\Phi_1(x)) - a_1 a_2 x, \Phi_2(\Phi_1(-x)) + a_1 a_2 x) > 0$ for $x > 0$ and $\lim_{x\to\infty} x^{-1} \max(\Phi_2(\Phi_1(x)), \Phi_2(\Phi_1(-x))) = \infty$.

Then, $\{U_n(V_n(t)) : 0 \le t \le M_0\}$ satisfies the LDP in $l_\infty[0, M_0]$ with speed ϵ_n^{-1} and rate function

$$I(z) = \begin{cases} \int_0^{M_0} \Psi_{2,1}(z'(t))\, dt, & \text{if } z(0) = 0 \text{ and } z \text{ is absolutely continuous} \\ \infty, & \text{else,} \end{cases}$$

where $\Psi_{2,1}$ is the conjugate of $\Phi_2 \circ \Phi_1$.

Proof. By Corollary 3.10 and Theorem 4.5, $\{U_n(V_n(t)) : 0 \le t \le M_0\}$ satisfies the LDP in $l_\infty[0, M_0]$ with speed ϵ_n^{-1} and rate function

$$\begin{array}{l} I(z) = \inf\{\int_0^\infty \Psi_1(\alpha'(t))\, dt + \int_0^{M_0} \Psi_2(\beta'(t))\, dt \\ : z = \alpha \circ \beta, \alpha(0) = 0 \text{ and } \beta(0) = 0\}. \end{array}$$

So, the rate function for the LDP of the finite dimensional distributions is

$$\begin{array}{rl} & I_{t_1,\dots,t_m}(u_1, \dots, u_m) \\ = & \inf\{\int_0^\infty \Psi_1(\alpha'(t))\, dt + \int_0^{M_0} \Psi_2(\beta'(t))\, dt \\ & : z = \alpha \circ \beta,\ \gamma(t_j) = u_j, 1 \le j \le m\} \\ = & \inf\{\sum_{j=1}^m \int_{v_{j-1}}^{v_j} \Psi_1(\alpha'(t))\, dt + \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \Psi_2(\beta'(t))\, dt : \\ & \beta(t_j) = v_j, \alpha(v_j) = u_j, 1 \le j \le m\}. \end{array}$$

By the Jensen's inequality

$$\begin{array}{rl} & \int_{v_{j-1}}^{v_j} \Psi_1(\alpha'(t))\, dt \geq (v_j - v_{j-1})\Psi_1((v_j - v_{j-1})^{-1} \int_{v_{j-1}}^{v_j} \alpha'(t)\, dt) \\ = & (v_j - v_{j-1})\Psi_1((u_j - u_{j-1})/(v_j - v_{j-1})), \end{array}$$

where we have inequality if α' is a constant. A similar inequality holds for Ψ_2. So,

$$\begin{array}{rl} (4.7) & I_{t_1,\ldots,t_m}(u_1, \ldots, u_m) \\ = & \inf\{\sum_{j=1}^m (v_j - v_{j-1})\Psi_1((v_j - v_{j-1})^{-1}(u_{j-1} - u_j)) \\ & + \sum_{j=1}^m (t_j - t_{j-1})\Psi_2((t_j - t_{j-1})^{-1}(v_j - v_{j-1})) : v_1, \ldots, v_m\} \\ = & \inf\{\sum_{j=1}^m (v_j - v_{j-1})\Psi_1((v_j - v_{j-1})^{-1}(u_{j-1} - u_j)) \\ & + \sum_{j=1}^m (t_j - t_{j-1})\Psi_2((t_j - t_{j-1})^{-1}(v_j - v_{j-1})) : v_1, \ldots, v_m\} \\ = & \sum_{j=1}^m \inf\{v\Psi_1(v^{-1}(u_{j-1} - u_j)) + (t_j - t_{j-1})\Psi_2((t_j - t_{j-1})^{-1}v) : v\} \\ = & \sum_{j=1}^m (t_j - t_{j-1}) \inf\{v\Psi_1(v^{-1}(t_j - t_{j-1})^{-1}(u_{j-1} - u_j)) + \Psi_2(v) : v\} \\ = & \sum_{j=1}^m (t_j - t_{j-1})\Psi_{2,1}((t_j - t_{j-1})^{-1}(u_{j-1} - u_j)). \end{array}$$

Observe that the minimax theorem (see for example Theorem 37.1.3 in Rockafellar, 1970)

$$\begin{array}{rl} & \inf_v\{v\Psi_1(v^{-1}x) + \Psi_2(v)\} = \inf_v \sup_b\{v(v^{-1}xb - \Phi_1(b)) + \Psi_2(v)\} \\ = & \sup_b \inf_v\{xb - v\Phi_1(b) + \Psi_2(v)\} = \sup_b\{xb - \sup_v\{v\Phi_1(b) - \Psi_2(v)\}\} \\ = & \sup_b\{xb - \Phi_2(\Phi_1(b))\} = \Psi_{2,1}(x) \end{array}$$

Since the rate function for the finite dimensional distributions is given by (4.7), Theorem 4.4 implies that the rate function is as claimed. □

The previous result is related with Lemma 2.1 on the Strassen class appearing in the law of the iterated logarithm of the iterated logarithm in Csáki, Földes, and Révész (1997).

5 The LDP Gaussian processes

In this section, we consider the large deviation principle of Gaussian processes. In the case of a sequence of Gaussian r.v.'s, we have the following theorem:

Theorem 5.1 *Let $\{X_n\}_{n=1}^\infty$ be a sequence of Gaussian r.v.'s with mean μ_n and variance σ_n^2. Let $\{\epsilon_n\}$ be a sequence of positive numbers converging to zero. Then, the following conditions are equivalent:*

(a) There are $\mu \in I\!\!R$ and $0 \leq a < \infty$ such that $\lim_{n\to\infty} \mu_n = \mu$ and $\lim_{n\to\infty} \epsilon_n^{-1}\sigma_n^2 = a$.

(b) $\{X_n\}_{n=1}^\infty$ satisfies the LDP with speed ϵ_n^{-1}.

Moreover, if either (a) or (b) holds, the rate function is $I(t) = \frac{(t-\mu)^2}{2a}$, if $a > 0$; $I(t) = 0$ if $t = \mu$ and $a = 0$; and $I(t) = \infty$ if $t \neq \mu$ and $a = 0$.

The proof of the previous theorem is omitted, since it follows from well known estimations on the tail of a standard normal r.v.

To obtain the rate function in the LDP of Gaussian processes, we use the results in Section 4 with $\Phi(x) = \Psi(x) = 2^{-1}x^2$. Suppose that there exists a measurable space $(S, \mathcal{S})$, a positive measure μ on S and a class of measurable functions $\{f(x,t) : t \in T\}$ such that the rate function for the finite dimensional distributions is

$$\begin{aligned} &I_{t_1,\dots,t_m}(u_1,\dots,u_m) = \inf\{\textstyle\int 2^{-1}\gamma^2(x)\,d\mu(x) : \\ &\textstyle\int f(x,t_j)\gamma(x)\,d\mu(x) = u_j, \text{ for each } 1 \le j \le m\}. \end{aligned}$$

The rate function for the stochastic process is

$$\begin{aligned} I(z) = \inf\Bigg\{\int 2^{-1}\gamma^2(x)\,d\mu(x) \\ : \int f(x,t)\gamma(x)\,d\mu(x) = z(t), \text{ for each } t \in T\Bigg\}, \end{aligned} \tag{5.1}$$

where $z \in l_\infty(T)$. Sometimes, it is preferable to write this rate function using reproducing kernel Hilbert spaces. In the previous situation,

$$R(s,t) = \int f(x,t)f(x,s)\,d\mu(x)$$

is a covariance function, i.e. for each $\lambda_1,\dots,\lambda_m \in \mathbb{R}$ and each $t_1,\dots,t_m \in T$,

$$\sum_{i,j=1}^m \lambda_i\lambda_j R(t_i,t_j) \ge 0.$$

Hence, there exists a mean–zero Gaussian process $\{Z(t) : t \in T\}$ such that $E[Z(s)Z(t)] = R(s,t)$ for each $s,t \in T$. Let $\mathcal{L}$ be the closed linear subspace of L_2, generated by $\{Z(t) : t \in T\}$. Let $\phi : \mathcal{L} \to l_\infty(T)$ defined by $\phi(\xi)(t) = E[Z(t)\xi]$. The reproducing kernel Hilbert space of the covariance function $R(s,t)$ is the Hilbert space $\{\phi(\xi) : \xi \in \mathcal{L}\}$ with respect to the inner product $<\phi(\xi_1), \phi(\xi_2)>= E[\xi_1\xi_2]$. The rate function in (5.1) can be written also as

$$I(z) = \inf\{2^{-1}E[\xi^2] : \xi \in \mathcal{L}, \phi(\xi) = z\}. \tag{5.2}$$

The next theorem gives necessary and sufficient conditions for the LDP for Gaussian processes.

Theorem 5.2 *Let $\{U_n(t) : t \in T\}$, $n \ge 1$, be a sequence of centered Gaussian processes. Let $\{\epsilon_n\}$ be a sequence of positive numbers such that $\epsilon_n \to 0$. Then, the following sets of conditions ((a) and (b)) are equivalent:*

(a.1) For each $s,t \in T$, $\epsilon_n^{-1}E[U_n(s)U_n(t)]$ converges as $n \to \infty$.

(a.2) (T,d) *is totally bounded, where*

$$d^2(s,t)=\lim_{n\to\infty}\epsilon_n^{-1}E[(U_n(s)-U_n(t))^2].$$

(a.3) $\sup_{t\in T}|U_n(t)|\xrightarrow{\Pr}0$.
(a.4) $\lim_{\eta\to 0}\limsup_{n\to\infty}\sup_{d(s,t)\le\eta}\epsilon_n^{-1}E[(U_n(s)-U_n(t))^2]=0$.
(b) $\{U_n(t):t\in T\}$ *satisfies the LDP in* $l_\infty(T)$ *with speed* ϵ_n^{-1}.

Moreover, if either (a) or (b) holds, the rate function is defined by (5.2) where

$$R(s,t)=\lim_{n\to\infty}\epsilon_n^{-1}E[U_n(s)U_n(t)].$$

Proof. Assume conditions (a). We apply Theorem 3.2. By condition (a.2), condition (a.1) in Theorem 3.2 is satisfied.

Given $t_1,\dots,t_m\in T$ and $\lambda_1,\dots,\lambda_m\in\mathbb{R}$, we have that

$$\epsilon_n\log(E[\exp(\epsilon_n^{-1}\sum_{j=1}^m\lambda_jU_n(t_j))])\to 2^{-1}E[(\sum_{j=1}^m\lambda_jZ(t_j))^2],$$

where $\{Z(t):\ t\in T\}$ is a centered Gaussian process with covariance $E[Z(s)Z(t)]=R(s,t)$, $s,t\in T$. The previous limit and the Ellis Theorem (Theorem 4.1) imply condition (a.2) in Theorem 3.2 with the rate function

$$I_{t_1,\dots,t_m}(u_1,\dots,u_m)=\sup_{\lambda_1,\dots,\lambda_m}\left\{\sum_{j=1}^m\lambda_ju_j-2^{-1}\sum_{j,k=1}^m\lambda_j\lambda_kE[Z(t_j)Z(t_k)]\right\}$$

By Lemma 4.2, this rate function can be expressed as

$$\inf\left\{2^{-1}E[\xi^2]:\xi\in\mathcal{L},E[\xi Z(t_j)]=u_j\text{ for each }1\le j\le m\right\}.$$

By the isoperimetric inequality for Gaussian processes (Sudakov and Cirel'son, 1974; and Borell, 1975),

$$\begin{aligned}&\Pr{}^*\{|\sup_{d(s,t)\le\eta}|U_n(s)-U_n(t)|-M_n|\ge u\}\\ \le\ &\exp\left(-\tfrac{u^2}{2\sup_{d(s,t)\le\eta}E[|U_n(s)-U_n(t)|^2]}\right),\end{aligned}$$

where M_n is the median of $\sup_{d(s,t)\le\eta}|U_n(s)-U_n(t)|$. This inequality and (a.4) imply (a.3) in Theorem 3.2. Therefore, (b) in Theorem 3.2 holds with the rate function

$$I(z)=\sup\{I_{t_1,\dots,t_m}(z(t_1),\dots,z(t_m)):t_1,\dots,t_m\in T,m\ge 1\},$$

where $I_{t_1,\dots,t_m}$ was defined above. By Theorem 4.4 this rate function can be expressed as in (5.2).

Assume condition (b). The contraction principle implies that for each $t_1, t_2 \in T$ and each $\lambda_1, \lambda_2 \in \mathbb{R}$, the LDP for $\lambda_1 U_n(t_1) + \lambda_2 U_n(t_2)$ with speed ϵ_n^{-1} holds. By Theorem 5.1, for each $t_1, t_2 \in T$ and each $\lambda_1, \lambda_2 \in \mathbb{R}$, $\epsilon_n^{-1} E[(\lambda_1 U_n(t_1) + \lambda_2 U_n(t_2))^2]$ converges. Therefore, condition (a.1) holds. Besides this implies that the rate function for the finite dimensional distributions is

$$\begin{aligned} & I_{t_1,\dots,t_m}(u_1, \dots, u_m) \\ = \; & \inf\left\{2^{-1}E[\xi^2] : \xi \in \mathcal{L}, E[\xi Z(t_j)] = u_j \text{ for each } 1 \le j \le m\right\}. \end{aligned}$$

So, $\rho_k(s,t) = \sup\{|u_2 - u_1| : I_{s,t}(u_1, u_2) \le k\} = d(s,t)(2k)^{1/2}$. So, by condition (a.1) in Theorem 3.2, (a.2) holds.

By condition (a.3) in Theorem 3.2, for each $\tau > 0$,

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \Pr{}^*\{\sup_{d(s,t) \le \eta} |U_n(t) - U_n(s)| \ge \tau\} = 0.$$

This and the fact that $U_n(t) \xrightarrow{\Pr} 0$, for each $t \in T$, implies (a.3).

Condition (a.3) in Theorem 3.2 also implies that for each $\tau > 0$,

$$\lim_{\eta \to 0} \sup_{d(s,t) \le \eta} \limsup_{n \to \infty} \epsilon_n \log \Pr\{|U_n(t) - U_n(s)| \ge \tau\} = -\infty,$$

which implies (a.4). $\square$

Large deviations for Gaussian processes have been considered by several authors. Schilder (1966) considered large deviations for the Brownian motion. In our notation, Chevet (1983) proved that if $\{\epsilon_n^{-1/2} U_n(t) : t \in T\}$ converges weakly to a Radon Gaussian process $\{Z(t) : t \in T\}$, then $\{U_n(t) : t \in T\}$ satisfies the LDP with speed ϵ_n^{-1}. The previous theorem generalizes Theorem 2 in this reference. It is easy to find examples which do not satisfy the conditions in the Chevet theorem. Let $\{g_k\}_{k=1}^\infty$ be a sequence of i.i.d.r.v.'s with standard normal distribution. Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers converging to infinity. Let $U_n(k) = a_n^{-1}(\log k)^{-1/2} g_k$. It follows from Theorem 5.2 that $\{U_n(k) : k \ge 1\}$, $n \ge 1$, satisfies the LDP in $l_\infty(\mathbb{N})$ with speed a_n^2. However, it is not true that $\{a_n U_n(k) : k \ge 1\}$ converges weakly to a Gaussian process in $l_\infty(\mathbb{N})$.

The previous theorem implies that if $\{B(t) : t \ge 0\}$ is a Brownian motion and $\{a_n\}$ is a sequence of real numbers such that $n^{-1/2} a_n \to 0$, then, for each $0 < M < \infty$,

$$\{a_n B(n^{-1}t) : 0 \le t \le M\}$$

satisfies the LDP in $l_\infty[0, M]$ with speed $n a_n^{-2}$ and rate function

$$I(z) = \begin{cases} \int_0^M 2^{-1}|z'(t)|^2\, dt, & \text{if } z(0) = 0 \\ & \text{and } z \text{ is absolutely continuous} \\ \infty, & \text{else} \end{cases} \tag{5.3}$$

(this result is due to Schilder, 1966). The next theorem considers centered Gaussian processes with stationary increments:

Theorem 5.3 *Let $\{X(t) : t \geq 0\}$ be a centered Gaussian process with stationary increments and $X(0) = 0$. Let $0 < M < \infty$. Let $\{a_n\}$ be a sequence of real numbers such that $a_n^2 E[X^2(n^{-1}M)] \to 0$. Then, the following sets of conditions are equivalent:*

(a.1) For some $0 < \alpha \leq 1$, for each $0 < t$,

$$\lim_{n\to\infty} \frac{E[X^2(n^{-1}t)]}{E[X^2(n^{-1}M)]} = t^{2\alpha}.$$

(a.2) $\sup_{0\leq t\leq M} a_n|X(n^{-1}t)| \xrightarrow{\Pr} 0.$

(b) $\{a_n X(n^{-1}t) : 0 \leq t \leq M\}$ satisfies the LDP with speed $a_n^{-2}(E[X^2(n^{-1}M)])^{-1}$.

Moreover, for $\alpha = 1/2$, the rate function is given by (5.3); if $0 < \alpha < 1$ and $\alpha \neq 1/2$, the rate function is given by

$$(5.4) \qquad I(z) = \inf\Big\{2^{-1}\tau_\alpha \int_{-\infty}^{\infty} \phi^2(x)\, dx : \\ \tau_\alpha \int_{-\infty}^{\infty} \phi(x)(|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2})\, dx = z(t), \text{ for each } 0 \leq t \leq M\Big\},$$

where $z \in l_\infty([0,1])$ and

$$\tau_\alpha = \left(\int_{-\infty}^{\infty} (|x-1|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2})^2\, dx\right)^{-1};$$

and if $\alpha = 1$, the rate function is

$$(5.5) \quad I(z) = \begin{cases} 2^{-1}a^2 & \textit{if for some } a,\ z(t) = at \textit{ for each } 0 \leq t \leq M, \\ \infty & \textit{else.} \end{cases}$$

Proof. Without loss of generality, we may assume that $M = 1$. Assume (a), we apply Theorem 5.2. Conditions (a.1)–(a.3) in Theorem 5.2 are assumed. By regular variation,

$$\lim_{\eta\to 0+} \limsup_{n\to\infty} \sup_{0<t\leq\eta} \frac{E[X^2(n^{-1}t)]}{E[X^2(n^{-1})]} = 0.$$

This implies condition (a.4) in Theorem 5.2.

Assume (b). Theorem 5.2 implies (a.2). By Theorem 5.1, for each $0 \leq s, t \leq 1$,

$$\lim_{n\to\infty} \frac{E[X(n^{-1}s)X(n^{-1}t)]}{E[X^2(n^{-1})]}$$

exists. By Theorem 1.9 in Bingham, Goldie and Teguels, 1975), $E[X^2(n^{-1}t)]$ is regularly varying as $t \to 0$. Hence, there exists an $\alpha \in I\!R$ such that for each $t > 0$,

$$\lim_{n\to\infty} \frac{E[X^2(n^{-1}t)]}{E[X^2(n^{-1})]} = t^{2\alpha}.$$

Since condition (a.4) in Theorem 5.2 holds, $\alpha > 0$. For $0 \le s < t$, we have that $\|X(t)\|_2 \le \|X(s)\|_2 + \|X(t) - X(s)\|_2$. Hence, we have that $t^\alpha \le s^\alpha + |t-s|^\alpha$. Taking $t = 2s$, we get that $\alpha \le 1$.

If $\alpha \ne 1/2$, by the change of variable $x = ty$, we get that

$$\int_{-\infty}^{\infty} (|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2})^2\, dx = |t|^{2\alpha}\tau_\alpha^{-1}.$$

Hence,

$$\begin{aligned} & R(s,t) = 2^{-1}(s^{2\alpha} + t^{2\alpha} - |s-t|^{2\alpha}) \\ = \quad & \tau_\alpha \textstyle\int_{-\infty}^{\infty} (|x-s|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2})(|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2})\, dx. \end{aligned}$$

We take the measure μ defined in $I\!R$ by $d\mu(x) = \tau_\alpha\, dx$ and

$$f(x,t) = |x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}.$$

From Theorem 4.4, we get that the rate function for the LDP when $0 < \alpha < 1$ and $\alpha \ne 1/2$, is given by (5.4).

If $\alpha = 1/2$, we have that the rate function is given by (5.3).

If $\alpha = 1$, $R(s,t) = st$. We apply Theorem 4.4 with $S = [0,1]$, $f(x,t) = t$ and μ equal to the Lebesgue measure, we get that the rate in (5.5). $\square$

A centered Gaussian process $\{B_\alpha(t) : t \ge 0\}$, it is a fractional Brownian motion of order α, $0 < \alpha < 1$, if its covariance given by

$$E[B_\alpha(s)B_\alpha(t)] = 2^{-1}(s^{2\alpha} + t^{2\alpha} - |s-t|^{2\alpha}), s, t \ge 0.$$

It is easy to see that Theorem 5.3 applies to the fractional Brownian motion of order α, if $a_n n^{-\alpha} \to 0$.

Theorems 3.10 and 4.6 allows to obtain LDP for compositions of Gaussian processes.

Theorem 5.4 *Let $\{B(t) : t \in I\!R\}$ be a Brownian motion. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \to \infty$ and $n^{-1}a_n^2 \to 0$. Let $0 < M < \infty$. Then,*

$$\left\{ \frac{a_n^{2-2^{-k+1}}}{n^{1-2^{-k+1}}} B^{(k)}(n^{-1}t) : 0 \le t \le M \right\},$$

where $B^{(k)} = B \circ \overset{(k)}{\cdots} \circ B$, *satisfies the LDP in* $l_\infty[0,M]$ *with speed* na_n^{-2} *and rate function*

$$(5.6)\quad I(z) = \begin{cases} \int_0^M \Psi_{(k)}(z'(t))\,dt, & \text{if } z(0)=0 \\ & \text{and } z \text{ is absolutely continuous} \\ \infty, & \text{else,} \end{cases}$$

where $\Psi_{(k)}(x) = \frac{2^{k+1}-2}{2^{k2^k/(2^k-1)}}|x|^{2^k/(2^k-1)}$.

Proof. We only consider the composition of two Brownian motions in detail. The general case is similar. We apply Theorem 4.6 with $\Phi_1(x) = \Phi_2(x) = 2^{-1}x^2$. By Theorem 5.3, for each $0 < M_1 < \infty$,

$$\{n^{-1/2}a_n^{3/2}B(a_n^{-1}t) : -M_1 \le t \le M_1\}$$

satisfies the LDP with speed na_n^{-2}. We also have that

$$\{a_n B(n^{-1}t) : 0 \le t \le M\}$$

satisfies the LDP with rate na_n^{-2}. This implies conditions (ii) and (iii) in Theorem 4.6. We need to obtain the LDP for the finite dimensional distributions (condition (i) in this theorem), given $0 \le t_1 < \cdots < t_m \le M_1$ and $0 \le s_1 < \cdots < s_m \le M_1$ and $0 \le r_1 < \cdots < r_m \le M$, consider

$$\begin{array}{l}(n^{-1/2}a_n^{3/2}B(-a_n^{-1}t_m), \dots, n^{-1/2}a_n^{3/2}B(-a_n^{-1}t_1) \\ , n^{-1/2}a_n^{3/2}B(a_n^{-1}s_1), \dots, n^{-1/2}a_n^{3/2}B(a_n^{-1}s_m) \\ , a_nB(n^{-1}r_1), \dots, a_nB(n^{-1}r_m)).\end{array}$$

Given $\lambda_1, \dots, \lambda_m, \tau_1 \dots, \tau_m, \nu_1 \dots, \nu_m \in \mathbb{R}$,

$$\begin{array}{rl} & n^{-1}a_n^2 \log(E[\exp(na_n^{-2}(\sum_{i=1}^m \lambda_i n^{-1/2}a_n^{3/2}B(-a_n^{-1}t_i) \\ & +\sum_{i=1}^m \tau_i n^{-1/2}a_n^{3/2}B(a_n^{-1}s_i) + \sum_{j=1}^m \nu_j a_n B(n^{-1}r_j))))]) \\ = & n^{-1}a_n^2 \log(E[\exp(\sum_{i=1}^m \lambda_i n^{1/2}a_n^{-1/2}B(-a_n^{-1}t_i) \\ & +\sum_{i=1}^m \tau_i n^{1/2}a_n^{-1/2}B(a_n^{-1}s_i) + \sum_{j=1}^m \nu_j n a_n^{-1} B(n^{-1}r_j))]) \\ = & 2^{-1}n^{-1}a_n^2 E\left[\left(\sum_{i=1}^m \lambda_i n^{1/2}a_n^{-1/2}B(-a_n^{-1}t_i)\right)^2\right. \\ & +\left(\sum_{i=1}^m \tau_i n^{1/2}a_n^{-1/2}B(a_n^{-1}s_i)\right)^2 \\ & +\left(\sum_{j=1}^m \nu_j n a_n^{-1}B(n^{-1}r_j)\right)^2 \\ & +2\sum_{i=1}^m \lambda_i n^{1/2}a_n^{-1/2}B(-a_n^{-1}t_i)\sum_{j=1}^m \tau_j n^{1/2}a_n^{-1/2}B(a_n^{-1}s_j) \\ & +2\sum_{i=1}^m \lambda_i n^{1/2}a_n^{-1/2}B(-a_n^{-1}t_i)\sum_{j=1}^m \nu_j n a_n^{-1}B(n^{-1}r_j) \\ & \left.+2\sum_{i=1}^m \tau_i n^{1/2}a_n^{-1/2}B(a_n^{-1}s_i)\sum_{j=1}^m \nu_j n a_n^{-1}B(n^{-1}r_j)\right] \\ \to & 2^{-1}\sum_{i=1}^m\sum_{j=1}^m (\lambda_i\lambda_j \min(t_i,t_j) + \tau_i\tau_j\min(s_i,s_j) + \nu_i\nu_j\min(r_i,r_j)) \\ = & \sum_{j=1}^m \Big(\Phi_1(\sum_{i=j}^m \lambda_i)(t_j - t_{j-1}) + \Phi_1(\sum_{i=j}^m \tau_i)(s_j - s_{j-1}) \\ & +\Phi_2(\sum_{i=j}^m \nu_i)(r_j - r_{j-1})\Big). \end{array}$$

We have that

$$\Psi_{2,1}(x) = \sup_y (xy - \Phi_2(\Phi_1(y))) = 2^{-5/3} \cdot 3|x|^{4/3}.$$

The rest of the conditions in Theorem 4.6 are trivially satisfied.

In the general case, we prove that

$$\left\{ \frac{a_n^{2-2^{-j+1}}}{n^{1-2^{-j+1}}} B\left(\frac{n^{1-2^{-j+2}}}{a_n^{2-2^{-j+2}}} t \right) : 0 \le t \le M \right\}, 1 \le j \le k,$$

satisfy the LDP jointly with speed na_n^{-2} and the rate in Theorem 4.6 corresponding to $\Phi(x) = 2^{-1}x^2$. By composition, we get that

$$\left\{ \frac{a_n^{2-2^{-k+1}}}{n^{1-2^{-k+1}}} B^{(k)}(n^{-1}t) : 0 \le t \le M \right\}$$

satisfies the LDP in $l_\infty[0, M]$ with speed na_n^{-2} and rate as in Theorem 4.6 with

$$\begin{aligned} &\Psi_{(k)}(x) = \sup_y (xy - \Phi \circ \overset{(k)}{\cdots} \circ \Phi(y)) = \sup_y (xy - 2^{-2^k+1} y^{2^k}) \\ = \ &\frac{2^{k+1}-2}{2^{k2^k/(2^k-1)}} |x|^{2^k/(2^k-1)}. \quad \square \end{aligned}$$

Next, we present a law of the iterated logarithm for the iterated Brownian motion.

Theorem 5.5 *Let $\{B(t) : t \in I\!\!R\}$ be a Brownian motion. Let $0 < M < \infty$. Then, with probability one,*

$$\left\{ \frac{n^{2^{-k}}}{(\ln \ln n)^{1-2^{-k}}} B^{(k)}(n^{-1}t) : 0 \le t \le M \right\} \tag{5.7}$$

is relatively compact in $l_\infty[0, M]$ and it limit set is

$$\begin{aligned} \{z : [0, M] \to I\!\!R : z(0) = 0, z \ \text{ is absolutely continuous} \\ \text{and } \int_0^M \frac{2^{k+1} - 2}{2^{k2^k/(2^k-1)}} |z'(t)|^{2^k/(2^k-1)} \, dt \le 1\}. \end{aligned} \tag{5.8}$$

In particular,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{n^{2^{-k}}}{(\ln \ln n)^{1-2^{-k}}} B^{(k)}(n^{-1}M) = \frac{2^k M^{2^{-k}}}{(2^{k+1}-2)^{(2^k-1)/2^k}} \text{ a.s.} \\ &\liminf_{n\to\infty} \frac{n^{2^{-k}}}{(\ln \ln n)^{1-2^{-k}}} B^{(k)}(n^{-1}M) = \frac{-2^k M^{2^{-k}}}{(2^{k+1}-2)^{(2^k-1)/2^k}} \text{ a.s.} \\ &\limsup_{n\to\infty} \sup_{0 \le t \le M} \frac{n^{2^{-k}}}{(\ln \ln n)^{1-2^{-k}}} B^{(k)}(n^{-1}t) = \frac{2^k M^{2^{-k}}}{(2^{k+1}-2)^{(2^k-1)/2^k}} \text{ a.s.} \end{aligned}$$

PROOF. By Theorem 4.2 in Arcones (1995), for each $0 < M < \infty$,

$$(Y_{n,1}(t), \dots, Y_{n,k}(t)) : |t| \le M\}$$

is relatively compact in $l_\infty[-M, M]$ and it limit set is

$$\{(\alpha_1, \dots, \alpha_k) : \alpha_j(0) = 0, \text{ for } 1 \le j \le k, \text{ and } \sum_{j=1}^k \int_{-M}^M 2^{-1}(\alpha_j'(t))^2\, dt \le 1\},$$

where

$$Y_{n,j}(t) = \left\{ \frac{n^{2^{-j}}}{(\ln \ln n)^{1-2^{-j}}} B^{(j)} \left(\frac{(\ln \ln n)^{1-2^{-j+1}} t}{n^{2^{-j+1}}} \right) : 0 \le t \le M \right\}.$$

By composing the stochastic processes, we have that, with probability one, the stochastic process in (5.7) is relatively compact in $l_\infty[-M, M]$ and it limit set is

$$\begin{aligned} \{\alpha_1 \circ \cdots \circ \alpha_k : \alpha_j(0) = 0, &\text{ for } 1 \le j \le k, \\ &\text{and } \textstyle\int_{-M}^M 2^{-1}(\alpha_1'(t))^2\, dt + \sum_{j=2}^k \int_{-\infty}^\infty 2^{-1}(\alpha_j'(t))^2\, dt \le 1\}. \end{aligned}$$

The proof of Theorem 4.6 implies that this set is the same one as in (5.8).

The second part of the theorem follows by noticing that

$$\begin{aligned} & \sup\{z(M) : z(0) = 0 \text{ and } \textstyle\int_0^M \frac{2^{k+1}-2}{2^{k 2^k/(2^k-1)}} |z'(t)|^{2^k/(2^k-1)}\, dt \le 1\} \\ = & -\inf\{z(M) : z(0) = 0 \text{ and } \textstyle\int_0^M \frac{2^{k+1}-2}{2^{k 2^k/(2^k-1)}} |z'(t)|^{2^k/(2^k-1)}\, dt \le 1\} \\ = & \sup\{z(t) : 0 \le t \le M, z(0) = 0 \text{ and } \textstyle\int_0^M \frac{2^{k+1}-2}{2^{k 2^k/(2^k-1)}} |z'(t)|^{2^k/(2^k-1)}\, dt \le 1\} \\ = & \frac{2^k M^{2^{-k}}}{(2^{k+1}-2)^{(2^k-1)/2^k}}. \end{aligned}$$

Observe that

$$z(M) \le \left(\int_0^M |z'(t)|^{2^k/(2^k-1)}\, dt \right)^{(2^k-1)/2^k} M^{1/2^k} \le \frac{2^k M^{2^{-k}}}{(2^{k+1}-2)^{(2^k-1)/2^k}}$$

and we have equality if z' is a constant. □

The law of the iterated logarithm for the composition of two independent Brownian motion was obtained by Deheuvels and Mason (1992) and Burdzy (1993). More general versions of the compact law of the iterated logarithm for the composition of two Brownian motions are in Csáki, Csörgö, Földes and Révész (1995) and Arcones (1995).

The next theorem gives the integrability of the iterated Brownian motion.

Theorem 5.6 *Let $\{B(t) : t \in I\!R\}$ be a Brownian motion. Then,*

(i) $\left\{ n^{-1+2^{-k}} B^{(k)}(t) : 0 \le t \le M \right\}$ satisfies the LDP in $l_\infty[0, M]$ with speed n and rate function in (5.6).

(ii) For each $0 < M < \infty$*,*

$$\lim_{\lambda\to\infty} \lambda^{-2^k/(2^k-1)} \log(\Pr\{|B^{(k)}(M)| \geq \lambda\}) = \frac{-(2^{k+1}-2)}{2^{k2^k/(2^k-1)} M^{1/(2^k-1)}}$$

and

$$\lim_{\lambda\to\infty} \lambda^{-2^k/(2^k-1)} \log(\Pr\{\sup_{0\leq t\leq M} |B^{(k)}(t)| \geq \lambda\}) = \frac{-(2^{k+1}-2)}{2^{k2^k/(2^k-1)} M^{1/(2^k-1)}}.$$

(iii) In particular,

$$\begin{array}{l} E[\exp(\lambda \sup_{0\leq t\leq M} |B^{(k)}(t)|^{2^k/(2^k-1)})] < \infty \text{ if } \lambda < \frac{2^{k+1}-2}{2^{k2^k/(2^k-1)} M^{1/(2^k-1)}} \\ E[\exp(\lambda \sup_{0\leq t\leq M} |B^{(k)}(t)|^{2^k/(2^k-1)})] = \infty \text{ if } \lambda > \frac{2^{k+1}-2}{2^{k2^k/(2^k-1)} M^{1/(2^k-1)}}. \end{array}$$

Proof. By the proof of argument in Theorem 5.4, we get that

$$\{n^{-2^{-1}+2^{-j}} B(n^{-2^{-(j-1)}} t) : 0 \leq t \leq M\},\ 1 \leq j \leq k,$$

satisfy the LDP jointly with speed n and rate function $\Phi(t) = 2^{-1}t^2$. Hence,

$$n^{-2^{-1}+2^{-k}} B(n^{-1/2} B(n^{1/2} \cdots (B(n^{-1/2} B(n^{-1}t) \cdots) : 0 \leq t \leq M\}$$

satisfies the LDP with speed n and rate function in (5.6). Using that $\{n^{1/2}B(n^{-1}t) : 0 \leq t \leq M\}$ has the same distribution of $\{B(t) : 0 \leq t \leq M\}$, we get that $\{n^{-1+2^{-k}} B^{(k)}(t) : 0 \leq t \leq M\}$ satisfies the LDP with speed n and rate function in (5.6).

(ii) follows from the fact that

$$\begin{array}{rl} & \inf\left\{\int_0^M \frac{2^{k+1}-2}{2^{k2^k/(2^k-1)}} |z'(t)|^{2^k/(2^k-1)}\, dt : z(0) = 0, \sup_{0\leq t\leq M} |z(t)| \geq 1\right\} \\ = & \frac{2^{k+1}-2}{2^{k2^k/(2^k-1)} M^{1/(2^k-1)}}. \end{array}$$

Observe that if $\sup_{0\leq t\leq M} |z(t)| \geq 1$ and $\int_0^M \frac{2^{k+1}-2}{2^{k2^k/(2^k-1)}} |z'(t)|^{2^k/(2^k-1)}\, dt < \infty$, then there exists a $0 < t_0 \leq M$ such that

$$1 \leq |z(t_0)| \leq \left(\int_0^M |z'(t)|^{2^k/(2^k-1)}\, dt\right)^{(2^k-1)/2^k} t_0^{1/2^k} \leq \frac{2^k (I(z))^{(2^k-1)/2^k} M^{2^{-k}}}{(2^{k+1}-2)^{(2^k-1)/2^k}},$$

with equality for $t_0 = M$ and z' constant.

(iii) follows immediately from (ii). □

Finally, we consider the iterated fractional Brownian motion.

Theorem 5.7 *Let* $\{B_\alpha(t) \in \mathbb{R}\}$ *be a fractional Brownian motion of order* α *with* $0 < \alpha < 1$ *and* $\alpha \neq 1/2$*. Let* $\{a_n\}$ *be a sequence of real numbers such that*

$$a_n \to \infty \text{ and } n^{-\alpha} a_n \to 0.$$

Then,
$$\{n^{(\alpha^k-\alpha)/(1-\alpha)}a_n^{(1-\alpha^k)/(1-\alpha)}B_\alpha^{(k)}(n^{-1}t) : 0 \le t \le M\}$$
satisfies the LDP in $l_\infty[0,M]$ *with speed* $n^{2\alpha}a_n^{-2}$ *and rate function*

$$I(\gamma) = \inf\Big\{2^{-1}\tau_\alpha \sum_{j=1}^k \int_{-\infty}^\infty \phi_j^2(x)\,dx : \gamma(t) = \beta_1\circ\cdots\circ\beta_k(t) \text{ for each } 0\le t\le M,\ \tau_\alpha\int_{-\infty}^\infty \phi_j(x)(|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2})\,dx = \beta_j(t) \text{ for each } t\in I\!R \text{ and each } 1\le j\le k\Big\}. \tag{5.9}$$

Proof. We only consider the composition of two processes. The general case is similar. We apply Theorem 3.10. By Theorem 5.3, for $0 < M_1 < \infty$,
$$\{n^{-\alpha}a_n^{\alpha+1}B_\alpha(a_n^{-1}t) : -M_1 \le t \le M_1\}$$
satisfies the LDP with speed $n^{2\alpha}a_n^{-2}$ and the rate function in (5.4). We also have that
$$\{a_nB_\alpha(n^{-1}t) : 0 \le t \le M\}$$
satisfies the LDP with rate $n^{2\alpha}a_n^{-2}$ and the rate function in (5.4). To prove the joint LPD of the two stochastic processes, we need to prove the LDP for the joint finite dimensional distributions. Given $-M_1 \le t_1 < \cdots < t_m \le M_1$ and $0 \le s_1 < \cdots < s_m \le M$, we need prove the LDP for
$$\begin{array}{l}(n^{-\alpha}a_n^{\alpha+1}B_\alpha(a_n^{-1}t_1),\dots,n^{-\alpha}a_n^{\alpha+1}B_\alpha(a_n^{-1}t_p)\\ ,a_nB_\alpha(n^{-1}s_1),\dots,a_nB_\alpha(n^{-1}s_q)).\end{array}$$
Given $\lambda_1,\dots,\lambda_p,\tau_1\,\dots,\tau_q \in I\!R$, we have that
$$\begin{array}{ll} & n^{-2\alpha}a_n^2\log(E[\exp(n^{2\alpha}a_n^{-2}(\sum_{i=1}^p \lambda_i n^{-\alpha}a_n^{\alpha+1}B_\alpha(a_n^{-1}t_i)\\ & +\sum_{j=1}^q \tau_j a_nB_\alpha(n^{-1}s_j))))])\\ = & 2^{-1}n^{2\alpha}a_n^{-2}E\Big[(\sum_{i=1}^p \lambda_i n^{-\alpha}a_n^{\alpha+1}B_\alpha(a_n^{-1}t_i))^2 + (\sum_{j=1}^q \tau_j a_nB_\alpha(n^{-1}s_j))^2\\ & +2\sum_{i=1}^p\sum_{j=1}^q \lambda_i\tau_j n^{-\alpha}a_n^{\alpha+1}B_\alpha(a_n^{-1}t_i)a_nB_\alpha(n^{-1}s_j)\Big]\\ \to & 2^{-1}\sum_{i=1}^p\sum_{j=1}^p \lambda_i\lambda_j E[B_\alpha(t_i)B_\alpha(t_j)]\\ & +2^{-1}\sum_{i=1}^q\sum_{j=1}^q \tau_i\tau_j E[B_\alpha(s_i)B_\alpha(s_j)],\end{array}$$
which implies that the LDP for the joint finite dimensional distributions. □

The methods used before for the Brownian motion give the following result for the fractional Brownian motion.

Theorem 5.8 *Let* $\{B_\alpha(t) \in I\!R\}$ *be a fractional Brownian motion of order* α*,* $0<\alpha<1$*,* $\alpha \ne 1/2$*. Let* $0 < M < \infty$*. Then, with probability one,*
$$\left\{\frac{n^{\alpha^k}}{(\ln\ln n)^{(1-\alpha^k)/(2(1-\alpha))}}B^{(k)}(n^{-1}t) : 0 \le t \le M\right\}$$

is relatively compact in $l_\infty[0,M]$ *and it limit set is*

$$\Big\{\beta_1\circ\cdots\circ\beta_k : \textstyle\sum_{j=1}^k 2^{-1}\tau_\alpha \int_{-\infty}^{\infty}\phi_j^2(x)\,dx \le 1$$
$$\tau_\alpha\int_{-\infty}^{\infty}\phi_j(x)(|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2})\,dx=\beta_j(t)$$
$$\text{for each } \ t\in I\!R \text{ and each } \ 1\le j\le k\Big\}\,.$$

In particular,

$$\limsup_{n\to\infty}\frac{n^{\alpha^k}}{(\ln\ln n)^{(1-\alpha^k)/(2(1-\alpha))}}B^{(k)}(n^{-1}M)=C_k(\alpha)M^{\alpha^k}\quad\text{a.s.}$$
$$\liminf_{n\to\infty}\frac{n^{\alpha^k}}{(\ln\ln n)^{(1-\alpha^k)/(2(1-\alpha))}}B^{(k)}(n^{-1}M)=C_k(\alpha)M^{\alpha^k}\quad\text{a.s.}$$
$$\limsup_{n\to\infty}\sup_{0\le t\le M}\frac{n^{\alpha^k}}{(\ln\ln n)^{(1-\alpha^k)/(2(1-\alpha))}}B^{(k)}(n^{-1}t)=C_k(\alpha)M^{\alpha^k}\quad\text{a.s.}$$

where

$$C_k(\alpha)=\left(\frac{2(1-\alpha)}{1-\alpha^k}\right)^{(1-\alpha^k)/(1-\alpha)}\alpha^{\frac{(k-1)\alpha^{k+1}-k\alpha^k+\alpha}{(1-\alpha)^2}}.$$

Proof. The first part follows similarly to that of the Brownian motion. We just need to show that

$$\sup\Big\{\beta_1\circ\cdots\circ\beta_k(M) : \textstyle\sum_{j=1}^k 2^{-1}\tau_\alpha \int_{-\infty}^{\infty}\phi_j^2(x)\,dx \le 1$$
$$\tau_\alpha\int_{-\infty}^{\infty}\phi_j(x)(|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2})\,dx=\beta_j(t)$$
$$\text{for each } \ t\in I\!R \text{ and each } \ 1\le j\le k\Big\}=C_k(\alpha).$$

We have that

$$\beta_k(M)\le\tau_\alpha\|\phi_k\|_2(\int_{-\infty}^{\infty}(|x-1|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2})^2\,dx)^{1/2}M^\alpha=\tau_\alpha^{1/2}\|\phi_k\|_2M^\alpha,$$

$$\beta_{k-1}(\beta_k(M))\le\tau_\alpha^{1/2}\|\phi_{k-1}\|_2(\tau_\alpha^{1/2}\|\phi_k\|_2)^\alpha M^{\alpha^2},$$

and by induction

$$\beta_1\circ\cdots\circ\beta_k(M)\le M^{\alpha^k}\prod_{j=1}^k(\tau_\alpha^{1/2}\|\phi_j\|_2)^{\alpha^{j-1}}.$$

Moreover, we have inequality if $\phi_k(x)=|x-M|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2}$ and

$$\phi_j(x)=(|x-\beta_{j+1}(M)|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2})\ \text{ for } 1\le j\le k-1.$$

Therefore,

$$C_k(\alpha)=\sup\{\prod_{j=1}^k|x_j|^{\alpha^{j-1}}:2^{-1}\sum_{j=1}^k x_j^2\le 1\}.$$

To find the supremum in the previous expression, we use the theorem of the multipliers of Lagrange. We have that

$$\alpha^{j-1} x_j^{-1} \prod_{j=1}^{k} |x_j|^{\alpha^{j-1}} = \lambda x_j,$$

for each $1 \leq j \leq k$, where $\lambda \in I\!R$. So, $x_j^2 = \alpha^{j-1} c$, where c is a constant. Since

$$1 = 2^{-1} \sum_{j=1}^{k} x_j^2 = 2^{-1} \sum_{j=1}^{k} \alpha^{j-1} c = 2^{-1}(1-\alpha^k)(1-\alpha)^{-1} c,$$

$c = 2(1-\alpha)(1-\alpha^k)^{-1}$ and $x_j^2 = \alpha^{j-1} 2(1-\alpha)(1-\alpha^k)^{-1}$. Thus,

$$\begin{aligned} C_k(\alpha) &= \sup\{\textstyle\prod_{j=1}^{k} (\alpha^{j-1} 2(1-\alpha)(1-\alpha^k)^{-1})^{\alpha^{j-1}} \\ &= \left(\frac{2(1-\alpha)}{1-\alpha^k}\right)^{(1-\alpha^k)/(1-\alpha)} \alpha^{\frac{(k-1)\alpha^{k+1} - k\alpha^k + \alpha}{(1-\alpha)^2}}. \quad \square \end{aligned}$$

Theorem 5.9 *Let $\{B_\alpha(t) \in I\!R\}$ be a fractional Brownian motion of order α, $0 < \alpha < 1$, $\alpha \neq 1/2$. Then,*

(i) $\left\{n^{-\alpha(1-\alpha^k)/(1-\alpha)} B^{(k)}(t) : 0 \leq t \leq M\right\}$ satifies the LDP in $l_\infty[0, M]$ with speed $n^{2\alpha}$ and rate function in (5.7).

(ii) For each $0 < M < \infty$,

$$\lim_{\lambda \to \infty} \lambda^{-2(1-\alpha)/(1-\alpha^k)} \log(\Pr\{|B_\alpha^{(k)}(M)| \geq \lambda\}) = D_k(\alpha) M^{\frac{-2\alpha^k(1-\alpha)}{1-\alpha^k}}$$

and

$$\lim_{\lambda \to \infty} \lambda^{-2(1-\alpha)/(1-\alpha^k)} \log(\Pr\{\sup_{0 \leq t \leq M} |B_\alpha^{(k)}(t)| \geq \lambda\}) = D_k(\alpha) M^{\frac{-2\alpha^k(1-\alpha)}{1-\alpha^k}},$$

where

$$D_k(\alpha) = \frac{1-\alpha^k}{2(1-\alpha)\alpha^{\frac{(k-1)\alpha^{k+1} - k\alpha^k + \alpha}{(1-\alpha)(1-\alpha^k)}}}.$$

PROOF. (i) follows similarly to the Brownian motion. As to (ii), by the arguments in the previous theorem, we have that

$$\begin{aligned} & \inf\{I_k(z) : \textstyle\sup_{0 \leq t \leq M} |z(t)| \geq 1\} \\ = \; & \inf\left\{\textstyle\sum_{j=1}^{k} 2^{-1} x_j^2 : M^{\alpha^k} \prod_{j=1}^{k} x_j^{\alpha^{j-1}} \geq 1\right\} \\ = \; & \frac{1-\alpha^k}{2(1-\alpha)\alpha^{\frac{(k-1)\alpha^{k+1} - k\alpha^k + \alpha}{(1-\alpha)(1-\alpha^k)}} M^{\frac{2\alpha^k(1-\alpha)}{1-\alpha^k}}}, \end{aligned}$$

where the infimum is attained when $x_j^2 = c\alpha^{j-1}$, for some constant c. $\square$

6 The LDP for Poisson processes

In this section, we present several results on the LDP for Poisson processes.

By the Cramèr theorem (see for example Theorem 1.26 in Deuschel and Stroock, 1989) if $\{X_n\}$ is a sequence of i.i.d.r.v.'s with Poisson distribution and mean n, then $\{n^{-1}X_n\}$ satisfies the LDP with speed n and rate function

$$h(x) = \sup_{\lambda \in \mathbb{R}} (x\lambda - (e^{\lambda} - 1)).$$

It is easy to see that

$$h(x) = \begin{cases} x \log\left(\frac{x}{e}\right) + 1 & \text{if } x \geq 0; \\ \infty & \text{if } x < 0. \end{cases} \tag{6.1}$$

We will use that if ξ is a Poisson r.v. with mean λ, then for $a, t > 0$,

$$\Pr\{\xi \geq a\} \leq e^{-ta} E[e^{t\xi}] = \exp(-ta + \lambda(e^t - 1)).$$

Taking the supremum over $t > 0$, we get that for $a > \lambda$,

$$\Pr\{\xi \geq a\} \leq \exp(-\lambda h(\lambda^{-1}a)). \tag{6.2}$$

Instead of dividing over n in $\{n^{-1}X_n\}$, we can divide over a sequence of real numbers of growing faster than n. The LDP in this case is given by the following theorem:

Theorem 6.1 *Let $\{X_n\}$ be a sequence of Poisson r.v.'s with $E[X_n] = n$ and let $\{a_n\}$ be a sequence of positive numbers such that $n^{-1}a_n \to \infty$. Then, $\{a_n^{-1}X_n\}$ satisfies the LDP in $[0, \infty)$ with speed $a_n \log(n^{-1}a_n)$ and rate function $I(t) = t$ for $t \geq 0$; $I(t) = \infty$ for $t < 0$.*

Proof. By (6.2), given $t > 0$, for n large enough,

$$\Pr\{a_n^{-1}X_n \geq t\} \leq \exp\left(-a_n t \log\left(\frac{a_n t}{n}\right) + t a_n - n\right).$$

Hence,

$$\begin{aligned} & \limsup_{n\to\infty} a_n^{-1}(\log(n^{-1}a_n))^{-1} \log(\Pr\{a_n^{-1}X_n \geq t\}) \\ \leq \; & -\limsup_{n\to\infty} \left(\tfrac{t\log(n^{-1}a_n t)}{\log(n^{-1}a_n)} + \tfrac{t}{\log(n^{-1}a_n)} - \tfrac{1}{n^{-1}a_n \log(n^{-1}a_n)}\right) = -t. \end{aligned}$$

This implies that for each closed set $F \subset [0, \infty)$,

$$\limsup_{n\to\infty} a_n^{-1}(\log(n^{-1}a_n))^{-1} \log(\Pr\{a_n^{-1}X_n \in F\}) \leq -\inf\{t : t \in F\}.$$

Let U be an open set of $[0,\infty)$ and let $t\in U$. Let $k_n=[a_nt]$. For n large enough, by the Stirling formula

$$\begin{array}{rl} & \Pr\{a_n^{-1}X_n\in U\}\ge \Pr\{X_n=k_n\}=e^{-n}\frac{n^{k_n}}{k_n!} \\ \simeq & e^{-n}n^{k_n}k_n^{-k_n}e^{k_n}(2\pi k_n)^{-1/2}\simeq e^{-n}n^{a_nt}(ta_n)^{-ta_n}e^{ta_n}(2\pi ta_n)^{-1/2} \\ = & e^{-n}(n^{-1}ta_n)^{-ta_n}e^{ta_n}(2\pi ta_n)^{-1/2}, \end{array}$$

So,

$$\liminf_{n\to\infty} a_n^{-1}(\log(n^{-1}a_n))^{-1}\log(\Pr\{a_n^{-1}X_n\in U\})\ge -t.$$

Therefore, the claim follows. □

Next, we consider the LDP for Poisson processes. The LDP for homogeneous Poisson processes has been considered by Lynch and Sethuraman (1987). We consider non–homogeneous Poisson processes.

Theorem 6.2 *Let $\{N(t):t\ge 0\}$ be a Poisson process with mean measure μ such that $\mu[0,\infty)=\infty$. Let $0<M<\infty$. Then, the following conditions are equivalent:*

(a) Either $\mu[0,x]$ is regularly varying at infinity with index $\alpha>0$ or $\lim_{n\to\infty}(\mu[0,Mn])^{-1}\mu[0,Mn)=0$.

(b) $\{(\mu[0,Mn])^{-1}N(tn):0\le t\le M\}$ satisfies the LDP in $l_\infty[0,M]$ with speed $\mu[0,Mn]$.

Moreover, if $\mu[0,x]$ is regularly varying at infinity with index $\alpha>0$ the rate function is

$$I(z)=\begin{cases}\int_0^M h(\alpha^{-1}t^{1-\alpha}M^\alpha z'(t))\alpha t^{\alpha-1}M^{-\alpha}\,dt & \text{if } z \text{ is absolutely continuous and } z(0)=0;\\ \infty & \text{elsewhere,}\end{cases}$$

(6.3)

where $h(x)$ is as (6.1). If $\lim_{n\to\infty}(\mu[0,M])^{-1}\mu[0,M)=0$, the rate function is

$$I(z)=\begin{cases} h(z(M)) & \text{if } z(t)=0 \text{ for } 0\le t<M\\ \infty & \text{else}\end{cases} \tag{6.4}$$

Proof. Let $\epsilon_n=(\mu[0,nM])^{-1}$. First, we prove that (a) implies (b). Suppose first that $\mu[0,x]$ is regularly varying at infinity with index $\alpha>0$. We apply Corollary 3.8. Given $0\le t_1<\cdots<t_m$ and $\lambda_1,\dots,\lambda_m\in\mathbb{R}$, we have that

$$\begin{array}{rl} & \epsilon_n\log(E[\exp(\sum_{j=1}^m\lambda_jN(t_jn))])=\epsilon_n\sum_{j=1}^m(e^{\sum_{i=j}^m\lambda_i}-1)\mu(t_{j-1}n,t_jn] \\ \to & \sum_{j=1}^m(\exp(\sum_{i=j}^m\lambda_i)-1)(t_j^\alpha-t_{j-1}^\alpha)M^{-\alpha} \\ = & \int_0^M(\exp(\sum_{j=1}^m\lambda_jI(0\le x\le t_j))-1)\alpha x^{\alpha-1}M^{-\alpha}\,dx. \end{array}$$

The previous limit and Theorem II.2 in Ellis (1984) imply that

$$(\epsilon_nN(t_1n),\dots,\epsilon_nN(t_mn))$$

satisfies the LDP with speed ϵ_n^{-1}, i.e. condition (ii) in Corollary 3.8 holds.

Given $0 \le s < t \le M$,

$$\begin{aligned} & \epsilon_n \log(E[\exp(\lambda_1 N(sn) + \lambda_2 N(tn))]) \\ \to \; & s^\alpha M^{-\alpha}(e^{\lambda_1+\lambda_2} - 1) + (t^\alpha - s^\alpha) M^{-\alpha}(e^{\lambda_2} - 1). \end{aligned}$$

Hence, the rate function for large deviations of $(\epsilon_n N(sn), \epsilon_n N(tn))$ is

$$\begin{aligned} & I_{s,t}(u_1, u_2) \\ = \; & \sup_{\lambda_1,\lambda_2} \left(\lambda_1 u_1 + \lambda_2 u_2 - s^\alpha M^{-\alpha}(e^{\lambda_1+\lambda_2} - 1) - (t^\alpha - s^\alpha) M^{-\alpha}(e^{\lambda_2} - 1)\right) \\ = \; & \sup_{\lambda_1,\lambda_2} \big((\lambda_1 + \lambda_2) u_1 + \lambda_2(u_2 - u_1) \\ & -s^\alpha M^{-\alpha}(e^{\lambda_1+\lambda_2} - 1) - (t^\alpha - s^\alpha) M^{-\alpha}(e^{\lambda_2} - 1)\big) \\ = \; & s^\alpha M^{-\alpha} h(s^{-\alpha} M^\alpha u_1) + (t^\alpha - s^\alpha) M^{-\alpha} h((t^\alpha - s^\alpha)^{-1} M^\alpha (u_2 - u_1)), \end{aligned}$$

if $0 \le u_1 \le u_2$; and $I_{s,t}(u_1, u_2) = \infty$, else. Let h_+ be h restricted to $[1, \infty)$. It is easy to see that h_+ is an increasing one–to–one transformation from $[1, \infty)$ into $[0, \infty)$. So, it has an inverse. We claim that

$$\begin{aligned} & \{u_2 - u_1 : I_{s,t}(u_1, u_2) \le k\} \\ \subset \; & [0, \max((t^\alpha - s^\alpha) M^{-\alpha}, (t^\alpha - s^\alpha) M^{-\alpha} h_+^{-1}((t^\alpha - s^\alpha)^{-1} M^\alpha k))]. \end{aligned} \tag{6.5}$$

This holds because if $u_2 - u_1 \ge (t^\alpha - s^\alpha) M^{-\alpha}$ and $I_{s,t}(u_1, u_2) \le k$, then

$$(t^\alpha - s^\alpha) M^{-\alpha} h_+((t^\alpha - s^\alpha)^{-1} M^\alpha (u_2 - u_1)) \le I_{s,t}(u_1, u_2) \le k.$$

We have that $\frac{h_+(x)}{x}$ is increasing and $\lim_{x\to\infty} \frac{h_+(x)}{x} = \infty$. This implies that $\frac{x}{h_+^{-1}(x)}$ is increasing and $\lim_{x\to\infty} \frac{x}{h_+^{-1}(x)} = \infty$. Hence, $\lim_{x\to 0} x h_+^{-1}(x^{-1}) = 0$. This limit and (6.5) implies condition (iii) in Corollary 3.8.

Now, suppose that $\lim_{n\to\infty} (\mu[0, Mn])^{-1} \mu[0, Mn) = 0$. We apply Theorem 3.2. Given $0 \le t_1 < \cdots < t_m = M$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, we have that

$$\begin{aligned} & \epsilon_n \log(E[\exp(\textstyle\sum_{j=1}^m \lambda_j N(t_j n))]) \\ = \; & \epsilon_n \textstyle\sum_{j=1}^m (e^{\sum_{i=j}^m \lambda_i} - 1) \mu(n t_{j-1}, n t_j] \to \exp(\lambda_m) - 1). \end{aligned}$$

So,

$$(\epsilon_n N(nt_1), \ldots, \epsilon_n N(nt_m))$$

satisfies the LDP with speed ϵ_n^{-1} and rate function

$$I_{t_1,\ldots,t_m}(u_1, \ldots, u_m) = \begin{cases} h(u_m) & \text{if } u_i = 0 \text{ for } 1 \le i \le m-1, \\ \infty & \text{elsewhere.} \end{cases} \tag{6.6}$$

So, for $0 \le s, t < M$, $\rho_k(s, t) = 0$ and for $0 \le t < M$, $\rho_k(t, M) = \sup\{u : h(u) \le k\}$. Hence, for each $k \ge 1$, (T, ρ_k) is totally bounded. For $0 < \eta < \rho(0, M)$,

$$\begin{aligned} & \Pr\{\sup_{\rho(s,t)\le\eta} \epsilon_n |N(tn) - N(sn)| \ge \tau\} \\ = \; & \Pr\{\sup_{0\le s,t<M} \epsilon_n |N(tn) - N(sn)| \ge \tau\} \\ = \; & \Pr\{\epsilon_n (N(Mn-) - N(0)) \ge \tau\}. \end{aligned}$$

Since $\lim_{n\to\infty}(\mu[0,Mn])^{-1}\mu[0,Mn)=0$, by Theorem 6.1 for each $\lambda>0$,

$$\begin{aligned}&\lim_{n\to\infty}\epsilon_n\log(\mu[0,Mn](\mu(0,Mn))^{-1})^{-1}\\&\times\log\left(\Pr\{\epsilon_n(N(Mn-)-N(0))\geq\lambda\}\right)=-\lambda.\end{aligned}$$

Hence, for each $\tau>0$,

$$\lim_{n\to\infty}\epsilon_n\log\left(\Pr\{\epsilon_n(N(Mn-)-N(0))\geq\tau\}\right)=-\infty.$$

Hence, conditions (a) in Theorem 3.2 hold.

Next, we prove (b) implies (a). First, we prove that for each $0\leq t\leq M$, $\{(\mu[0,n])^{-1}\mu[0,nt]\}$ converges. We prove this by contradiction. Suppose that there are $0\leq c_1<c_2<\infty$ and subsequences n'_k and n''_k such that $(\mu[0,n_{k'}])^{-1}\mu[0,tn'_k]\to c_1$ and $(\mu[0,n''_k])^{-1}\mu[0,tn''_k]\to c_2$. This implies that LDP of $\{\epsilon_nN(tn)\}$ with speed ϵ_n^{-1} has two rates. Therefore, for each $0\leq t\leq M$,

$$b(t)=\lim_{n\to\infty}(\mu[0,Mn])^{-1}\mu[0,tn]$$

exists.

Now, we make two cases according to whether $b(t)>0$ for each $0<t<M$ or not. Suppose that there exists a $0<t_0<M$ such that $b(t_0)=0$. Since b is nondecreasing for each $0\leq t\leq t_0$, $b(t_0)=0$. For $0\leq s,t\leq M$

$$(\mu[0,Mn])^{-1}\mu[0,M^{-1}stn]=(\mu[0,sn])^{-1}\mu[0,M^{-1}stn](\mu[0,Mn])^{-1}\mu[0,sn].$$

Hence, $0\leq s,t\leq M$

$$b(M^{-1}st)=b(t)b(s). \tag{6.7}$$

Hence, for $t_0<t<M$, there exists a positive integer k such that $M^{-(k-1)}t^k<t_0$. By (6.7), we have that $(b(t))^k=b(M^{-(k-1)}t^k)=0$. So, $b(t)=0$. This implies that for $0\leq s,t<M$,

$$I_{s,t}(u,v)=\begin{cases}0 & \text{if } u=v\\ \infty & \text{else.}\end{cases} \tag{6.8}$$

Hence, for each $k\geq 1$ and each $0\leq s,t<M$, $\rho_k(s,t)=0$. So, the asymptotic equicontinuity condition implies that for each $\tau>0$,

$$\lim_{n\to\infty}\epsilon_n\log\left(\Pr\{\epsilon_n(N(Mn-)-N(0))\geq\tau\}\right)=\infty.$$

This implies that $(\mu[0,n])^{-1}\mu[0,Mn)\to 0$.

If for each $0<t\leq m$, $b(t)>0$, by Theorem 1.9.2 in Bingham, Goldie and Teugels (1987), $\mu[0,x]$ is regularly varying. If it is regularly varying of order $\alpha>0$, we are done. If $\mu[0,x]$ is slowly varying at infinity, then for each $0<s<t$, and $\lambda_1,\lambda_2\in\mathbb{R}$,

$$\epsilon_n\log(E[\exp(\lambda_1N(sa_n)+\lambda_2N(ta_n))])\to e^{\lambda_1+\lambda_2}-1+e^{\lambda_2}-1.$$

Hence, the rate function for large deviations of $(N(sa_n), N(ta_n))$ is

$$I_{s,t}(u_1, u_2) = \sup\{\lambda_1 u_1 + \lambda_2 u_2 - (e^{\lambda_1+\lambda_2} - 1 + e^{\lambda_2} - 1) : \lambda_1, \lambda_2 \in I\!R\}$$
$$= h(u_1) + h(u_2 - u_1).$$

If $\{\epsilon_n N(tn) : 0 \le t \le M\}$ satisfied the LDP with speed ϵ_n^{-1}, then by Theorem 3.2 $([0,1], \rho_k)$ would be totally bounded, where

$$\begin{aligned} & \rho_k(s,t) := \sup\{|u_2 - u_1| : I_{s,t}(u_1, u_2) \le k\}) \\ = \ & \sup\{|u_2 - u_1| : h(u_1) + h(u_2 - u_1) \le k\}, \end{aligned}$$

in contradiction.

By the Theorem 4.4, the rate functions are given by (6.3) and (6.4). □

Next, we consider the case when the normalizing constant is of bigger order than the mean.

Theorem 6.3 *Let $\{N(t) : t \ge 0\}$ be a Poisson process with mean measure μ, let $0 < M < \infty$ and let $\{a_n\}$ be a sequence of positive numbers converging to infinity. Suppose that:*

(i) $\mu[0, x]$ is regularly varying at infinity of order $\alpha \ge 0$.

(ii) $\frac{\mu[0,nM]}{a_n} \to 0$.

Then, $\{a_n^{-1} N(tn) : 0 \le t \le M\}$ does not satisfy the LDP in $l_\infty[0, M]$ with speed $a_n \log\left(\frac{a_n}{\mu[0,nM]}\right)$.

Proof. We claim that given $0 < s < t < M$, $\{(a_n^{-1} N(sn), a_n^{-1} N(tn))\}$ satisfies the LDP with speed $a_n \log\left(\frac{a_n}{\mu[0,n]}\right)$ and rate function $I_{s,t}(u_1, u_2) = u_2$ for $0 \le u_1 \le u_2$; $I_{s,t}(u_1, u_2) = \infty$ else. This claim implies the theorem, since by Theorem 3.2, if $\{a_n^{-1} N(tn) : 0 \le t \le M\}$ satisfied the LDP in $l_\infty[0, M]$, then for each $k > 0$, $([0, M], \rho_k)$ would be totally bounded, where $\rho_k(s,t) = \sup\{|u_2 - u_1| : I_{s,t}(u_1, u_2) \le k\} = k$. But, this condition does not hold.

By the contraction principle it suffices to show that

$$(U_n, V_n) := (a_n^{-1} N(sn), a_n^{-1}(N(tn) - N(sn)))$$

satisfies the LDP with rate $I^{(2)}(u_1, u_2) = u_1 + u_2$, if $u_1, u_2 \ge 0$, and $I^{(2)}(u_1, u_2) = \infty$, else. By regular variation,

$$\lim_{n\to\infty} \frac{\mu[0, ns]}{a_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{a_n \log\left(\frac{a_n}{\mu[0,ns]}\right)}{a_n \log\left(\frac{a_n}{\mu[0,n]}\right)} = 1.$$

Hence, by Theorem 6.1, U_n satisfies the LDP with rate $I^{(1)}(t) = t$, if $t \ge 0$, and $I^{(1)}(t) = \infty$, else. Similarly, we get that $\{V_n\}$ satisfies the LDP with rate $I^{(1)}$. Since U_n and V_n are independent, we have that for each open O set in $I\!R^2$,

$$\begin{aligned}
&\liminf_{n\to\infty} \left(a_n \log\left(\tfrac{a_n}{\mu[0,n]}\right)\right)^{-1} \log\left(\Pr\{(U_n, V_n) \in O\}\right) \\
\geq\ & -\{I^{(1)}(u) + I^{(1)}(v) : (u, v) \in O\}.
\end{aligned}$$

To check the condition for closed sets, it suffices to prove that for each $t > 0$

$$\limsup_{n\to\infty} \left(a_n \log\left(\frac{a_n}{\mu[0,n]}\right)\right)^{-1} \log(\Pr\{U_n + V_n \geq t\}) \leq -t.$$

But, $U_n + V_n = a_n^{-1} N(tn)$ satisfies the LDP with rate $I^{(1)}$. $\square$

In the situation of the previous theorem, although, the LDP does not hold in $l_\infty[0, M]$, it does in a set of measures. Let $\mathcal{M}_+([0, M], w)$ be the set of positive measures on $[0, M]$ with the weak topology. As it is well known, this topology is defined as follows: $\mu_n \xrightarrow{\mathrm{w}} \mu$ if for each continuous function f on $[0, M]$, $\int_0^M f(x) d\mu_n \to \int_0^M f(x)\, d\mu(x)$. Given a Poisson process $\{N(t) : t \geq 0\}$, let $\{T_j\}$ be the jumps of this process. Given $0 < M < \infty$, we have the random measure $\mu_n = a_n^{-1} \sum_{T_j \leq nM} \delta_{n^{-1}T_j}$ in $[0, M]$.

Theorem 6.4 *Let $\{N(t) : t \geq 0\}$ be a Poisson process with mean measure μ and let $\{a_n\}$ be a sequence of positive numbers converging to infinity. Suppose that:*

(i) $\mu[0, x]$ is regularly varying at infinity of order $\alpha > 0$.

(ii) $\frac{\mu[0,nM]}{a_n} \to 0$.

Then, $\{\mu_n\}$ satisfies the LDP in $\mathcal{M}_+([0, M], w)$ with speed $a_n \log\left(\frac{a_n}{\mu[0,n]}\right)$ and rate function $I(\nu) = \nu[0, M]$.

Proof. Since $a_n^{-1} N(Mn)$ satisfies the LDP, given a closed set $F \subset (\mathcal{M}_+([0, M]), w)$,

$$\begin{aligned}
& \left(a_n \log\left(\tfrac{a_n}{\mu[0,n]}\right)\right)^{-1} \log \Pr\{\mu_n \in F\} \\
\leq\ & \left(a_n \log\left(\tfrac{a_n}{\mu[0,n]}\right)\right)^{-1} \log \Pr\{\mu_n([0, M]) \geq \inf\nolimits_{\mu\in F} \mu([0, M])\} \\
=\ & \left(a_n \log\left(\tfrac{a_n}{\mu[0,n]}\right)\right)^{-1} \log \Pr\{a_n^{-1} N(Mn) \geq \inf\nolimits_{\mu\in F} \mu([0, M])\} \\
\to\ & -\inf\nolimits_{\mu\in F} \mu([0, M]).
\end{aligned}$$

Given an open set $G \subset \mathcal{M}_+([0, M])^d, w)$ and $\nu_0 \in G$, there are $\delta > 0$ and $0 \leq t_1 < \ldots < t_m \leq 1$ such that

$$\{\nu : |\nu[0, t_1] - \nu_0[0, t_1]| < \delta, \sup_{2\leq i\leq m} |\nu(t_{i-1}, t_i] - \nu_0(t_{i-1}, t_i]| \leq \delta\} \subset G.$$

Hence,

$$\Pr\{\mu_n \in G\} \geq \Pr\{\sup_{1\leq i\leq m} |a_n^{-1}(N(nt_i) - N(nt_{i-1})) - p_i]| \leq \delta\},$$

where $t_0 = 0$, $p_1 = \nu_0[0, t_1]$ and $p_i = \nu_0(t_{i-1}, t_i]$, for $2 \leq i \leq m$. By an argument in the previous theorem

$$\{(a_n^{-1}(N(nt_1) - N(nt_0)), \dots, a_n^{-1}(N(nt_m) - N(nt_{m-1}))\}$$

satisfies the LDP with rate $I(u_1 \dots, u_m) = \sum_{j=1}^m u_j$, $u_j \geq 0$, for each $1 \leq j \leq m$, $I(u_1 \dots, u_m) = \infty$, else. Hence,

$$\liminf_{n\to\infty} \left(a_n \log \left(\frac{a_n}{\mu[0,n]} \right) \right)^{-1} \log \Pr\{\mu_n \in G\} \geq -\sum_{i=1}^m (p_i - \delta)^+.$$

Therefore, the claim follows. □

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