Large and moderate deviations of empirical processes with nonstandard rates [1]

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Abstract

We discuss the large and moderate deviations of a type of empirical processes whose finite dimensional distributions do not satisfy the Cramér condition. Nonstandard speeds and rate functions appear.

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1 Introduction

We consider the large and moderate deviations of a particular type of empirical processes whose finite dimensional distributions do not satisfy the Cramér condition. The speed and rate functions in the large and moderate deviations of these empirical processes are different from the usual ones. The large and moderate deviations of empirical processes in the standard situation were considered by Wu (1994) and Arcones (2001b).

Given a sequence of r.v.'s $\{U_n\}$ with values in a metric space S, a sequence of positive numbers $\{\epsilon_n\}_{n=1}^{\infty}$ such that $\epsilon_n \to 0$, and a function $I : S \to [0, \infty]$, it is said that $\{U_n\}$ satisfies the LDP with speed ϵ_n^{-1} and with good rate function I if:

(i) For each $0 \le c < \infty$, $\{z \in l_\infty(T) : I(z) \le c\}$ is a compact set of S.

(ii) For each set $A \in l_\infty(T)$,

$$-I(A^o) \le \liminf_{n\to\infty} \epsilon_n \log(\Pr\{U_n \in A\})$$

and

$$\limsup_{n\to\infty} \epsilon_n \log(\Pr\{U_n \in A\}) \le -I(\bar{A}),$$

where $I(B) = \inf\{I(x) : x \in B\}$.

General references on large deviations are Bahadur (1971), Varadhan (1984), Deuschel and Stroock (1989) and Dembo and Zeitouni (1998). Given a sequence of nondegenerate i.i.d.r.v's $\{X_i\}_{i=1}^{\infty}$ such that $E[e^{\lambda |X_1|}] < \infty$ for some $\lambda > 0$, $n^{-1}\sum_{n=1}^{\infty} X_i$, $n \ge 1$, satisfy the large deviation principle with speed n and rate $I(z) = \sup_{\lambda \in I\!R} \left(\lambda z - E[e^{\lambda X_1}]\right)$ (Cramér, 1937, and Chernoff, 1952). The moderate deviations principle says that in the situation above, for any sequence of positive real numbers $\{a_n\}$ such that $a_n \to \infty$ and $n^{-1/2} a_n \to 0$, $a_n^{-1} n^{-1/2} \sum_{n=1}^{\infty} (X_i - E[X_i])$, $n \ge 1$, satisfy the large deviation principle with speed a_n^2 and with rate function $I(z) = 2^{-1}\sigma^{-2} z^2$ (Petrov, 1965; see also Petrov, 1996, Theorem 5.23)

It is known (see for example Nagaev, 1978 and Mikosch and Nagaev, 1998) that if the Cramér condition fails other speeds and rate functions appear in the large and moderate deviations of sums of r.v.'s. We give an example of this for empirical processes. We present the following result:

Theorem 1.1 *Let* $\{\xi_{i,j}\}_{i,j=1}^{\infty}$ *be a double sequence of symmetric i.i.d.r.v.'s with* $\Pr\{|\xi_{i,j}| \ge u\} = e^{-u^p}$, *for each* $u > 0$, *where* $0 < p \le 1$. *Let* $\{b_n\}_{n=1}^{\infty}$ *be a sequence of positive numbers such that* $\frac{n}{b_n^{2-p}} \to 0$. *For each* $t \in T$, *let* $\{x_j(t)\}_{j=1}^{\infty}$ *be a sequence of real numbers such that* $\sum_{j=1}^{\infty} |(x_j(t)|^2 < \infty$. *Then, the following conditions are equivalent:*

(a.1) (T, d) *is totally bounded, where* $d(s,t) = \sup_{j \ge 1} |x_j(s) - x_j(t)|$.

(a.2) $b_n^{-1} \sup_{t \in T} |\sum_{i=1}^{n} \sum_{j=1}^{\infty} x_j(t) \xi_{i,j}| \xrightarrow{\Pr} 0$.

(b) $\{b_n^{-1}\sum_{i=1}^n\sum_{j=1}^\infty x_j(t)\xi_{i,j} : t\in T\}$ *satisfies the LDP with speed* b_n^p *and rate funtion*

$$I(z)=\inf\{\sum_{i=1}^\infty |\gamma_i|^p : \sum_{i=1}^\infty x_i(t)\gamma_i = z(t) \text{ for each } t\in T\}.$$

If $b_n=n$ and $0<p<1$, the previous theorem presents an empirical process for which the large deviations hold with speed n^p. This speed n^p is slower than the usual one n. If $\{b_n\}$ is a sequence of positive numbers such that $\frac{n}{b_n^{2-p}}\to 0$ and $\frac{b_n}{n}\to 0$, then previous theorem is considering the moderate deviations of certain empirical processes. The speed of the standard moderate deviations is $n^{-1}b_n^2$. The speed of the LDP in the previous theorem is b_n^p, for $0<p<1$, which is of smaller order of magnitude than $n^{-1}b_n^2$. These speeds are so, because the considered empirical processes do not satisfy the Cramér condition.

The condition in Theorem 1.1: for each $t\in T$, $\sum_{j=1}^\infty |(x_j(t)|^2<\infty$, is needed. By the three series theorem $\sum_{i=1}^\infty x_j(t)\xi_{1,j}$ converges a.s. if and only if $\sum_{j=1}^\infty x_j^2(t)<\infty$ (see for example Theorem 4 in Chow and Teicher, 1978).

Stochastic processes similar to the ones in Theorem 1.1 have been studied by several authors (see for example, Talagrand, 1991, 1994).

Theorem 1.1 is related with the moderate deviations of sums of i.i.d.r.v.'s with values in a separable space B. Chen (1991), Ledoux (1992), Wu (1994) and Arcones (2001b) have studied this problem. The conditions imposed in these papers to obtain the moderate deviations in this papers are not satisfied. In particular, Ledoux (1992) assumed that there are $c\ge 1$ and $M>0$ such that for each $u>0$,

$$nb_n^{-2}\log\left(\Pr\{\|X\|\ge ub_n\}\right)\le -M^{-1}u^2. \tag{1.1}$$

In the situation considered in Theorem 1.1,

$$\lim_{\lambda\to\infty}\lambda^{-p}\log\left(\Pr\left\{\sup_{t\in T}|\sum_{j=1}^\infty x_j(t)\xi_{i,j}|\ge\lambda\right\}\right)=-(\sup_{t\in T}\sup_{j\ge 1}|x_j(t)|^p)^{-1}$$

(see for example Theorem 3.11 in Arcones, 2001c). So, under the conditions in Theorem 1.1, (1.1) does not hold:

$$\lim_{n\to\infty}nb_n^{-2}\log(n\Pr\{\sup_{t\in T}|\sum_{j=1}^\infty x_j(t)\xi_{i,j}|\ge b_n\})=0.$$

c will denote an universal constant that may vary from line to line. Given a sequence of real numbers $a=\{a_k\}$, we denote $|a|_\infty=\sup_{k\ge 1}|a_k|$.

2 Proofs.

First, we consider the one dimensional case:

Theorem 2.1 *Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of symmetric i.i.d.r.v.'s. Let $0 < p \le 1$. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\frac{n}{b_n^{2-p}} \to 0$ and $\frac{b_n}{b_{n+1}} \to 1$. Let $a > 0$. Then, the following conditions are equivalent:*

(a) $\lim_{t\to\infty} t^{-p} \log(\Pr\{|X| \ge t\}) = -a$.

(b) $\{b_n^{-1} \sum_{j=1}^n X_j\}$ *satisfies the LDP with speed* b_n^p *and rate function* $I(t) = a|t|^p$.

Proof. Without loss of generality, we may assume that $a = 1$. We only consider the case $0 < p < 1$, the case $p = 1$ is similar.

Suppose (a). It suffices to prove that for each $t > 0$,

$$\lim_{n\to\infty} b_n^{-p} \log(\Pr\{|\sum_{j=1}^n X_j| \ge tb_n\}) = -t^p. \tag{2.1}$$

By the contraction principle (see for example Theorem 4.4. in Ledoux and Talagrand, 1991)

$$\Pr\{|X| \ge tb_n\} \le \Pr\{|\sum_{j=1}^n X_j| \ge tb_n\}.$$

Hence,

$$\liminf_{n\to\infty} b_n^{-p} \log(\Pr\{|\sum_{j=1}^n X_j| \ge tb_n\}) \ge -t^p. \tag{2.2}$$

Given $0 < \lambda < t^{p-1}$ and $1 > \delta > 0$, we have that, for n large enough,

$$\begin{aligned} &\Pr\{|\textstyle\sum_{j=1}^n X_j| \ge tb_n\} \\ \le\ & n\Pr\{|X| \ge tb_n\} + \Pr\{|\textstyle\sum_{j=1}^n X_j I(|X_j| \le tb_n)| \ge tb_n\} \\ \le\ & ne^{-(1-\delta)t^p b_n^p} + 2e^{-\lambda t b_n^p} E[e^{\lambda b_n^{p-1} \sum_{j=1}^n X_j I(|X_j| \le tb_n)}] \\ =\ & ne^{-(1-\delta)t^p b_n^p} + 2e^{-\lambda t b_n^p + n\log M_n}, \end{aligned} \tag{2.3}$$

where $M_n = E[e^{\lambda b_n^{p-1} X I(|X| \le tb_n)}]$. We claim that

$$b_n^{-p} n \log M_n \simeq b_n^{-p} n(M_n - 1) \to 0. \tag{2.4}$$

By a change of variables,

$$
\begin{aligned}
& M_n = \int_0^\infty \Pr\{e^{\lambda b_n^{p-1} X I(|X|\le t b_n)} \ge u\}\, du = \int_{-\infty}^\infty \Pr\{\lambda b_n^{p-1} X \ge u, |X| \le t b_n\} e^u\, du \\
= \quad & \int_{-\infty}^{-t\lambda b_n^p} \Pr\{|X| \le t b_n\} e^u\, du + \int_{-t\lambda b_n^p}^0 \Pr\{t b_n \ge X \ge \lambda^{-1} b_n^{1-p} u\} e^u\, du \\
& + \int_0^{t\lambda b_n^p} \Pr\{t b_n \ge X \ge \lambda^{-1} b_n^{1-p} u\} e^u\, du \\
= \quad & \Pr\{|X| \le t b_n\} e^{-t\lambda b_n^p} + \int_{-t\lambda b_n^p}^0 e^u\, du \\
& - \int_{-t\lambda b_n^p}^0 \Pr\{X \le \lambda^{-1} b_n^{1-p} u\} e^u\, du - \int_{-t\lambda b_n^p}^0 \Pr\{X \ge t b_n\} e^u\, du \\
& + \int_0^{t\lambda b_n^p} \Pr\{X \ge \lambda^{-1} b_n^{1-p} u\} e^u\, du - \int_0^{t\lambda b_n^p} \Pr\{X \ge t b_n\} e^u\, du \\
= \quad & \Pr\{|X| \le t b_n\} e^{-t\lambda b_n^p} + (1 - e^{-t\lambda b_n^p}) + \int_0^{t\lambda b_n^p} \Pr\{X \ge \lambda^{-1} b_n^{1-p} u\} (e^u - e^{-u})\, du \\
& - \Pr\{X \ge t b_n\} (e^{t\lambda b_n^p} - e^{-t\lambda b_n^p}) \\
=: \quad & I + II + III - IV.
\end{aligned}
$$

Given $1 - \lambda t^{1-p} > \epsilon > 0$, we have that for n large enough,

$$b_n^{-p} n I \le b_n^2 e^{-t\lambda b_n^p} \to 0,$$

$$b_n^{-p} n |II - 1| \le b_n^2 e^{-t\lambda b_n^p} \to 0,$$

and

$$b_n^{-p} n IV \le b_n^2 e^{-(1-\epsilon) t^p b_n^p + t\lambda b_n^p} \to 0.$$

By the change of variables, $u b_n^{1-p} = x$, we also have that

$$
\begin{aligned}
& b_n^{-p} n III \le 2^{-1} b_n^{-p} n \int_0^{t\lambda b_n^p} e^{-(1-\epsilon)\lambda^{-p} b_n^{p(1-p)} u^p} (e^u - e^{-u})\, du \\
= \quad & 2^{-1} b_n^{-1} n \int_0^{t\lambda b_n} e^{-(1-\epsilon)\lambda^{-p} x^p} (e^{b_n^{p-1} x} - e^{-b_n^{p-1} x})\, du \\
= \quad & 2^{-1} b_n^{-1} n \int_0^{b_n^{1-p}} e^{-(1-\epsilon)\lambda^{-p} x^p} (e^{b_n^{p-1} x} - e^{-b_n^{p-1} x})\, dx \\
& + 2^{-1} b_n^{-1} n \int_{b_n^{1-p}}^{t\lambda b_n} e^{-(1-\epsilon)\lambda^{-p} x^p} (e^{b_n^{p-1} x} - e^{-b_n^{p-1} x})\, dx \\
\le \quad & c b_n^{-1} n \int_0^{b_n^{1-p}} e^{-(1-\epsilon)\lambda^{-p} x^p} b_n^{p-1} x\, dx + 2^{-1} b_n^{-1} n \int_{b_n^{1-p}}^{t\lambda b_n} e^{-(1-\epsilon)\lambda^{-p} x^p + b_n^{p-1} x}\, dx \\
\le \quad & c n b_n^{p-2} + 2^{-1} b_n^{-1} n \int_{b_n^{1-p}}^{t\lambda b_n} e^{-(1-\epsilon)\lambda^{-p} x^p + t^{1-p} \lambda^{1-p} x^p}\, dx \to 0.
\end{aligned}
$$

(2.4) follows from the previous estimations.

It follows from (2.3) and (2.4) that

$$\limsup_{n\to\infty} b_n^{-p} \log(\Pr\{|\sum_{j=1}^n X_j| \ge t b_n\}) \le -\lambda t.$$

Letting $\lambda \to t^{p-1}$, we get that

$$\limsup_{n\to\infty} b_n^{-p} \log(\Pr\{|\sum_{j=1}^n X_j| \ge t b_n\}) \le -t^p. \tag{2.5}$$

(2.2) and (2.5) imply (2.1).

Suppose (b). By the arguments in Lemma 2.1 in Arcones (2001b), given $t > 0$ for each $t^p > \delta > 0$,

$$\limsup_{n\to\infty} b_n^{-p} \log\left(n \Pr\{|X| \geq (t+\delta)b_n\}\right) \leq -(t^p - \delta). \tag{2.6}$$

This implies that for $t > 0$,

$$\limsup_{n\to\infty} b_n^{-p} \log\left(\Pr\{|X| \geq tb_n\}\right) \leq -t^p.$$

By the contraction principle (see for example Theorem 4.4. in Ledoux and Talagrand, 1991), for each $t > 0$,

$$\Pr\{|X_1| \geq t\} \leq \Pr\{|\sum_{i=1}^{n} X_i| \geq t\},$$

we have that for each $t > 0$,

$$\liminf_{n\to\infty} b_n^{-p} \log\left(\Pr\{|X| \geq tb_n\}\right) \geq -t^p. \tag{2.7}$$

(2.6) and (2.7) imply (a). □

Under the conditions in (a) in Theorem 3.1, by Theorem 3.6 in Arcones (2001c), $\{n^{-1}\sum_{j=1}^{\infty} x_j(t)\xi_{1,j} : t \in T\}$ satisfies the LDP with speed n^p and rate funtion

$$I(z) = \inf\{\sum_{i=1}^{\infty} |\gamma_i|^p : \sum_{i=1}^{\infty} x_i(t)\gamma_i = z(t) \text{ for each } t \in T\}.$$

This implies that for each $t \in T$,

$$\lim_{u\to\infty} u^{-p} \log\left(\Pr\{|\sum_{j=1}^{\infty} x_j(t)\xi_{1,j}| \geq u\}\right) = -u^p / \sup_{1\leq j} |x_j(t)|^p. \tag{2.8}$$

So, the processes which are considering satisfy (a) in Theorem 3.1.

We will need the following lemma:

Lemma 2.2 *Let $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}$ be two sequences of r.v.'s. Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\epsilon \to 0$. Suppose that*

(i) Suppose that for each $n \geq 1$, X_n and Y_n are independent.

(ii) $\{X_n\}_{n=1}^{\infty}$ satisfies the LDP with speed $\{\epsilon_n\}_{n=1}^{\infty}$ and good rate function I_1.

(iii) $\{Y_n\}_{n=1}^{\infty}$ satisfies the LDP with speed $\{\epsilon_n\}_{n=1}^{\infty}$ and good rate function I_2.

(iv) For each $i = 1, 2$, I_i is continuous in $\{x : I_i(x) < \infty\}$.

Then, $\{(X_n, Y_n)\}_{n=1}^{\infty}$ satisfies the LDP with speed $\{\epsilon_n\}_{n=1}^{\infty}$ and good rate function $I(u, v) = I_1(u) + I_2(v)$.

PROOF. Let F be a closed set of $\mathbb{R}^2$. Let $c = \inf\{I(u,v) : (u,v) \in F\}$. Let $C = F \cap \{(u,v) : I_1(u) \le c, I_2(v) \le c\}$. C is a compact set. Given $\delta > 0$ and $0 < t < c$, let

$$U_t = \{(u,v) : I_1(u) > t - \delta, I_2(v) > c - t - \delta\}.$$

Then, $C \subset \cup_{0<t<c} U_t$. By compactness, there are $t_1, \dots, t_m$ such that $C \subset \cup_{j=1}^m U_{t_j}$. Hence,

$$F \subset \{(u,v) : I_1(u) > c\} \cup \{(u,v) : I_2(v) > c\} \cup \cup_{j=1}^m U_{t_j}.$$

This implies, using hypotheses (i)–(iv), that

$$\limsup_{n\to\infty} \epsilon_n \log \Pr(\{(X_n, Y_n) \in F\}) \le -c.$$

It is obvious that for each open set U,

$$\liminf_{n\to\infty} \epsilon_n \log(\Pr\{(X_n, Y_n) \in U\}) \ge -\inf\{I(u,v) : (u,v) \in U\},$$

and that I is a good rate function. $\square$

PROOF OF THEOREM 1.1. Assume (a). We use Theorem 3.1 in Arcones (2001a). Next, we obtain the LDP of the finite dimensional distributions. We need to prove that for each $t_1, \dots, t_m \in T$,

$$(b_n^{-1} \sum_{i=1}^n \sum_{j=1}^\infty x_j(t_1)\xi_{i,j}, \dots, b_n^{-1} \sum_{i=1}^n \sum_{j=1}^\infty x_j(t_m)\xi_{i,j}).$$

satisfies the LDP with speed b_n^p and rate function

$$I(u_1, \dots, u_m) = \inf\{\sum_{i=1}^\infty |\gamma_i|^p : \sum_{i=1}^\infty x_i(t_l)\gamma_i = u_l \text{ for each } 1 \le l \le m\}. \tag{2.9}$$

To do that, we use Lemma 2.2 in Arcones (2001c) with

$$X_{n,k} = (b_n^{-1} \sum_{i=1}^n \sum_{j=1}^k x_j(t_1)\xi_{i,j}, \dots, b_n^{-1} \sum_{i=1}^n \sum_{j=1}^k x_j(t_m)\xi_{i,j}).$$

and

$$X_n = (b_n^{-1} \sum_{i=1}^n \sum_{j=1}^\infty x_j(t_1)\xi_{i,j}, \dots, b_n^{-1} \sum_{i=1}^n \sum_{j=1}^\infty x_j(t_m)\xi_{i,j}).$$

By Theorem 2.1. and Lemma 1.2,

$$(b_n^{-1} \sum_{i=1}^n \xi_{i,1}, \dots, b_n^{-1} \sum_{i=1}^n \xi_{i,k})$$

satisfies the LDP with speed b_n^p and rate function $I(\gamma_1, \ldots, \gamma_k) = \sum_{j=1}^k |\gamma_j|^p$. Hence, by the contraction principle for each $k \geq 1$, $\{X_{n,k}\}_{n=1}^\infty$ satisfies the LDP with speed b_n^p and rate function

$$I_k(u_1, \ldots, u_m) = \inf\{\sum_{i=1}^k |\gamma_i|^p : \sum_{i=1}^k x_i(t_l)\gamma_i = u_l \text{ for each } 1 \leq l \leq m\}. \tag{2.10}$$

Hence, condition (i) in Lemma 2.2 in Arcones (2001c) holds. By Theorem 2.1 and (2.8), given $\tau > 0$,

$$b_n^{-p} \log \left(\Pr\{|X_{n,k} - X_k| \geq \tau\}\right) \leq b_n^{-p} \log \left(\sum_{l=1}^m \Pr\left\{ |b_n^{-1} \sum_{i=1}^n \sum_{j=k+1}^\infty x_j(t_l)\xi_{i,1}| \geq \tau/m \right\} \right)$$

$$\to \frac{-(\tau/m)^p}{\max_{1\leq l\leq m} \sup_{k+1\leq j} |x_j(t_l)|^p}, \text{ as } n \to \infty.$$

So, condition (ii) in Lemma 2.2 in Arcones (2001c) holds. We have that

$$\inf\{I_k(u_1, \ldots, u_k) : |u_1, \ldots, u_k)|_\infty \geq m\} = m^p / \max_{1\leq l\leq m} \max_{1\leq i\leq k} |x_i(x_l)|^p,$$

where I_k is as in (2.10). This implies condition (iii) in Lemma 2.2 in Arcones (2001c). To end the proof of the LDP for the finite dimensional distributions, we need to prove that

$$\lim_{\delta\to 0} \liminf_{k\to\infty} I_k(B_\infty((u_1, \ldots, u_m), \delta)) = I(u_1, \ldots, u_m),$$

where I is as in (2.9) and I_k is as in (2.10). First we prove that

$$\lim_{\delta\to 0} \liminf_{k\to\infty} I_k(B_\infty((u_1, \ldots, u_m), \delta)) \geq I(u_1, \ldots, u_m). \tag{2.11}$$

We may assume that the left hand side of (2.11) is finite. Suppose that $\{a_i^{(k_j)} : 1 \leq i \leq k_j\}_{j=1}^\infty$ satisfies that for each $1 \leq l \leq m$, $\sum_{i=1}^{k_j} a_i^{(k_j)} x_i(t_l) \to u_l$ as $j \to \infty$ and

$$\sum_{i=1}^{k_j} |a_i^{(k_j)}|^p \to \lim_{\delta\to 0} \liminf_{k\to\infty} I_k(B_\infty(u_1, \ldots, u_m), \delta)).$$

There is a finite constant c such that $\sum_{i=1}^{k_j} |a_i^{(k_j)}|^p \leq c$ for each $j \geq 1$. By taking subsequences, we may assume that for each $i \geq 1$, $a_i^{(k_j)} \to a_i$ for some a_i. By the Fatou's lemma

$$\sum_{i=1}^\infty |a_i|^p \leq \lim_{\delta\to 0} \liminf_{k\to\infty} I_k(B_\infty(u_1, \ldots, u_m), \delta)).$$

To end the proof of (2.11), it suffices to show that for each $1 \leq l \leq m$,

$$\sum_{i=1}^{k_j} a_i^{(k_j)} x_i(t_l) \to \sum_{i=1}^{\infty} a_i x_i(t_l). \tag{2.12}$$

Given $\epsilon > 0$, there exists a positive integer i_0 such that $\sum_{i=i_0+1}^{\infty} |a_i|^p \leq \epsilon/3$ and for $i \geq i_0$ $|x_i(t_l)| \leq \epsilon/(3c^p)$. Then, for k large,

$$|\sum_{i=1}^{i_0} a_i^{(k_j)} x_i(t_l) - \sum_{i=1}^{i_0} a_i x_i(t_l)| \leq \epsilon/3.$$

We also have that

$$|\sum_{i=i_0+1}^{k_j} a_i^{(k_j)} x_i(t_l)| \leq ((\epsilon/(3c^p)) \sum_{i=i_0+1}^{k_j} |a_i^{(k_j)}| \leq (\epsilon/(3c^p)) \left(\sum_{i=i_0+1}^{k_j} |a_i^{(k_j)}|^p \right)^{1/p} \leq \epsilon/3$$

and

$$|\sum_{i=i_0+1}^{k_j} a_i x_i(t_l)| \leq \epsilon/3.$$

Hence, (2.12) follows.

We also have that

$$I(u_1, \ldots, u_m) \geq \lim_{\delta \to 0} \liminf_{k \to \infty} I_k(B_\infty(u_1, \ldots, u_m), \delta)). \tag{2.13}$$

We may assume that $I(u_1, \ldots, u_m) < \infty$. Given a sequence $\{a_i\}_{i=1}^\infty$ such that $\sum_{i=1}^\infty a_i x_i(t_l) = u_l$, for each $1 \leq l \leq m$, we have that for each $1 \leq l \leq m$, $\sum_{i=1}^k a_i x_i(t_l) \to u_l$, as $k \to \infty$. So, (2.13) holds. Therefore, by Lemma 2.2 in Arcones (2001c), the large deviation principle of the finite dimensional distributions hold with the claimed rate.

To get tightness, we apply Lemma 2.5 in Arcones (2001b). By Theorem 3.6 in Arcones (2001c), $\{n^{-1} \sum_{j=1}^\infty x_j(t)\xi_{1,j} : t \in T\}$ satisfies the LDP with speed n^p and rate funtion

$$I(z) = \inf\{\sum_{i=1}^\infty |\gamma_i|^p : \sum_{i=1}^\infty x_i(t)\gamma_i = z(t) \text{ for each } t \in T\}.$$

This implies that for each $u > 0$,

$$\begin{aligned} & \limsup_{n\to\infty} b_n^{-p} \log(\Pr\{\sup_{|x(s)-x(t)|_\infty \leq \eta} |\textstyle\sum_{j=1}^\infty (x_j(t) - x_j(t))\xi_{1,j}| \geq b_n u\} \\ = \;& -\inf\{\textstyle\sum_{i=1}^\infty |\gamma_i|^p : \sum_{i=1}^\infty x_i(t)\gamma_i = z(t), \sum_{i=1}^\infty x_i(s)\gamma_i = z(s), \\ & \qquad \text{and } \sup_{|x(s)-x(t)|_\infty \leq \eta} |z(s) - z(t)| \geq u\} \end{aligned}$$

Now, given $u > \epsilon > 0$, if $|x(s) - x(t)|_\infty \le \eta$ and $u - \epsilon \le |z(s) - z(t)|$, then

$$u - \epsilon \le |\textstyle\sum_{i=1}^\infty (x_i(s) - x_i(t))\gamma_i| \le |x(s) - x(t)|_\infty (\sum_{i=1}^\infty |\gamma_i|^p)^{1/p} \le \eta(\sum_{i=1}^\infty |\gamma_i|^p)^{1/p}.$$

So,

$$\begin{array}{c}\inf\{\sum_{i=1}^\infty |\gamma_i|^p : \sum_{i=1}^\infty x_i(t)\gamma_i = z(t), \sum_{i=1}^\infty x_i(s)\gamma_i = z(s),\\ \text{and } \sup_{|x(s)-x(t)|_\infty \le \eta} |z(s) - z(t)| \ge u\} \ge (u/\eta)^p\end{array}$$

This implies that for each $u > 0$,

$$\limsup_{n\to\infty} b_n^{-p} \log\left(\Pr\{ Y_1^{(\eta)} \ge b_n u\}\right) \le -(u/\eta)^p,$$

where

$$Y_1^{(\eta)} = \sup_{|x(s)-x(t)|_\infty \le \eta} |\sum_{j=1}^\infty (x_j(s) - x_j(t))\xi_{1,j}|.$$

Hence, condition (i) in Lemma 2.5 in Arcones (2001b) holds. To prove condition (ii) in Lemma 2.5 in Arcones (2001b), we need to prove that for each $0 < \lambda, M < \infty$,

$$\lim_{\eta\to 0} \limsup_{n\to\infty} n b_n^{-p} E[(e^{\lambda_n^{p-1} Y_1^{(\eta)}} - 1) I(M b_n \ge Y_1^{(\eta)} \ge b_n^{1-p})] = 0.$$

Take $\tau > M^{1-p}\lambda$ and η such that

$$\limsup_{t\to\infty} t^{-p} \log(\Pr\{Y_1^{(\eta)} \ge t\}) < -\tau.$$

By the change of variables $1 + u = e^{b_n^{p-1} x}$, for n large enough,

$$\begin{aligned}
& n b_n^{-p} E[(e^{\lambda b_n^{p-1} Y_1^{(\eta)}} - 1) I(M b_n \ge Y_1^{(\eta)} \ge b_n^{1-p})] \\
= \;& n b_n^{-p} \textstyle\int_0^\infty \Pr\{e^{\lambda b_n^{p-1} Y_1^{(\eta)}} - 1 \ge u,\ M b_n \ge Y_1^{(\eta)} \ge b_n^{1-p}\}\, du \\
= \;& n b_n^{-p} \textstyle\int_0^\infty \Pr\{\lambda Y_1^{(\eta)} \ge x,\ M b_n \ge Y_1^{(\eta)} \ge b_n^{1-p}\} e^{x b_n^{p-1}} b_n^{p-1}\, dx \\
\le \;& n b_n^{-p} e^\lambda \Pr\{Y_1^{(\eta)} \ge b_n^{1-p}\} + n b_n^{-p} \textstyle\int_{\lambda b_n^{1-p}}^{M\lambda b_n} \Pr\{\lambda Y_1^{(\eta)} \ge x\} e^{x b_n^{1-p}} b_n^{p-1}\, dx \\
\le \;& n b_n^{-p} e^{\lambda - \tau b_n^{p(1-p)}} + n b_n^{-1} \textstyle\int_{\lambda b_n^{1-p}}^{M\lambda b_n} e^{-\tau\lambda^{-p} x^p + x b_n^{1-p}}\, dx \\
\le \;& n b_n^{-p} e^{\lambda - \tau b_n^{p(1-p)}} + n b_n^{-1} \textstyle\int_{\lambda b_n^{1-p}}^{M\lambda b_n} e^{-(\tau\lambda^{-p} - M^{1-p}\lambda^{1-p}) x^p}\, dx \to 0.
\end{aligned}$$

Condition (iii) in in Lemma 2.5 in Arcones (2001b) holds by symmetry. Condition (iv) in Theorem 2.5 follows from (a.2) and the Hoffmann–Jørgensen inequality. We claim that

$$\sup_{t\in T} \sum_{j=1}^\infty x_j^2(t) < \infty. \tag{2.14}$$

We have that $\sup_{t\in T} |\sum_{j=1}^\infty x_j(t)\xi_{1,j}| = O_p(1)$. By the contraction principle for sums of Rademacher r.v.'s (see for example Theorem 4.4. in Ledoux and Talagrand, 1991),

$$\sup_{t\in T} |\sum_{j=1}^\infty x_j(t)\xi_{1,j} I(|\xi_{1,j} x_j(t)| \le 1)| = O_p(1).$$

Now the Hoffmann–Jørgensen inequality (see for example Proposition 6.8 in Ledoux and Talagrand, 1991) implies that

$$E[\sup_{t\in T}|\sum_{j=1}^{\infty} x_j(t)\xi_{1,j}I(|x_j(t)\xi_{1,j}|\le 1)|^2]<\infty.$$

Let $a=\sup_{t\in T}\sup_{j\ge 1}|x_j(t)|$. Hence,

$$\sup_{t\in T}\sum_{j=1}^{\infty} x_j^2(t)E[\xi_{1,1}^2 I(a|\xi_{1,1}|\le 1)]\le \sup_{t\in T}\sum_{j=1}^{\infty} E[x_j^2(t)\xi_{1,1}^2 I(|\xi_{1,j}x_j(t)|\le 1)]<\infty$$

and (2.14) holds. (2.14) implies condition (v) in Lemma 2.5 in Arcones (2001b).

Assume (b). Since for each $\tau>0$,

$$\inf\{I(z):\sup_{t\in T}|z(t)|\ge \tau\}=\tau^p,$$

(a.1) holds. Theorem 3.1 in Arcones (2001a) implies that for each $k>0$, (t,ρ_k) is totally bounded, where

$$d_k(s,t)=\inf\{|u_2-u_1|:I_{s,t}(u_1,u_2)\le k\}$$

It is easy to see that

$$d_k(s,t)=\inf\{\sum_{j=1}^{\infty}(x_j(s)-x_j(t))\gamma_j:\sum_{j=1}^{\infty}|\gamma_j|^p\le k\}=|x(s)-x(t)|_\infty k^{1/p}.$$

So, (a.2) follows. □

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