Moderate deviations of empirical processes

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Abstract. We give necessary and sufficient conditions for the moderate deviations of empirical processes and of sums of i.i.d. random vectors with values in a separable Banach space. Our approach is based in a characterization of the large deviation principle using the large deviations of the finite dimensional distributions plus an asymptotic exponential equicontinuity condition.

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1. Introduction

We study the moderate deviations for different types of sequences of empirical processes $\{U_n(t) : t \in T\}$, where T is an index set. We also consider the moderate deviations of sums of i.i.d. random vectors with values in a separable Banach space. Our results are stated as functional large deviations with a Gaussian rate function.

General references on (functional) large deviations are Bahadur [4]; Varadhan [23]; Deuschel and Stroock [9] and Shwartz and Weiss [21]. We consider stochastic processes as elements of $l_\infty(T)$, where T is an index set. $l_\infty(T)$ is the Banach space consisting of the bounded functions defined in T with the norm $\|x\|_\infty = \sup_{t\in T} |x(t)|$. We will use the following definition:

Definition 1.1. Given a sequence of stochastic processes $\{U_n(t) : t \in T\}$, a sequence of positive numbers $\{\epsilon_n\}_{n=1}^\infty$ such that $\epsilon_n \to 0$, and a function $I : l_\infty(T) \to [0, \infty]$, we say that $\{U_n(t) : t \in T\}$ satisfies the LDP (large deviation principle) with speed ϵ_n^{-1} and with a good rate function I if:

(i) For each $0 \le c < \infty$, $\{z \in l_\infty(T) : I(z) \le c\}$ is a compact set of $l_\infty(T)$.

(ii) For each set $A \in l_\infty(T)$,

$$\begin{aligned} &-I(A^o) \le \liminf_{n\to\infty} \epsilon_n \log(\Pr{}_*\{\{U_n(t) : t \in T\} \in A\}) \\ \le\; &\limsup_{n\to\infty} \epsilon_n \log(\Pr{}^*\{\{U_n(t) : t \in T\} \in A\}) \le -I(\bar{A}), \end{aligned}$$

where for $B \subset l_\infty(T)$, $I(B) = \inf\{I(z) : z \in B\}$.

By Theorem 3.2 in Arcones [1], this definition is equivalent to the large deviations of the finite dimension distributions plus an asymptotic equicontinuity

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condition. This will allow us to obtain necessary and sufficient conditions for the moderate deviations of the considered stochastic processes.

We consider stochastic processes $\{U_n(t) : t \in T\}$ satisfying the large deviation principle with a Gaussian rate function. This rate function is related with a covariance function on T. By a covariance function R on T, we mean a function $R : T \times T \to \mathbb{R}$ such that for each $s, t \in T$ $R(s,t) = R(t,s)$, and for each $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ and each $t_1, \dots, t_m \in T$, $\sum_{j,k=1}^m \lambda_j \lambda_k R(t_j, t_k) \geq 0$. By Theorem II.3.1 in Doob [10], a function $R : T \times T \to \mathbb{R}$ is a covariance function if and only if there exists a (Gaussian) process $\{Z(t) : t \in T\}$ with mean zero and covariance given by $E[Z(s)Z(t)] = R(s,t)$, for each $s, t \in T$. In the considered situations, the rate function of the LDP of $\{(U_n(t_1), \dots, U_n(t_m))\}$ is

$$I_{t_1,\dots,t_m}(u_1, \dots, u_m) = \sup_{\lambda_1,\dots,\lambda_m} \left(\sum_{j=1}^m \lambda_j u_j - 2^{-1} \sum_{j,k=1}^{\infty} \lambda_j \lambda_k R(t_j, t_k) \right), \tag{1.1}$$

where $u_1, \dots, u_m \in \mathbb{R}$. This is the rate function of the LDP of the finite dimensional distributions of a Gaussian process. If $\{Z(t) : t \in T\}$ is a Gaussian process with mean zero and covariance function R, then for each $t_1, \dots, t_m$, $(n^{-1/2}Z(t_1), \dots, n^{-1/2}Z(t_m))$ satisfies the LDP with speed n and the rate function in (1.1).

For sums of i.i.d.r.v.'s, the moderate deviations can be defined as follows. Let $\{X_j\}_{j=1}^\infty$ be a sequence of nondegerate i.i.d.r.v.'s such that for some $\lambda > 0$, $E[e^{\lambda|X_1|}] < \infty$. Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers such that $a_n \to \infty$ and $a_n^{-1} n^{1/2} \to \infty$, it follows from the results in Petrov [19] that for each $t \geq 0$,

$$\lim_{n\to\infty} a_n^{-2} \log(\Pr\{|\sum_{j=1}^n (X_j - E[X_j])| \geq t n^{1/2} a_n\}) = -t^2/(2\mathrm{Var}(X_1)).$$

Cramér [8] obtained the previous result assuming the extra condition that $a_n^{-1} n^{1/2} \log n \to \infty$. We obtain necessary and sufficient conditions for the moderate deviations of sums of i.i.d.r.v.'s which apply to r.v.'s which may not have finite second moment. In particular, we obtain that $\{n^{-1/2} a_n^{-1} \sum_{j=1}^n X_j\}$ satisfies the LDP with speed a_n^2 and a Gaussian rate if and only if $E[X] = 0$, $E[X^2] < \infty$ and

$$\lim_{n\to\infty} a_n^{-2} \log(n \Pr\{|X| \geq n^{1/2} a_n\}) = -\infty.$$

We also give necessary and sufficient conditions for the moderate deviations with a Gaussian rate of empirical processes and for sums of i.i.d.r.v.'s with values in a separable Banach space. The partial sums processes obtained from the processes above satisfy the LDP under the same conditions as the regular sums do. Moderate deviations for empirical processes have been studied by Borovkov and Mogul'skiĭ[5, 6], Ledoux [15] and Wu [24, 25], among other authors. In other situations, moderate deviations may have not a Gaussian rate.

We will use the usual multivariate notation. For example, given $u = (u_1, \dots, u_d)' \in \mathbb{R}^d$ and $v = (v_1, \dots, v_d)' \in \mathbb{R}^d$, $u'v = \sum_{j=1}^d u_j v_j$ and $|u| =$

$(\sum_{j=1}^n u_j^2)^{1/2}$. Whenever, we consider a sequence of i.i.d.r.v.'s $\{X_j\}$, X will denote a copy of X_1. c will denote an arbitrary constant which may vary from occurrence to occurrence.

2. Moderate deviations of empirical processes

The basis of our work is the following theorem:

Theorem 2.1. *(Theorem 3.2 in Arcones [1]) Let $\{U_n(t) : t \in T\}$ be a sequence of stochastic processes, let $\{\epsilon_n\}$ be a sequence of positive numbers that converges to zero. Let $I : l_\infty(T) \to [0,\infty]$ and let $I_{t_1,\dots,t_m} : \mathbb{R}^m \to [0,\infty]$ be a function, where $t_1,\dots,t_m \in T$. Let d be a pseudometric in T. Consider the conditions:*

(a.1) (T,d) is totally bounded.

(a.2) For each $t_1,\dots,t_m \in T$, $(U_n(t_1),\dots,U_n(t_m))$ satisfies the LDP with speed ϵ_n and good rate function $I_{t_1,\dots,t_m}$.

(a.3) For each $\tau > 0$,

$$\lim_{\eta\to 0}\limsup_{n\to\infty} \epsilon_n \log\left(\Pr{}^*\left\{\sup_{d(s,t)\le\eta} |U_n(t)-U_n(s)| \ge \tau\right\}\right) = -\infty.$$

(b) $\{U_n(t) : t \in T\}$ satisfies the LDP in $l_\infty(T)$ with speed ϵ_n^{-1} and with good rate function I.

If the set of conditions (a) is satisfied for some pseudometric d, then (b) holds with

$$I(z) = \sup\{I_{t_1,\dots,t_m}(z(t_1),\dots,z(t_m)) : t_1,\dots,t_m \in T, m \ge 1\}.$$

If (b) is satisfied, then the set of conditions (a) holds with

$$I_{t_1,\dots,t_m}(u_1,\dots,u_m) = \inf\{I(z) : z \in l_\infty(T), (z(t_1),\dots,z(t_m)) = (u_1,\dots,u_m)\}$$

and the pseudometric $\rho(s,t) = \sum_{k=1}^\infty k^{-2}\min(\rho_k(s,t),1)$, where $\rho_k(s,t) = \sup\{|u_2 - u_1| : I_{s,t}(u_1,u_2) \le k\}$.

First, we see how to express the rate function I, when the rate function for the finite dimensional distributions is given by (1.1).

Theorem 2.2. *Let T be a parameter set and let R be covariance function on T. Let $\{f(\cdot,t) : t \in T\}$ be a class of measurable functions on the same measure space $(\Omega,\mathcal{F},\mu)$ such that for each $t \in T$, $\int f(x,t)\,d\mu(x) = 0$ and $\int (f(x,t))^2\,d\mu(x) < \infty$, and for each $s,t \in T$, $\int f(x,s)f(x,t)\,d\mu(x) = R(s,t)$. Then,*

(i) For each $t_1,\dots,t_m \in T$, and each $u_1,\dots,u_m \in \mathbb{R}$,

$$\begin{aligned} & \sup\{\textstyle\sum_{j=1}^m \lambda_j u_j - 2^{-1}\sum_{j,k=1}^\infty \lambda_j\lambda_k R(t_j,t_k) : \lambda_1,\dots,\lambda_m\} \\ = \ & \inf\{2^{-1}\textstyle\int \gamma^2(x)\,d\mu(x) : \\ & \gamma \in L_2,\ \textstyle\int \gamma(x)f(x,t_j)\,d\mu(x) = z(t_j) \text{ for each } 1 \le j \le m\}. \end{aligned}$$

Besides, if the infimum above is finite, there exists a function γ attaining the infimum.

(ii) If $\{f(\cdot,t) : t \in T\}$ is a separable subset of L_2, then for each $z \in l_\infty(T)$,

$$\begin{aligned} &\sup\{I_{t_1,\dots,t_m}(z(t_1),\dots,z(t_m)) : t_1,\dots,t_m \in T, m \geq 1\} \\ = \ &\inf\{2^{-1}\textstyle\int \gamma^2(x)\, d\mu(x) : \gamma \in L_2,\ \int \gamma(x) f(x,t)\, d\mu(x) = z(t) \text{ for each } t \in T\}, \end{aligned}$$

where

$$\begin{aligned} &I_{t_1,\dots,t_m}(u_1,\dots,u_m) = \inf\{2^{-1}\textstyle\int \gamma^2(x)\, d\mu(x) : \\ &\gamma \in L_2,\ \textstyle\int \gamma(x) f(x,t_j)\, d\mu(x) = z(t_j) \text{ for each } 1 \leq j \leq m\}. \end{aligned}$$

As before, if the infimum above is finite, there exists a function γ attaining the infimum.

Proof. Part (i) follows from Lemma 4.1 in Arcones [2] with $\Phi(x) = \Psi(x) = 2^{-1}x^2$, $x \in \mathbb{R}$. Since a bounded set of L_2 is weakly compact, the infimum in part (i) is attained.

Let

$$I^{(1)}(z) := \sup\{I_{t_1,\dots,t_m}(z(t_1),\dots,z(t_m)) : t_1,\dots,t_m \in T, m \geq 1\}$$

and let

$$I^{(2)}(z) := \inf\{2^{-1}\int \gamma^2(x)\, d\mu(x) : \int \gamma(x) f(x,t)\, d\mu(x) = z(t) \text{ for each } t \in T\}.$$

It is obvious that for each $z \in l_\infty(T)$, $I^{(2)}(z) \geq I^{(1)}(z)$. Next, we show that $I^{(1)}(z) \geq I^{(2)}(z)$. We may assume that $I^{(2)}(z) < \infty$. Take a sequence $\{t_n\}$ in T such that

$$I^{(1)}(z) = \lim_{n\to\infty} I_{t_1,\dots,t_n}(z(t_1),\dots,z(t_n)).$$

Given $r_1,\dots,r_m,s_1,\dots,s_k \in T$, we have that

$$I_{r_1,\dots,r_m}(z(r_1),\dots,z(r_m)) \leq I_{r_1,\dots,r_m,s_1,\dots,s_k}(z(r_1),\dots,z(r_m),z(s_1),\dots,z(s_k)).$$

Hence, we may assume that $\{f(\cdot,t_n) : n \geq 1\}$ is dense in L_2. Let $\gamma_n \in L_2$ be such that

$$2^{-1}\int \gamma_n^2(x)\, d\mu(x) = I_{t_1,\dots,t_n}(z(t_1),\dots,z(t_n)).$$

Then, there exists a subsequence γ_{n_k} and $\gamma \in L_2$ such that γ_{n_k} converges weakly to γ. This implies that for each $m \geq 1$,

$$z(t_m) = \lim_{k\to\infty} \int \gamma_{n_k}(x) f(x,t_m)\, d\mu(x) = \int \gamma(x) f(x,t_m)\, d\mu(x).$$

Since $I^{(2)}(z) < \infty$, $z : (T, \|\cdot\|_2) \to \mathbb{R}$ is continuous. From this and the fact that $\{f(\cdot,t_m) : m \geq 1\}$ is dense in L_2, we get that $z(t) = \int \gamma(x) f(x,t)\, d\mu(x)$, for each $t \in T$. □

Given a covariance function R, there exists a Gaussian process $\{Z(t) :\ t \in T\}$ with mean zero and covariance R. Let $\mathcal{L}$ be the closed vector space of L_2 generated by $\{Z(t):\ t\in T\}$. If $\{U_n(t): t\in T\}$ satisfies the LDP with speed ϵ_n^{-1} and for each $t_1,\dots,t_m\in T$, the rate function of the LDP of $\{(U_n(t_1),\dots,U_n(t_m))\}$ is

$$I_{t_1,\dots,t_m}(u_1,\dots,u_m)=\sup_{\lambda_1,\dots,\lambda_m}\left(\sum_{j=1}^m \lambda_j u_j-2^{-1}\sum_{j,k=1}^m \lambda_j\lambda_k R(t_j,t_k)\right), \tag{2.1}$$

then, by theorems 2.1 and 2.2, the rate function of the LDP of $\{U_n(t): t\in T\}$ is

$$I(z)=\inf\{2^{-1}E[\gamma^2]:\gamma\in\mathcal{L}, E[\gamma Z(t)]=z(t) \text{ for each } t\in T\}. \tag{2.2}$$

It follows that if $\sup_{t\in T}R(t,t)>0$, then for each $\lambda\ge 0$,

$$\inf\{I(z):\sup_{t\in T}|z(t)|\ge\lambda\}=\frac{\lambda^2}{2\sup_{t\in T}R(t,t)}.$$

If $\sup_{t\in T}R(t,t)=0$, then

$$I(z)=\begin{cases}0 & \text{if } \sup_{t\in T}|z(t)|=0\\ \infty & \text{if } \sup_{t\in T}|z(t)|>0.\end{cases}$$

So, if $\sup_{t\in T}R(t,t)>0$, then for each $\lambda\ge 0$,

$$\lim_{n\to\infty}\epsilon_n\log\left(\Pr\{\sup_{t\in T}|U_n(t)|\ge\lambda\}\right)=-\frac{\lambda^2}{2\sup_{t\in T}R(t,t)}.$$

If $\sup_{t\in T}R(t,t)=0$, then for each $\lambda>0$,

$$\lim_{n\to\infty}\epsilon_n\log\left(\Pr\{\sup_{t\in T}|U_n(t)|\ge\lambda\}\right)=-\infty.$$

We also have that

$$\begin{aligned}&\rho_k(s,t):=\sup\{|u_2-u_1|: I_{s,t}(u_1,u_2)\le k\}\\ &=\sup\{|E[\gamma(Z(s)-Z(t))]|:\gamma\in\mathcal{L}, 2^{-1}E[\gamma^2]\le k\}=(2k)^{1/2}\|Z(s)-Z(t)\|_2.\end{aligned}$$

So, the LDP implies that $\{Z(t):\ t\in T\}$ is a totally bounded set of L_2.

The rates above appear in the large deviations of Gaussian processes. If $\{Z(t):\ t\in T\}$ is a Gaussian process with mean zero and covariance R, then the finite dimensional distributions of $\{n^{-1/2}Z(t):\ t\in T\}$ satisfy the LDP with speed n and the rate function in (2.1). If $\sup_{t\in T}|Z(t)|<\infty$ a.s., then $\{n^{-1/2}Z(t):\ t\in T\}$ satisfy the LDP with speed n and with the rate in (2.2).

To get the LDP for the finite dimensional distributions, we will apply the following lemma.

Lemma 2.3. *Let $\{X_{n,j}: 1\le j\le n\}$ be a triangular array of independent r.v.'s with values in $\mathbb{R}^d$ and mean zero. Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers converging to infinity. Suppose that:*

(i) The following limit exists and it is finite: $\lim_{n\to\infty}\sum_{j=1}^n E[X_{n,j}X_{n,j}']=:\Sigma$.

(ii) There exists a constant τ such that for each $1\le j\le n$, $|X_{n,j}|\le\tau a_n^{-1}$ a.s.

(iii) For each $\delta>0$, $a_n^{-2}\sum_{j=1}^n\Pr\{|X_{n,j}|\ge\delta a_n^{-1}\}\to 0$.

Then, $a_n^{-1}\sum_{j=1}^n X_{n,j}$ *satisfies the LDP with speed* a_n^2 *and rate function* $I(u) = \sup_\lambda(\lambda' u - 2^{-1}\lambda\Sigma\lambda')$.

Proof. By Theorem II.2 in Ellis [12], it suffices to prove that for each $\lambda \in \mathbb{R}^d$,

$$\lim_{n\to\infty} a_n^{-2}\log(E[\exp(a_n\sum_{j=1}^n \lambda' X_{n,j})]) = 2^{-1}\lambda'\Sigma\lambda.$$

First, we prove that by a Taylor expansion, we have that

$$a_n^{-2}\sum_{j=1}^n E[\exp(a_n\lambda' X_{n,j}) - 1] \to 2^{-1}\lambda'\Sigma\lambda. \tag{2.3}$$

Since $a_n|X_{n,j}|$, $1 \le j \le n$, are uniformly bounded, we need to prove that

$$a_n^{-2}\sum_{j=1}^n E[|a_n\lambda' X_{n,j}|^3] \to 0.$$

We have that for any $\delta > 0$,

$$a_n^{-2}\sum_{j=1}^n E[|a_n\lambda' X_{n,j}|^3]$$

$$= a_n^{-2}\sum_{j=1}^n E[|a_n\lambda' X_{n,j}|^3 I_{a_n|X_{n,j}|\le\delta}] + a_n^{-2}\sum_{j=1}^n E[|a_n\lambda' X_{n,j}|^3 I_{a_n|X_{n,j}|>\delta}]$$

$$\le \delta|\lambda|\sum_{j=1}^n E[\lambda' X_{n,j}X_{n,j}'\lambda] + |\lambda|^3\tau^3 a_n^{-2}\sum_{j=1}^n \Pr\{|X_{n,j}| \ge \delta a_n^{-1}\}.$$

Hence, $\limsup_{n\to\infty} a_n^{-2}\sum_{j=1}^n E[|a_n\lambda' X_{n,j}|^3] \le \delta|\lambda|\lambda'\Sigma\lambda$. Since δ is arbitrary, (2.3) follows.

Again, using that $a_n|X_{n,j}|$, $1 \le j \le n$, are uniformly bounded, we have that

$$\begin{aligned}
& a_n^{-2}\sum_{j=1}^n |\log(E[\exp(a_n\lambda' X_{n,j})]) - E[\exp(a_n\lambda' X_{n,j}) - 1]| \\
\le\ & c a_n^{-2}\sum_{j=1}^n |E[\exp(a_n\lambda' X_{n,j}) - 1]|^2 \\
=\ & c a_n^{-2}\sum_{j=1}^n |E[\exp(a_n\lambda' X_{n,j}) - 1 - a_n\lambda' X_{n,j}]|^2 \\
\le\ & c a_n^{-2}\sum_{j=1}^n (E[|a_n\lambda' X_{n,j}|^2])^2 \\
\le\ & c a_n^{-2}\sum_{j=1}^n E[|a_n\lambda' X_{n,j}|^4] \to 0.
\end{aligned}$$

□

Next, we consider the moderate deviations of sums of real valued i.i.d.r.v.'s. We present a general theorem which applies to r.v.'s which may not have finite second moment.

Theorem 2.4. *Let* $\{X_j\}_{j=1}^\infty$ *be a sequence of i.i.d.r.v.'s. Let* $\{a_n\}_{n=1}^\infty$ *and let* $\{c_n\}_{n=1}^\infty$ *be two sequences of real numbers such that* $a_n \nearrow \infty$ *and* $a_n^{-1}c_n^{-1}n \nearrow \infty$ *and* $\{n^{-1}c_n^2\}$ *is nondecreasing. Then, the following sets of conditions ((a), (b), (c)) are equivalent:*

(a.1) $\lim_{n\to\infty} a_n^{-2} \log(n \Pr\{|X| \ge c_n a_n\}) = -\infty$.
(a.2) $a_n^{-1} c_n^{-1} n E[XI(|X| \le a_n c_n)] \to 0$.
(a.3) $c_n^{-2} n \mathrm{Var}(XI(|X| \le a_n^{-1} c_n))$ *converges to a finite limit* σ^2.
(b) $\{c_n^{-1} a_n^{-1} \sum_{j=1}^n X_j\}$ *satisfies the LDP with speed* a_n^2 *and a rate function* I *such that* $\lim_{|\lambda|\to\infty} \lambda^{-1} I(\lambda) = \infty$ *and for each* $\delta > 0$, $\inf\{I(z) : |z| \ge \delta\} > 0$.
(c) $\{c_n^{-1} a_n^{-1} \sum_{j=1}^{[nu]} X_j : 0 \le u \le 1\}$ *satisfies the LDP in* $l_\infty([0,1])$ *with speed* a_n^2 *and a rate function* I *such that* $\lim_{\lambda\to\infty} \lambda^{-1} \inf\{I(z) : \sup_{0\le u\le 1} |z(u)| \ge \lambda\} = \infty$ *and for each* $\delta > 0$, $\inf\{I(z) : \sup_{0\le u\le 1} |z(u)| \ge \delta\} > 0$.
Moreover, the rate function in (b) is given by

$$I(z) = \frac{z^2}{2\sigma^2}, \text{ if } \sigma^2 > 0. \tag{2.4}$$

and

$$I(z) = \begin{cases} 0 & \text{if } z = 0 \\ \infty & \text{if } z \ne 0 \end{cases} \tag{2.5}$$

if $\sigma^2 = 0$.
The rate function in (c) is given by

$$I(z) = \begin{cases} 2^{-1}\sigma^{-2} \int_0^1 (z'(u))^2\, du & \text{if } z(0) = 0 \text{ and } z \text{ is absolutely continuous} \\ \infty & \text{else} \end{cases} \tag{2.6}$$

if $\sigma^2 > 0$, *and*

$$I(t) = \begin{cases} 0 & \text{if } \sup_{0\le u\le 1} |z(u)| = 0 \\ \infty & \text{if } \sup_{0\le u\le 1} |z(u)| > 0 \end{cases} \tag{2.7}$$

if $\sigma^2 = 0$.

Proof. First, we prove that (a) implies (b). Observe that if $n^{-1/2} c_n \to M < \infty$, then $E[X^2] < \infty$ and $E[X] = 0$. If $n^{-1/2} c_n \to \infty$, then $E[X^2] = \infty$. For each $\delta > 0$,

$$a_n^{-2} \log(\Pr\{|c_n^{-1} a_n^{-1} \sum_{j=1}^n X_j I_{|X_j| \ge a_n c_n}| \ge \delta\}) \le a_n^{-2} \log(n \Pr\{|X| \ge a_n c_n\}) \to -\infty.$$

Next, we prove that for each $\delta > 0$,

$$a_n^{-2} \log \Pr\{|c_n^{-1} a_n^{-1} \sum_{j=1}^n X_j I_{a_n^{-1} c_n \le |X_j| < a_n c_n}| \ge \delta\} \to -\infty. \tag{2.8}$$

We have that for each $\lambda > 0$,

$$\begin{aligned} & a_n^{-2} \log \Pr\{|c_n^{-1} a_n^{-1} \textstyle\sum_{j=1}^n X_j I_{a_n^{-1} c_n \le |X_j| < a_n c_n}| \ge \delta\} \\ \le\ & -\lambda\delta + n a_n^{-2} \log E[\exp(\lambda c_n^{-1} a_n |X| I_{a_n^{-1} c_n \le |X_j| < a_n c_n})]. \end{aligned}$$

Hence, if we show that for each $0 < \lambda < \infty$,

$$na_n^{-2}E[\exp(\lambda c_n^{-1}a_n|X|)I_{a_n^{-1}c_n\le|X_j|<a_nc_n}-1]\to 0, \tag{2.9}$$

then (2.8) follows by letting $\lambda\to\infty$. Given $1<M<\infty$,

$$\begin{aligned}
& na_n^{-2}E[\exp(\lambda c_n^{-1}a_n|X|)I_{a_n^{-1}c_n\le|X|<a_nc_n}-1]\\
= \; & na_n^{-2}\int_0^\infty \Pr\{\exp(\lambda c_n^{-1}a_n|X|)I_{a_n^{-1}c_n\le|X|<a_nc_n}-1\ge t\}\,dt\\
\le \; & (e^M-1)na_n^{-2}\Pr\{|X|\ge a_n^{-1}c_n\}\\
& +na_n^{-2}\int_{e^M-1}^\infty \Pr\{\exp(\lambda c_n^{-1}a_n|X|)\ge t+1, a_nc_n\ge|X|\}\,dt =: I+II.
\end{aligned}$$

By the change of variables $t+1=e^u$, we have that

$$II\le a_n^{-2}n\int_M^{\lambda a_n^2}\Pr\{|X|\ge u\lambda^{-1}c_na_n^{-1}\}e^u\,du.$$

Take $m=m(n,u)$ such that $a_mc_m\le u\lambda^{-1}c_na_n^{-1}<a_{m+1}c_{m+1}$. Observe $m(n,1)\to\infty$ as $n\to\infty$. So, given $c>1+2\lambda$, for m large

$$m\Pr\{|X|\ge a_mc_m\}\le -e^{-ca_m^2}.$$

Hence, for $u\ge1$, and n large enough,

$$\begin{aligned}
& na_n^{-2}\Pr\{|X|\ge u\lambda^{-1}c_na_n^{-1}\}e^u\\
\le \; & nu^{-2}\lambda^2c_n^{-2}a_{m+1}^2c_{m+1}^2\Pr\{|X|\ge c_ma_m\}e^u\\
\le \; & n\lambda^2c_n^{-2}a_{m+1}^2c_{m+1}^2m^{-1}e^{-ca_m^2+u}.
\end{aligned}$$

We also have that for $m\ge3$, $a_{m+1}^2c_{m+1}^2\le a_m^2c_m^2(m+1)^2m^{-2}\le 2a_m^2c_m^2$. Since $u<\lambda a_n^2$, $a_mc_m\le u\lambda^{-1}c_na_n^{-1}<a_nc_n$, for n large, $m\le n$. So, $c_n^{-2}c_m^2nm^{-1}\le1$,

$$nc_n^{-2}a_{m+1}^2c_{m+1}^2m^{-1}\le 2nc_n^{-2}a_m^2c_m^2m^{-1}\le 2a_m^2$$

and

$$u\le\lambda c_n^{-1}a_na_{m+1}c_{m+1}\le 2\lambda c_n^{-1}a_na_mc_m\le 2\lambda a_m^2.$$

From these estimations, for n large enough,

$$\begin{aligned}
na_n^{-2}\Pr\{|X|\ge\lambda^{-1}ua_n^{-1}c_n\}e^u &\le 2\lambda^2a_m^2e^{-ca_m^2+u}\\
&\le 2\lambda^2e^{-(c-1)a_m^2+u}\le 2\lambda^2e^{-((c-1)2^{-1}\lambda^{-1}-1)u}.
\end{aligned} \tag{2.10}$$

Hence,

$$\begin{aligned}
&\limsup_{n\to\infty} na_n^{-2}\int_M^{\lambda a_n^2}\Pr\{|X|\ge\lambda^{-1}c_na_n^{-1}u\}e^u\,du\\
&\le 2\lambda^2((c-1)2^{-1}\lambda^{-1}-1)^{-1}e^{-((c-1)2^{-1}\lambda^{-1}-1)M}.
\end{aligned}$$

Since M can be made arbitrarily large, $II\to0$. By the previous estimations with $u=\lambda$, we get that $I\le(e^M-1)2\lambda^2e^{-(c-1)a_m^2+\lambda}\to0$. Hence, (2.9) follows.

It follows from (2.9) that

$$na_n^{-1}c_n^{-1}E[|X|I_{a_n^{-1}c_n\le|X_j|<a_nc_n}]\to 0$$

By the previous estimations, it suffices to show that $\{c_n^{-1}a_n^{-1}\sum_{j=1}^n(X_jI_{|X_j|<a_n^{-1}c_n}-E[X_jI_{|X_j|<a_n^{-1}c_n}])\}$ satisfies the LDP with speed

a_n^2 and rate function $I(t) = \sup_{u\in\mathbb{R}}(ut - 2^{-1}u^2\sigma^2)$. This follows from Lemma 2.3. Therefore, we got that (a) implies (b).

Next, we prove that (b) implies (a). Let $J(t) = \inf\{I(u) : |u| \geq t\}$. Then, for each $t > 0$,

$$\limsup_{n\to\infty} a_n^{-2} \log(\Pr\{a_n^{-1}c_n^{-1}|\sum_{j=1}^n X_j| \geq t\}) \leq -J(t).$$

By Lemma 2.1 in Arcones [3], for each $t > 0$,

$$\limsup_{n\to\infty} a_n^{-2} \log(n \Pr\{|X| \geq 3a_n c_n t\}) \leq -J(t).$$

Given $t > 1$, $m = m(n,t)$ such that $a_m c_m \leq 3^{-1}t^{-1}c_n a_n < a_{m+1}c_{m+1}$. Observe that for each $t > 0$, $m(n,t) \to \infty$ as $n \to \infty$. Since $t > 1$, $m \leq n$. For n large,

$$a_n c_n \leq 3ta_{m+1}c_{m+1} \leq 6ta_m c_m \leq 6ta_n c_m \leq 6tn^{-1/2}m^{1/2}a_n c_n.$$

So, $m^{-1}n \leq 36t^2$. We also have that

$$a_n^2 \leq 36t^2 c_n^{-2}a_m^2 c_m^2 \leq 36t^2 n^{-1}a_m^2 m \leq 36t^2 a_n^{-1}c_n^{-1}a_m^3 c_m \leq 12ta_m^2.$$

From these estimations, for n large enough,

$$\begin{aligned} n\Pr\{|X| \geq a_n c_n\} &\leq n\Pr\{|X| \geq 3ta_m c_m\} \\ \leq \quad m^{-1}ne^{-2^{-1}J(t)a_m^2} &\leq 36t^2 e^{-2^{-5}t^{-1}J(t)a_n^2}. \end{aligned} \tag{2.11}$$

Since, $\lim_{t\to\infty} t^{-1}J(t) = \infty$, (a.1) follows.

Since for each $\delta > 0$, $\inf\{I(t) : |t| \geq \delta\} > 0$, $c_n^{-1}a_n^{-1}\sum_{j=1}^n X_j \xrightarrow{\Pr} 0$. By the necessary and sufficient conditions for the weak law of the large numbers (see for example Corollary 10.1.3 in Chow and Teicher [7]), (a.2) follows.

Since (a.1) and (a.2) hold, by the proof (a) implies (b), we have that the sum of the r.v.'s for $|X_j| \geq a_n^{-1}c_n$ is asymptotically exponentially negligible. So, we have that for each $t > 0$,

$$\limsup_{n\to\infty} a_n^{-2} \log(\Pr\{|c_n^{-1}a_n^{-1}\sum_{j=1}^n (X_j I_{|X_j|<a_n^{-1}c_n} - E[X_j I_{|X_j|<a_n^{-1}c_n}])| \geq t\}) \leq -z(t).$$

Let $\{X_n'\}$ be an independent copy of $\{X_n\}$. Then, for each $t > 0$,

$$\limsup_{n\to\infty} a_n^{-2} \log(\Pr\{|c_n^{-1}a_n^{-1}\sum_{j=1}^n (X_j I_{|X_j|<a_n^{-1}c_n} - X_j' I_{|X_j'|<a_n^{-1}c_n})| \geq 2t\}) \leq -z(t).$$

By Proposition 6.14 in Ledoux and Talagrand [16],

$$\begin{aligned} &E[\exp(\lambda c_n^{-1}a_n|\sum_{j=1}^n (X_j I_{|X_j|<a_n^{-1}c_n} - X_j' I_{|X_j'|<a_n^{-1}c_n})|)] \\ &\qquad \leq e^{2\lambda a_n^2 t} + 2e^{2\lambda a_n^2(t+a_n^{-2})-2z(t)a_n^2} \times \\ &E[\exp(\lambda c_n^{-1}a_n|\sum_{j=1}^n (X_j I_{|X_j|<a_n^{-1}c_n} - X_j' I_{|X_j'|<a_n^{-1}c_n})|)]. \end{aligned} \tag{2.12}$$

Hence, for $0<\lambda<2^{-1}t^{-1}z(t)$, and n large enough.

$$E[\exp(\lambda c_n^{-1}a_n|\sum_{j=1}^{n}(X_jI_{|X_j|<a_n^{-1}c_n}-X_j'I_{|X_j'|<a_n^{-1}c_n})|)]\le 2e^{2\lambda a_n^2 t}.$$

Now, Lemma 10.2.1 in Chow and Teicher [7] implies that for $0<\lambda<2^{-1}$,

$$\begin{aligned}&\exp(\lambda^2a_n^2s_n^2g(-2\lambda)(1-2\lambda^2))\\ \le\;& E[\exp(\lambda c_n^{-1}a_n\textstyle\sum_{j=1}^{n}(X_jI_{|X_j|<a_n^{-1}c_n}-X_j'I_{|X_j'|<a_n^{-1}c_n}))],\end{aligned}$$

where $g(x)=x^{-2}(e^x-1-x)$ and $s_n^2=c_n^{-2}n\mathrm{Var}(XI_{|X|<a_n^{-1}c_n})$. Hence, $\limsup_{n\to\infty}s_n^2<\infty$. If $\lim_{n\to\infty}s_n^2$ does not exist, we have sequences such that (a.1)–(a.3) are satisfied with different σ^2. But, this implies that there are different rates functions for the LDP of the whole sequence, in contradiction. Therefore, (a.3) holds. Hence, (b) implies (a).

The contraction principle implies that (c) implies (b).

Finally, we prove that (b) implies (c). We use Theorem 2.1. Let

$$U_n(u)=c_n^{-1}a_n^{-1}\sum_{j=1}^{[nu]}(X_jI_{|X_j|<a_n^{-1}c_n}-E[X_jI_{|X_j|<a_n^{-1}c_n}]),0\le u\le 1.$$

We need to prove that $\{U_n(u):0\le u\le 1\}$ satisfies the LDP in $l_\infty([0,1])$ with speed a_n^2. The convergence of the finite dimensional distributions follows from Lemma 2.3. The corresponding covariance function is

$$R(u_1,u_2)=\lim_{n\to\infty}\mathrm{Cov}(U_n(u_1),U_n(u_2))=\min(u_1,u_2)\sigma^2. \tag{2.13}$$

To prove (a.3) in Theorem 2.1, it suffices to prove that for each $\tau,M>0$, there exists a function $\pi:[0,1]\to[0,1]$ with finite range such that

$$\limsup_{n\to\infty}a_n^{-2}\log\left(\Pr\{\sup_{0\le u\le 1}|U_n(u)-U_n(\pi(u))|\ge\tau\}\right)\le -M. \tag{2.14}$$

Let

$$U_n'(u)=c_n^{-1}a_n^{-1}\sum_{j=1}^{[nu]}(X_j'I_{|X_j'|<a_n^{-1}c_n}-E[X_j'I_{|X_j'|<a_n^{-1}c_n}]),0\le u\le 1.$$

Given a positive integer m, let $\pi:[0,1]\to[0,1]$ given by $\pi(u)=\frac{[mu]}{m}$. By symmetrization (see Lemma 1.2.1 in Giné and Zinn [13]) and the Lévy 's inequality, we have that

$$\begin{aligned}&\Pr\{\sup_{0\le u\le 1}|U_n(u)-U_n(\pi(u))|\ge\tau\}\\ =\;&\Pr\{\max_{1\le j\le m}\max_{(j-1)/m<u<j/m}|U_n(u)-U_n((j-1)/m)|\ge\tau\}\\ \le\;& m\Pr\{\max_{0<u<1/m}|U_n(u)|\ge\tau\}\\ \le\;& 2m\Pr\{\max_{0<u<1/m}|U_n(u)-U_n'(u)|\ge 2^{-1}\tau\}\\ \le\;& 4m\Pr\{|U_n(1/m)-U_n'(1/m)|\ge 2^{-1}\tau\}\\ \le\;& 8m\Pr\{|U_n(1/m)|\ge 2^{-2}\tau\}\end{aligned}$$

By the Prokhorov's inequality (Prokhorov [20]),

$$\limsup_{n\to\infty} a_n^{-2}\log(\Pr\{|U_n(1/m)|\geq 2^{-2}\tau\} \leq -2^{-3}\tau\operatorname{arcsinh}(2^{-2}\sigma^{-2}m\tau).$$

Hence, (2.13) follows.

The rate function in (b) is $I(z) = \sup_{t\in R}(zt - t^2\sigma^2/2)$, which is given by (2.3) and (2.4).

The rate function in (c) is determined by the covariance function in (2.13). The class of functions $\{\sigma I(x\leq t) : 0\leq t\leq 1\}$ in the probability space $([0,1],\mathcal{B},dx)$ has this covariance function. So, the rate function is

$$I(z) = \inf\{2^{-1}\int_0^1 \gamma^2(x)\,dx : z(t) = \sigma\int_0^1 \gamma(x)I(x\leq t)\,dx \text{ for each } 0\leq t\leq 1\}.$$

This rate function is given by (2.5) and (2.6). □

To obtain (a) implies in (b) in the previous theorem it suffices that $a_n\to\infty$, $a_n^{-1}c_n^{-1}n\to\infty$ and $\inf_{n\geq 1} n^{-1}c_n^2 > 0$.

Next, we will present two examples, where Theorem 2.4 applies to r.v.'s with infinite second moment.

Let $\{X_j\}_{j=1}^\infty$ be a sequence of symmetric i.i.d.r.v.'s with $\Pr\{|X|\geq t\} = t^{-2}$, for each $t\geq 1$. Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers such that $a_n\to\infty$ and $a_n^{-2}\log\log n\to\infty$. By Theorem 2.4, $\{n^{-1/2}(\log n)^{-1/2}a_n^{-1}\sum_{j=1}^n X_j\}$ satisfies the LDP with speed a_n^2 and rate function $I(t) = 2^{-1}t^2$. Observe that $\{n^{-1/2}(\log n)^{-1/2}\sum_{j=1}^n X_j\}$ converges in distribution to a normal r.v. with mean zero and variance 1.

Given $2 > p > 1$, let $\{X_j\}_{j=1}^\infty$ be a sequence of symmetric i.i.d.r.v.'s with $\Pr\{|X|\geq t\} = t^{-p}$, for each $t\geq 1$. Let $\{a_n\}_{n=1}^\infty$ and $\{c_n\}_{n=1}^\infty$ be two sequences of real numbers such that $a_n\to\infty$ and $a_n^{-1}c_n^{-1}n\to\infty$ and $a_n^{-2}\log(n^{-1}c_n^p)\to\infty$. Then, Theorem 2.4 gives that $\{c_n^{-1}a_n^{-1}\sum_{j=1}^n X_j\}$ satisfies the LDP with speed a_n^2 and the rate function I in (2.4). It is well known that $n^{-1/p}\sum_{j=1}^n X_j$ converges in distribution to a symmetric stable r.v. of order p.

To get (b) implies (a), we need that the rate function $I(t)$ satisfies $\lim_{|t|\to\infty} t^{-1}I(t) = \infty$. There are examples of sequence of i.i.d.r.v.'s $\{X_n\}$ such that $\{a_n^{-1}n^{-1/2}\sum_{j=1}^n X_j\}$ satisfies the LDP with speed a_n^2 and a rate function $I(t) = a|t|^p$, for some $a > 0$ and some $0 < p < 1$ (see for example, Nagaev [17, 18]).

When $c_n = n^{1/2}$, the conditions in the previous theorem simplify.

Theorem 2.5. *Let $\{X_j\}_{j=1}^\infty$ be a sequence of i.i.d.r.v.'s. Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers such that $a_n\nearrow\infty$ and $n^{-1/2}a_n\searrow 0$. Then, the following conditions are equivalent:*

(a) $E[X] = 0$, $E[X^2] < \infty$ and

$$\lim_{n\to\infty} a_n^{-2}\log(n\Pr\{|X|\geq n^{1/2}a_n\}) = -\infty. \tag{2.15}$$

(b) $\{a_n^{-1}n^{-1/2}\sum_{j=1}^n X_j\}$ *satisfies the LDP with speed* a_n^2 *and a rate* I *such that* $\lim_{|\lambda|\to\infty}\lambda^{-1}I(\lambda)=\infty$.

(c) $\{a_n^{-1}n^{-1/2}\sum_{j=1}^{[nu]} X_j : 0 \le u \le 1\}$ *satisfies the LDP in* $l_\infty([0,1])$ *with speed* a_n^2 *and a rate function* I *such that*

$$\lim_{\lambda\to\infty}\lambda^{-1}\inf\{I(z): |z(1)|\ge\lambda\}=\infty.$$

Moreover, the rate functions in (b) and (c) are given by (2.3)–(2.5).

Proof. Assume (a). Condition (a) implies condition (a) in Theorem 2.4. So, (b) and (c) follow. (c) implies (b) trivially.

Assume (b). The proof is similar to that of Theorem 2.4. The difference is that we do not assume that the rate function satisfies that for each $\delta > 0$, $\inf\{I(z) : |z| \ge \delta\} > 0$. The argument in the proof of Theorem 2.4 implies that

$$\lim_{n\to\infty} a_n^{-2}\log(n\Pr\{|X|\ge n^{1/2}a_n\}) = -\infty.$$

So, $\lim_{t\to\infty} t^2\Pr(|X|\ge t)=0$. In particular, for each $0<p<2$, $E[|X|^p]<\infty$ and $\lim_{t\to\infty} t^{2-p}E[|X|^pI(|X|\ge t)]=0$.

Since for M large enough, $\inf\{I(z) : |z| \ge M\} > 0$, $a_n^{-1}n^{-1/2}\sum_{j=1}^n X_j$ is bounded in probability. This implies that $n^{-1}\sum_{j=1}^n X_j \xrightarrow{\Pr} 0$. Since $E[|X|]<\infty$, we have that $E[X]=0$. Hence,

$$\lim_{n\to\infty} n^{1/2}a_n^{-1}E[XI(|X|\le a_nn^{1/2}] = \lim_{n\to\infty} n^{1/2}a_n^{-1}E[XI(|X|> a_nn^{1/2})]=0.$$

we got that conditions (a.1) and (a.2) in Theorem 2.4 hold. Proceeding as in the proof of Theorem 2.4, we get that $E[X^2]<\infty$. $\square$

To obtain (a) implies in (b) in the previous theorem it suffices that $a_n\to\infty$ and $n^{-1/2}a_n\to 0$.

By the results in Cramér [8] and Petrov [19], if for some $\lambda>0$, $E[e^{\lambda|X|}]<\infty$, then (b) in Corollary 2.4 holds. But, Corollary 2.4 applies to r.v.'s whose moment generating function is not defined in a neighborhood of zero. It follows from Corollary 2.4 that if X is symmetric and for some $0<p<1$,

$$\lim_{t\to\infty} t^{-p}\log(\Pr\{|X|\ge t\})=-1,$$

and $\{a_n\}$ a sequence of positive numbers such that $n^{-p/2}a_n^{2-p}\to\infty$ and $n^{-1/2}a_n\to 0$, then $\{a_n^{-1}n^{-1/2}\sum_{j=1}^n X_j\}$ satisfies the LDP with speed a_n^2 and with rate function $I(t)=\frac{t^2}{2E[X^2]}$. It is easy to see that if for some $\lambda>0$, $E[e^{\lambda|X|}]<\infty$, then (2.14) holds.

We will obtain the LDP for empirical processes, using a general lemma dealing with triangular array of empirical processes. Let $(\Omega_n,\mathcal{A}_n,Q_n)$, $n\ge 1$, be a sequence of probability spaces and let $(S_{n,j},\mathcal{S}_{n,j})$, $1\le j\le k_n$, $n\ge 1$, be measurable spaces. For each $n\ge 1$, let $\{X_{n,j} : 1\le j\le k_n\}$ be independent r.v.'s defined on $(\Omega_n,\mathcal{A}_n,Q_n)$ and with values in $(S_{n,j},\mathcal{S}_{n,j})$, $1\le j\le k_n$. Let $f_{n,j}(\cdot,t) : (S_{n,j},\mathcal{S}_{n,j})\to\mathbb{R}$ be a measurable function for each $1\le j\le k_n$, each $n \geq 1$ and each $t \in T$. Let $U_n(t) := \sum_{j=1}^{k_n} f_{n,j}(X_{n,j}, t)$. To avoid measurability problems, we will assume that $\Omega_n := \prod_{j=1}^{k_n} S_{n,j}$, $\mathcal{A}_n := \prod_{j=1}^{k_n} \mathcal{S}_{n,j}$, $Q_n := \prod_{j=1}^{k_n} Q_{n,j}$ and that for each $1 \leq j \leq k_n$, $\{f_{n,j}(x,t) : t \in T\}$ is an image admissible Suslin class of functions (see page 80 in Dudley [11]). By an abuse of notation, we denote by Pr to Q_n.

Lemma 2.6. *Let d be a pseudometric in T. Assume that:*

(i) (T,d) is totally bounded.

(ii) For each $0 < M < \infty$,

$$\lim_{n\to\infty} \epsilon_n \log(\sum_{j=1}^{k_n} \Pr\{F_{n,j}(X_{n,j}) > M\}) = -\infty,$$

where $F_{n,j}(x) = \sup_{t\in T} |f_{n,j}(x,t)|$.

(iii) For each $0 < a, M, \lambda < \infty$,

$$\lim_{n\to\infty} \epsilon_n \log \left(E[\exp(\epsilon_n^{-1}\lambda \sum_{j=1}^{k_n} F_{n,j}(X_{n,j}) I_{M \geq F_{n,j}(X_{n,j}) > a\epsilon_n})] \right) = 0.$$

(iv) For some $a > 0$,

$$E[\sup_{t\in T} |\sum_{j=1}^{k_n} (f_{n,j}(X_{n,j}, t) I_{F_{n,j}(X_{n,j}) \leq a\epsilon_n} - E[f_{n,j}(X_{n,j}, t) I_{F_{n,j}(X_{n,j}) \leq a\epsilon_n}])|] \to 0.$$

(v) For some $a > 0$,

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \sup_{d(s,t)\leq\eta} \epsilon_n^{-1} \sum_{j=1}^{k_n} \mathrm{Var}((f_{n,j}(X_{n,j}, s) - f_{n,j}(X_{n,j}, t)) I_{F_{n,j}(X_{n,j}) \leq a\epsilon_n}) = 0.$$

(vi) For a fixed $0 < a < \infty$, for each $s, t \in T$,

$$\lim_{n\to\infty} \epsilon_n^{-1} \sum_{j=1}^{k_n} \mathrm{Cov}(f_{n,j}(X_{n,j}, s) I_{F_{n,j}(X_{n,j}) \leq a\epsilon_n}, f_{n,j}(X_{n,j}, t) I_{F_{n,j}(X_{n,j}) \leq a\epsilon_n})$$

exists.

Then, for each $a > 0$,

$$\{\sum_{j=1}^{k_n} (f_{n,j}(X_{n,j}, t) - E[f_{n,j}(X_{n,j}, t) I_{F_{n,j}(X_{n,j}) \leq a\epsilon_n}]) : t \in T\}$$

satisfies the LDP with speed ϵ_n and the rate function in (2.2) with respect to the covariance function

$$R(s,t) := \lim_{n\to\infty} \epsilon_n^{-1} \sum_{j=1}^{k_n} \mathrm{Cov}(f_{n,j}(X_{n,j}, s) I_{F_{n,j}(X_{n,j}) \leq a\epsilon_n}, f_{n,j}(X_{n,j}, t) I_{F_{n,j}(X_{n,j}) \leq a\epsilon_n}), s, t \in T.$$

Proof. First, we prove that condition (iii) implies that if (iv)–(vi) hold for some $a > 0$, then they hold for each $a > 0$. Note that condition (iii) implies that for each $0 < a, M, \lambda < \infty$,

$$\lim_{n\to\infty} \epsilon_n E[\exp(\epsilon_n^{-1}\lambda \sum_{j=1}^{k_n} F_{n,j}(X_{n,j}) I_{M\ge F_{n,j}(X_{n,j})>a\epsilon_n}) - 1] = 0.$$

This implies that for each $0 < a, M < \infty$,

$$\lim_{n\to\infty} E[\sum_{j=1}^{k_n} F_{n,j}(X_{n,j}) I_{M\ge F_{n,j}(X_{n,j})>a\epsilon_n})] = 0 \tag{2.16}$$

and

$$\lim_{n\to\infty} E[\sum_{j=1}^{k_n} F_{n,j}^2(X_{n,j}) I_{M\ge F_{n,j}(X_{n,j})>a\epsilon_n})] = 0.$$

Hence, we may assume that (iv)–(vi) hold for each $a > 0$.

Fixed $a > 0$, we use Theorem 2.1 to prove that $\{U_n(t) : t \in T\}$ satisfies the LDP, where

$$U_n(t) := \sum_{j=1}^{k_n} (f_{n,j}(X_{n,j}, t) - E[f_{n,j}(X_{n,j}, t) I_{F_{n,j}(X_{n,j})\le a\epsilon_n}]).$$

Condition (i) implies (a.1) in Theorem 2.1.

Let

$$V_n(t) := \sum_{j=1}^{k_n} f_{n,j}(X_{n,j}, t) I_{F_{n,j}(X_{n,j})>a\epsilon_n}.$$

First, we prove that $\{V_n(t) : t \in T\}$ is uniformly exponentially asymptotically negligible, i.e. for each $\tau > 0$,

$$\lim_{n\to\infty} \epsilon_n \log\left(\Pr\{\sup_{t\in T} |V_n(t)| \ge \tau\}\right) = -\infty. \tag{2.17}$$

By (ii) and (iii), we may take $\lambda, M > 0$ such that $\lambda\tau \ge 4c$,

$$\limsup_{n\to\infty} \epsilon_n \log(\sum_{j=1}^{k_n} \Pr\{F_{n,j}(X_{n,j}) > M\}) < -c$$

and

$$\limsup_{n\to\infty} \epsilon_n \log E[\exp(\epsilon_n^{-1}\lambda \sum_{j=1}^{k_n} F_{n,j}(X_{n,j}) I_{M\ge F_{n,j}(X_{n,j})>a\epsilon_n})] < 4^{-1}\lambda\tau.$$

Then, for n large enough,

$$\begin{aligned} &\epsilon_n \log(\Pr\{\sup_{t\in T} |\textstyle\sum_{j=1}^{k_n} f_{n,j}(X_{n,j}, t) I_{F_{n,j}(X_{n,j})>M}| \ge 2^{-1}\tau\}) \\ &\le \epsilon_n \log(\textstyle\sum_{j=1}^{k_n} \Pr\{F_{n,j}(X_{n,j}) > M\}) \le -c, \end{aligned}$$

and

$$\begin{aligned} & \Pr\{\sup_{t\in T}|\sum_{j=1}^{k_n} f_{n,j}(X_{n,j},t)I_{M>F_{n,j}(X_{n,j})\geq a\epsilon_n}|\geq 2^{-1}\tau\} \\ \leq\ & \Pr\{\sum_{j=1}^{k_n} F_{n,j}(X_{n,j})I_{M>F_{n,j}(X_{n,j})\geq a\epsilon_n}\geq 2^{-1}\tau\} \\ \leq\ & e^{-2^{-1}\epsilon_n^{-1}\lambda\tau}E[\exp(\epsilon_n^{-1}\lambda\sum_{j=1}^{k_n} F_{n,j}(X_{n,j})I_{M>F_{n,j}(X_{n,j})\geq a\epsilon_n})]\leq e^{-c\epsilon_n^{-1}}. \end{aligned}$$

Hence,

$$\limsup_{n\to\infty}\epsilon_n\log\left(\Pr\{\sup_{t\in T}|V_n(t)|\geq\tau\}\right)\leq -c$$

and (2.17) follows.

By (2.17), to get the LDP for the finite dimensional distributions, we need to prove that for each $t_1,\ldots,t_m\in T$, $(W_n(t_1),\ldots,W_n(t_m))$ satisfies the LDP, where

$$W_n(t)=\sum_{j=1}^{k_n}(f_{n,j}(X_{n,j},t)I_{F_{n,j}(X_{n,j})\leq a\epsilon_n}-E[f_{n,j}(X_{n,j},t)I_{F_{n,j}(X_{n,j})\leq a\epsilon_n}]).$$

We claim that Lemma 2.3 implies the LDP for the finite dimensional distributions of the process $\{W_n(t):\ t\in T\}$. Condition (i) in Lemma 2.3 follows from (vi). Condition (ii) in Lemma 2.3 is obviously satisfied. As to condition (iii) in Lemma 2.3, given $t\in T$ and $0<\delta<a$,

$$\begin{aligned} & \epsilon_n\sum_{j=1}^{k_n}\Pr\{|f_{n,j}(X_{n,j},t)I_{F_{n,j}(X_{n,j})\leq\epsilon_n a}-E[f_{n,j}(X_{n,j},t)I_{F_{n,j}(X_{n,j})\leq\epsilon_n a}]|\geq\epsilon_n\delta\} \\ & \leq\epsilon_n\sum_{j=1}^{k_n}\Pr\{|f_{n,j}(X_{n,j},t)I_{2^{-2}\epsilon_n\delta<F_{n,j}(X_{n,j})\leq\epsilon_n a} \\ & -E[f_{n,j}(X_{n,j},t)I_{2^{-2}\epsilon_n\delta<F_{n,j}(X_{n,j})\leq\epsilon_n a}]|\geq 2^{-1}\epsilon_n\delta\} \\ & \leq 4\delta\sum_{j=1}^{k_n}E[F_{n,j}(X_{n,j},t)I_{2^{-2}\epsilon_n\delta<F_{n,j}(X_{n,j})\leq\epsilon_n a}]\to 0, \end{aligned}$$

by (2.16).

By (2.17), to prove (a.3) in Theorem 2.1, it suffices to consider $\{Z_n(t):t\in T\}$, where

$$Z_n(t)=\sum_{j=1}^{k_n}(f_{n,j}(X_{n,j},t)I_{F_{n,j}(X_{n,j})<a\epsilon_n}-E[f_{n,j}(X_{n,j},t)I_{F_{n,j}(X_{n,j})<a\epsilon_n}]).$$

It is easy to see that by Theorem 1.4 in Talagrand [22], condition (v) imply that for each $\tau>0$,

$$\begin{aligned} & \lim_{\eta\to 0}\limsup_{n\to\infty}\epsilon_n\log(\Pr\{|\sup_{d(s,t)\leq\eta}|Z_n(s)-Z_n(t)| \\ & -E[\sup_{d(s,t)\leq\eta}|Z_n(s)-Z_n(t)|]|\geq\tau\})=-\infty. \end{aligned}$$

Since condition (iv) implies that $E[\sup_{t\in T}|Z_n(t)|]|\to 0$, (a.3) in Theorem 2.1 follows. $\square$

Next, we present necessary and sufficient conditions for moderate deviations of empirical processes. We will use the following set–up. Let $(S,\mathcal{S},\nu)$ be a probability space. Let $\Omega=S^{\mathbb{N}}$, $\mathcal{A}=\mathcal{S}^{\mathbb{N}}$, and $Q=\nu^{\mathbb{N}}$. Let X_n be the n–th projection from Ω into S. Then, $\{X_n\}_{n=1}^{\infty}$ is a sequence of i.i.d.r.v.'s with values in S. Let $\{f(\cdot,t):t\in T\}$ be an image admissible Suslin class of measurable functions from

S into $\mathbb{R}$. F will denote an envelope of $\{f(\cdot,t): t\in T\}$, i.e. a measurable function on S, such that $F(X)\ge \sup_{t\in T}|f(X,t)|$ a.s.

Theorem 2.7. *With the notation above, let $\{a_n\}_{n=1}^\infty$ and $\{c_n\}_{n=1}^\infty$ be two sequences of real numbers such that $a_n \nearrow \infty$ and $a_n^{-1}c_n^{-1}n\nearrow\infty$ and $\{n^{-1}c_n^2\}$ is nondecreasing. Then, the following sets of conditions ((a), (b) and (c)) are equivalent:*

(a.1) $\lim_{n\to\infty} a_n^{-2}\log(n\Pr\{F(X)\ge a_nc_n\})=-\infty$.

(a.2) $a_n^{-1}c_n^{-1}\sup_{t\in T}|\sum_{j=1}^n f(X_j,t)|\xrightarrow{\Pr}0$.

(a.3) For each $s,t\in T$, the following limit exists:

$$\lim_{n\to\infty} nc_n^{-2}\mathrm{Cov}(f(X,s)I_{F(X)<a_n^{-1}c_n}, f(X,t)I_{F(X)<a_n^{-1}c_n})$$

(a.4) (T,d) is totally bounded, where

$$d^2(s,t)=\lim_{n\to\infty} nc_n^{-2}\mathrm{Var}((f(X,s)-f(X,t))I_{F(X)<a_n^{-1}c_n}).$$

(a.5) $\lim_{\eta\to0}\lim_{n\to\infty}\sup_{d(s,t)\le\eta} nc_n^{-2}\mathrm{Var}((f(X,s)-f(X,t))I_{F(X)<a_n^{-1}c_n})=0$.

(b) $\{a_n^{-1}c_n^{-1}\sum_{j=1}^n f(X_j,t): t\in T\}$ satisfies the LDP in $l_\infty(T)$ with speed a_n^2 and a rate function I such that $\lim_{\lambda\to\infty}\lambda^{-1}\inf\{I(z):\sup_{t\in T}|z(t)|\ge\lambda\}=\infty$ and for each $\delta>0$, $\inf\{I(z):\sup_{t\in T}|z(t)|\ge\delta\}>0$.

(c) $\{c_n^{-1}a_n^{-1}\sum_{j=1}^{[nu]} f(X_j,t): 0\le u\le 1, t\in T\}$ satisfies the LDP in $l_\infty([0,1]\times T)$ with speed a_n^2 and a rate function I such that

$$\lim_{\lambda\to\infty}\lambda^{-1}\inf\{I(z):\sup_{t\in T}|z(1,t)|\ge\lambda\}=\infty$$

and for each $\delta>0$, $\inf\{I(z):\sup_{t\in T}|z(1,t)|>\delta\}>0$.

Moreover, in (b) the rate function is given by (2.2) with respect to the covariance function

$$R(s,t)=\lim_{n\to\infty} nc_n^{-2}E[f(X,s)f(X,t)I_{F(X)\le a_nc_n}].$$

In (c), the rate function in (c) is

$$\begin{aligned} I(\alpha)=\ &\inf\{2^{-1}\textstyle\int_0^1\int_\Omega \xi^2(v,\omega)\,d\nu(\omega)\,dv:\ \text{where } \xi:[0,1]\times\Omega\to\mathbb{R}\\ &\text{is a measurable function such that}\\ &\alpha(u,t)=\textstyle\int_0^u\int_\Omega \xi(v,\omega)Z(t,\omega)\,d\nu(\omega)\,dv,\ \text{for each } u\in[0,1], t\in T\},\end{aligned}$$

where $(\Omega,\mathcal{F},\nu)$ is probability space, $\{Z(t,\omega): t\in T\}$ is a Gaussian process defined on Ω with zero means and covariance given by

$$\int_\Omega Z(s,\omega)Z(t,\omega)\,d\nu(\omega)=\lim_{n\to\infty} nc_n^{-2}E[f(X,s)f(X,t)I_{F(X)\le a_nc_n}],$$

for each $s,t\in T$.

Proof. To prove that (a) implies (b) we apply Lemma 2.6. Conditions (i), (ii), (v) and (vi) in Lemma 2.6 are obviously satisfied. By the arguments used in (2.9), we have that for each $0<a,M,\lambda<\infty$,

$$na_n^{-2}E[(\exp(\lambda c_n^{-1}a_nF(X))-1)\,I_{aa_n^{-1}c_n<F(X)\le a_nc_nM}]\to 0. \tag{2.18}$$

This implies condition (iii) in Lemma 2.6. From (a.1) and (2.18), we have that for each $a > 0$,

$$\sup_{t\in T} a_n^{-1}c_n^{-1}\sum_{j=1}^{n} F(X_j)I_{F(X_j)>aa_n^{-1}c_n} \overset{\Pr}{\to} 0.$$

This limit and (a.2) imply that for each $a > 0$,

$$\sup_{t\in T} a_n^{-1}c_n^{-1}|\sum_{j=1}^{n} f(X_j)I_{F(X_j)\le a_n^{-1}c_n}| \overset{\Pr}{\to} 0.$$

So, by the Hoffmann–Jørgensen inequality (see for example Proposition 6.8 in Ledoux and Talagrand [16]), condition (iv) in Lemma 2.6 follows.

Next, we prove that (b) implies (a). From Lemma 2.1 in Arcones [3], for each $\lambda > 0$,

$$\limsup_{n\to\infty} a_n^{-2}\log(n\Pr\{F(X)\ge 2\lambda a_n c_n\}) \le -\inf\{I(z) : |z|_\infty \ge \lambda\}.$$

This estimation and the argument in (2.11) imply (a.1). Since for each $\delta > 0$, $\inf\{I(z) : \sup_{t\in T}|z(t)| \ge \delta\} > 0$, (a.2) holds. Since (a.1) holds, the stochastic process

$$\{a_n^{-1}c_n^{-1}\sum_{j=1}^{n}(f(X_j,t)I_{F(X_j)<a_n^{-1}c_n} - E[f(X_j,t)I_{F(X_j)<a_n^{-1}c_n}]) : t\in T\}$$

satisfies the LDP. This fact and Theorem 2.4 imply (a.3). By the remark after Theorem 2.2, (a.4) holds. (a.5) follows from the argument in (2.12).

Finally, we prove that (a) implies (c). We use Theorem 2.1. By (2.18), it suffices to prove that $\{U_n(u,t) : 0\le u\le 1, t\in T\}$ satisfies the LDP, where

$$U_n(u,t) := c_n^{-1}a_n^{-1}\sum_{j=1}^{[nu]}(f(X_j,t)I_{F(X_j)<a_n^{-1}c_n} - E[f(X_j,t)I_{F(X_j)<a_n^{-1}c_n}]).$$

The LDP for the finite dimensional distributions follows from Lemma 2.3 applied to the independent random vectors

$$\begin{aligned}X_{n,j} &= (Y_j^{(1,1)},Y_j^{(1,2)},\dots,Y_j^{(1,m)},\dots,Y_j^{(m,1)},Y_j^{(m,2)},\dots,Y_j^{(m,m)})\\ &\quad -E[(Y_j^{(1,1)},Y_j^{(1,2)},\dots,Y_j^{(1,m)},\dots,Y_j^{(m,1)},Y_j^{(m,2)},\dots,Y_j^{(m,m)})], 1\le j\le n\end{aligned}$$

where

$$Y_j^{(k,l)} = c_n^{-1}f(X_j,t_l)I(F(X_j,t_l) < a_n^{-1}c_n, j\le [nu_k]),$$

$t_1,\dots,t_m \in T$ and $u_1,\dots,u_m \in [0,1]$. Observe that the (k,l) coordinate of $\sum_{j=1}^{n} X_{n,j}$ is

$$c_n^{-1}\sum_{j=1}^{[nu_k]}\left(f(X_j,t_l)I_{F(X_j)<a_n^{-1}c_n} - E[f(X_j,t_l)I_{F(X_j)<a_n^{-1}c_n}]\right).$$

We have that the covariance of

$$(c_n^{-1}\sum_{j=1}^{[nu_1]} f(X_j,t_1)I_{F(X_j)<a_n^{-1}c_n}, c_n^{-1}\sum_{j=1}^{[nu_2]} f(X_j,t_2)I_{F(X_j)<a_n^{-1}c_n})$$

converges to

$$\min(u_1,u_2)\lim_{n\to\infty} nc_n^{-2}\mathrm{Cov}(f(X,t_1)I_{F(X)<a_n^{-1}c_n}, f(X,t_2)I_{F(X)<a_n^{-1}c_n}).$$

Next, we prove the asymptotic exponential equicontinuity. Given $\tau, M>0$, there exists a function $\pi_2 : T\to T$ with finite range such that

$$\text{(2.19)} \qquad \limsup_{n\to\infty} a_n^{-2}\log(\Pr\{\sup_{t\in T}|V_n(1,t)-V_n(1,\pi_2(t))|\ge 2^{-3}\tau\})\le -M.$$

Let $\{X_j'\}$ be a independent copy of $\{X_j\}$. Let

$$V_n'(u,t) := c_n^{-1}a_n^{-1}\sum_{j=1}^{[nu]}(f(X_j',t)I_{F(X_j')<a_n^{-1}c_n} - E[f(X_j',t)I_{F(X_j')<a_n^{-1}c_n}]).$$

By symmetrization (see Lemma 1.2.1 in Giné and Zinn [13]) and the Lévy 's inequality, we have that

$$\begin{aligned}
&\text{(2.20)} \qquad \Pr\{\sup_{t\in T,\ 0\le u\le 1}|V_n(u,t)-V_n(u,\pi_2(t))|\ge 2^{-1}\tau\}\\
&\le 2\Pr\{\sup_{t\in T,\ 0\le u\le 1}|V_n(u,t)-V_n(u,\pi_2(t))-(V_n'(u,t)-V_n'(u,\pi_2(t)))|\ge 2^{-2}\tau\}\\
&\le 4\Pr\{\sup_{t\in T}|V_n(1,t)-V_n(1,\pi_2(t))-(V_n'(1,t)-V_n'(1,\pi_2(t)))|\ge 2^{-2}\tau\}\\
&\le 8\Pr\{\sup_{t\in T}|V_n(1,t)-V_n(1,\pi_2(t))|\ge 2^{-3}\tau\}.
\end{aligned}$$

Let m be a positive integer such that $M\le 2^{-4}\tau\mathrm{arcsinh}(2^{-3}(\sup_{t\in\pi_2(T)}R(t,t))^{-2}m\tau)$. Let $\pi_2(u)=\frac{[mu]}{m}$, $0\le u\le 1$. Using the the Prokhorov's inequality as in the proof of the Theorem 2.4, we have that

$$\begin{aligned}
\text{(2.21)}\ &\limsup_{n\to\infty} a_n^{-2}\log(\Pr\{\sup_{t\in\pi_2(T),\ 0\le u\le 1}|V_n(u,t)-V_n(\pi_1(u),t)|\ge 2^{-1}\tau\})\\
&\le -2^{-4}\tau\mathrm{arcsinh}\left(2^{-3}(\sup_{t\in\pi_2(T)}R(t,t))^{-2}m\tau\right)\le -M.
\end{aligned}$$

From (2.19)–(2.21), we get that

$$\limsup_{n\to\infty} a_n^{-2}\log(\Pr\{\sup_{t\in T,\ 0\le u\le 1}|V_n(u,t)-V_n(\pi_1(u),\pi_2(t))|\ge\tau\})\le -M.$$

□

When $c_n = n^{1/2}$, the previous theorem gives the following:

Theorem 2.8. *Let $\{a_n\}$ be a sequence of real numbers such that $a_n \nearrow \infty$ and $n^{-1/2}a_n \searrow 0$.*

Then, the following sets of conditions are equivalent:

(a.1)

$$\lim_{n\to\infty} a_n^{-2}\log(n\Pr\{\sup_{t\in T}|f(X,t)-E[f(X,t)]|\geq n^{1/2}a_n\})=-\infty.$$

(a.2) $\sup_{t\in T}|a_n^{-1}n^{-1/2}\sum_{j=1}^n(f(X_j,t)-E[f(X_j,t)])|\xrightarrow{\Pr}0.$
(a.3) For each $t\in T$, $E[f^2(X,t)]<\infty$.
(a.4) (T,d) *is totally bounded, where*

$$d^2(s,t)=\mathrm{Var}(f(X,s)-f(X,t)).$$

(b) $\{a_n^{-1}n^{-1/2}\sum_{j=1}^n(f(X_j,t)-E[f(X_j,t)]) : t\in T\}$ *satisfies the LDP in* $l_\infty(T)$ *with speed* a_n^2 *and a rate function* I *such that*

$$\lim_{\lambda\to\infty}\lambda^{-1}\inf\{I(z):\sup_{t\in T}|z(t)|\geq\lambda\}=\infty$$

and for each $\delta>0$, $\inf\{I(z):\sup_{t\in T}|z(t)|\geq\delta\}>0.$

(c) $\{a_n^{-1}n^{-1/2}\sum_{j=1}^{[nu]}(f(X_j,t)-E[f(X_j,t)]) : 0\leq u\leq 1, t\in T\}$ *satisfies the LDP in* $l_\infty([0,1]\times T)$ *with speed* a_n^2 *and a rate function* I *such that* $\lim_{\lambda\to\infty}\lambda^{-1}\inf\{I(z):\sup_{t\in T}|z(1,t)|\geq\lambda\}=\infty$ *and for each* $\delta>0$, $\inf\{I(z):\sup_{t\in T}|z(1,t)|>\delta\}>0.$

Moreover, the rate function in (b) is

$$I(z)=\inf\{2^{-1}E[\alpha^2(X)]:z(t)=E[\alpha(X)(f(X,t)-E[f(X,t)])]\\ \text{for each } t\in T\}.$$

Moreover, the rate function in (c) is

$$\begin{aligned}I(z)= \ &\inf\{2^{-1}\textstyle\int_0^1 E[\gamma^2(u,X)]\,du:\ \text{where }\gamma:[0,1]\times S\to\mathbb{R}\\ &\text{is a measurable function such that for each } u\in[0,1], t\in T,\\ &z(u,t)=\textstyle\int_0^u E[\gamma(v,X)(f(X,t)-E[f(X,t)])]\,dv\}.\end{aligned}$$

Ledoux [15] and Wu [24] proved the part (a) implies (b) in the previous theorem assuming a little different conditions. For example, they assume that for some $c>1$ and some $|\delta|<1/2$, $a_{nk}\leq ck^\delta a_n$, for each $n,k\geq 1$. These conditions do not apply to sequences that very close to n. For example, it does not apply to sequences of the form $a_n=n(\log n)^{-\alpha}$, where $\alpha>0$, for example. Condition (a.4) above is stated differently in Ledoux [15] and Wu [24].

For classical empirical processes Theorem 2.8 gives the following:

Corollary 2.9. *Let* $\{X_j\}_{j=1}^\infty$ *be a sequence of i.i.d.r.v.'s. Let* $\{a_n\}$ *be a sequence of real numbers such that* $a_n\to\infty$ *and* $n^{-1/2}a_n\to 0$. *Then,*

(i) $\{a_n^{-1}n^{-1/2}\sum_{j=1}^n(I(X_j\leq t)-P\{X_j\leq t\}) : t\in\mathbb{R}\}$ *satisfies the LDP in* $l_\infty(\mathbb{R})$ *with speed* a_n^2 *and rate function*

$$I(z)=\inf\{2^{-1}E[\alpha^2(X)]:z(t)=E[\alpha(X)(I(X\leq t)-P(X\leq t))]\text{ for each } t\in T\}.$$

(ii) $\{a_n^{-1}n^{-1/2}\sum_{j=1}^{[nu]}(I(X_j \le t) - P\{X_j \le t\}) : t \in \mathbb{R}, 0 \le u \le 1\}$ *satisfies the LDP in* $l_\infty([0,1]\times\mathbb{R})$ *with speed* a_n^2 *and rate function*

$$\begin{aligned} I(z) = \ & \inf\{2^{-1}\int_0^1 E[\gamma^2(u,X)]\,du : \text{ where } \gamma : [0,1]\times\mathbb{R} \text{ is a measurable function} \\ & \text{such that for each } (u,t) \in [0,1]\times\mathbb{R}, \\ & z(u,t) = \int_0^u E[\gamma(v,X)(I(X \le t) - P(X \le t))]\,dv\}. \end{aligned}$$

In the previous corollary, if X has a positive density f_X, then the rate function in part (i) can be written as

$$I(z) = \begin{cases} \int_{-\infty}^{\infty} \frac{(z'(t))^2}{2f_X(t)}\,dt & \text{if } z \text{ is absolutely continuous and } \lim_{|t|\to\infty} z(t) = 0. \\ \infty & \text{else .} \end{cases}$$

By taking T equal to the unit ball of a Banach space, the previous theorems give necessary and sufficient conditions for the moderate deviations of Banach space valued random vectors. In particular, we have:

Theorem 2.10. *Let* $\{X_j\}_{j=1}^\infty$ *be a sequence of i.i.d.r.v.'s with values in a separable Banach space* B. *Let* $\{a_n\}_{n=1}^\infty$ *and* $\{c_n\}_{n=1}^\infty$ *be two sequences of real numbers such that* $a_n \nearrow \infty$ *and* $a_n^{-1}c_n^{-1}n \nearrow \infty$ *and* $\{n^{-1}c_n^2\}$ *is nondecreasing. Then, the following sets of conditions ((a), (b) and (c)) are equivalent:*

(a.1) $\lim_{n\to\infty} a_n^{-2}\log(n\Pr\{|X| \ge c_n a_n\}) = -\infty$.

(a.2) $a_n^{-1}c_n^{-1}\sum_{j=1}^n X_j \xrightarrow{\Pr} 0$.

(a.3) For each $f_1, f_2 \in B^*$, *the following limit exists:*

$$\lim_{n\to\infty} nc_n^{-2}E[f_1(X)f_2(X)I_{|X|\le a_n^{-1}c_n}],$$

where B^* *is the dual of* B.

(a.4) (B_1^*, d) *is totally bounded, where*

$$d^2(f_1,f_2) = \lim_{n\to\infty} nc_n^{-2}\mathrm{Var}((f_1(X) - f_2(X))I_{|X|\le a_n^{-1}c_n})$$

and B_1^* *is the unit ball of the dual of* B.

(a.5)

$$\lim_{\eta\to 0}\lim_{n\to\infty}\sup_{f_1,f_2\in B_1^*, d(f_1,f_2)\le\eta} nc_n^{-2}\mathrm{Var}((f_1(X) - f_2(X))I_{|X|\le a_n^{-1}c_n}) = 0.$$

(b) $\{c_n^{-1}a_n^{-1}\sum_{j=1}^n X_j\}$ *satisfies the LDP in* $l_\infty(B)$ *with speed* a_n^2 *and a good rate function such that* $\lim_{\lambda\to\infty}\lambda^{-1}\inf\{I(z) : |z| \ge \lambda\} = \infty$ *and for each* $\delta > 0$, $\inf\{I(z) : |z| \ge \delta\} > 0$.

(c) $\{c_n^{-1}a_n^{-1}\sum_{j=1}^{[nu]} X_j : 0 \le u \le 1\}$ *satisfies the LDP in* $l_\infty([0,1], B)$ *with speed* a_n^2 *and a good rate function such that* $\lim_{\lambda\to\infty}\lambda^{-1}\inf\{I(z) : |z(1)| \ge \lambda\} = \infty$ *and for each* $\delta > 0$, $\inf\{I(z) : |z(1)| \ge \delta\} > 0$.

When $c_n = n^{1/2}$, the conditions (a) in the previous theorem simplify considerably:

Theorem 2.11. *Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of i.i.d.r.v.'s with values in a separable Banach space B. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $a_n \nearrow \infty$ and $n^{-1/2}a_n \searrow 0$. Then, the following sets of conditions ((a), (b) and (c)) are equivalent:*

(a.1) $\lim_{n\to\infty} a_n^{-2} \log(n \Pr\{|X| \ge a_n n^{1/2}\}) = -\infty$.

(a.2) $a_n^{-1} n^{-1/2} \sum_{j=1}^{n} (X_j - E[X_j]) \xrightarrow{\Pr} 0$.

(a.3) $\{f(X) - E[f(X)] : f \in B_1^*\}$ *is a totally bounded set of* L_2.

(b) $\{a_n^{-1} n^{-1/2} \sum_{j=1}^{n} (X_j - E[X_j])\}$ *satisfies the LDP in* $l_\infty(B)$ *with speed* a_n^2 *and a good rate function such that* $\lim_{t\to\infty} \lambda^{-1} \inf\{I(z) : |z| \ge \lambda\} = \infty$ *and for each* $\delta > 0$, $\inf\{I(z) : |z| \ge \delta\} > 0$.

(c) $\{a_n^{-1} n^{-1/2} \sum_{j=1}^{[nu]} (X_j - E[X_j]) : 0 \le u \le 1\}$ *satisfies the LDP in* $l_\infty([0,1], B)$ *with speed* a_n^2 *and a good rate function such that* $\lim_{\lambda\to\infty} \lambda^{-1} \inf\{I(z) : |z(1)| \ge \lambda\} = \infty$ *and for each* $\delta > 0$, $\inf\{I(z) : |z(1)| \ge \delta\} > 0$.

Hu and Lee [14] obtained the LDP for the stochastic processes in (c) in the previous theorem assuming that for some $\lambda > 0$, $E[e^{\lambda |X|}] < \infty$.

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