Linear Algebra- Practice Second Exam

1. Consider the following matrix operations. If the operation is valid, find the result. If the operation is not valid, explain why.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 3 & 1 \\ 2 & 5 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 \\ -2 & 1 \\ 1 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & 3 \\ 3 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 2 \\ -4 & 1 \end{pmatrix}$$

   (a) $A + B$
   (b) $B + A$
   (c) $C + D$
   (d) $D + C$
   (e) AB
   (f) BA
   (g) CD
   (h) DC

2. Determine if the following sets of vectors form a basis of $\mathcal{P}_2$. Explain why or why not.

   (a) $\{1 + x, 2x - x^2\}$
   (b) $\{2 - x + x^2, -1 + 5x - 3x^2, 1 + 4x - 2x^2\}$
   (c) $\{1 + x, 1 - x, x^2\}$

3. Consider the following matrix A and its equivalent reduced echelon form E:

$$A = \begin{pmatrix} 1 & 3 & 4 & 1 & -2 \\ 2 & 2 & 0 & 1 & -1 \\ -1 & 1 & 4 & 1 & -4 \\ 3 & -1 & -8 & 1 & 0 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   (a) Give a basis for the column space of A.
   (b) Give a basis for the row space of A.
   (c) Give a basis for the null space of A.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $T(\vec{x}) = A\vec{x}$ for some matrix A.

   (a) What is the size of the matrix A?
   (b) State the Rank/Nullity Theorem.
   (c) If T is onto, what is the rank of A?
   (d) If T is one-to-one, what is the rank of A?

5. Determine if each of the following is a subspace of $\mathbb{R}^3$. If it is a subspace, explain why. If not, explain why not.

   (a) $\left\{ \begin{pmatrix} x \\ 1 \\ 0 \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}$

   (b) $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x + y + z = 0 \right\}$

6. True or False:

   (a) If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a linearly dependent set of vectors, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set of vectors.

   (b) If V is a vector space with dimension n, then every basis of V contains n vectors.

   (c) If $\vec{v}_1$ is not a linear combination of $\vec{v}_2$ and $\vec{v}_3$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set of vectors.

   (d) If $T : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one, then the dimension of range of T is 3.

7. The set S of vector is defined below, with vectors $\vec{s}_1$ through $\vec{s}_4$.

$$S = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \{\vec{s}_1, \vec{s}_2, \vec{s}_3, \vec{s}_4\}$$

   (a) Show that that S is linearly dependent by finding a linear combination of $\{\vec{s}_1, \vec{s}_2, \vec{s}_3, \vec{s}_4\}$ that is equal to the zero vector.

   (b) Find a linearly independent subset of S.