MATH 304 MIDTERM EXAM I SAMPLE PROBLEMS

Level: EASY

1. What is the graph of a (nontrivial) linear equation in two variables? Describe the geometric (graphical) meaning of the solution set of a linear system of two equations in two variables.

2. Describe the relationship between the consistency of a linear system, its number of free variables and its number of solutions.

3. Below is the Echelon Form of a linear system of 4 equations in 4 variables. How many leading variables and how many free variables are there in the system? Explain how you determined your answer.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

4. The first three columns describe a function. Check all the boxes that corespond to a true statement; justify your answers.

$f(x)$	Domain	Codomain	One-to-one	Onto	Bijective
x^2	$[0,\infty)$	$[0,\infty)$			
x^2	$(-1,1)$	$(-1,1)$			
$2x-1$	$\mathbb{R}$	$\mathbb{R}$			
$2x-1$	$[0,1]$	$[-1,1]$			
x^3	$\mathbb{N}$	$\mathbb{Z}$			
$2x^2$	$\mathbb{N}$	$\mathbb{N}$			

5. Suppose we are given that every student in this section is currently enrolled in at least 3 and at most 4 courses. A, B, and C are three students in this section. Additionally, we know:
(i) A and B are both taking a computer science course together;

(ii) A is taking one course more than B and C each;

(iii) Two people from $\{A, B, C\}$ are taking a course in macroeconomics together;

(iv) At least one of A, B, C is taking a course in psychology.

Based on this information, can you deduce who is taking which courses? If yes, list the courses for each student. If no, explain why.

6. Consider the following statement and a sketch of its proof. Is this proof mathematically correct? Justify your answer.

Stetement: "$\forall n \in \mathbb{N}$, $1 + 2 + \ldots + n = \frac{n(n+1)}{2} + 10$"

"Sketch of the proof":
Assume that this formula is true for some $n \in \mathbb{N}$. We will show that it is true for $n + 1$ as follows:

$$\begin{aligned} 1 + 2 + \ldots + n + (n+1) &= (1 + 2 + \ldots + n) + (n+1) \\ &= \frac{n(n+1)}{2} + 10 + (n+1) \\ &= (n+1)\left(\frac{n}{2} + 1\right) + 10 \\ &= \frac{(n+1)(n+2)}{2} + 10 \\ &= \frac{(n+1)((n+1)+1)}{2} + 10 \end{aligned}$$

By the Principle of Mathematical Induction, this proves the statement for all natural numbers.

7. a) Create A, an augmented matrix for a homogeneous system of two equations in three unknowns. The choice for entries a_{ij} is up to you.
b) Describe the augmented column (the right-hand-side column) of RREF(A) and explain your reasoning.

8. Let A be a 2×5 augmented matrix with two pivots.
a) If A is the matrix of a homogeneous system, is it possible for the system to have the trivial solution? why?
b) If A is the matrix of a homogeneous system, is it possible for the system to have a non-trivial solution? why?
c) If A is the matrix of a homogeneous system, is it possible for the system to have no solutions at all? i.e. can the system be inconsistent? why?

9. Determine if the following homogeneous systems have non-trivial solutions.

a) $\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$ b) $\begin{cases} x_1 + x_2 = 0 \\ x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$

10. Which of the following matrices are in Reduced Row Echelon Form? Why or why not?

$$\begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

11. Solve $A\mathbf{x} = 0$ for $A = \begin{pmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & 6 \end{pmatrix}$.

12. Consider the system

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -3 & -3 \end{array}\right)$$

Find two different vectors that are the solution to this system. Are there only two solutions?

13. Find the points of intersection of the coordinate axes and the plane $2x + 3y + 4z = 10$.

14. Given points $A = (1, 1, 2, 3)$ and $B = (3, 0, 2, 5)$, find the vector $\vec{AB}$ and its length.

15. Describe how to add vectors in $\mathbb{R}^n$ algebraically and geometrically.

16. Find the angle between two vectors $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -2 \\ 7 \\ 5 \end{bmatrix}$ (Simplify by evaluating the arccos function, if possible.)

17. Is it true that every matrix has exactly one Echelon form? Why or why not?

Level: MEDIUM

18. Let P_1 be the plane through the origin, orthogonal to $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and P_2 be the plane orthogonal to $\begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$. What is the line of intersection between P_1 and P_2?

19. For what values of h is the system $\begin{cases} x_1 + x_2 &= 0 \\ x_2 + x_3 &= h \\ x_1 - x_3 &= 1 \end{cases}$ consistent?

20. Suppose that A is a 3×3 matrix, $\mathbf{b}$ is a vector, and the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. What is the maximum number of pivots A can have? Why?

21. For a given vector $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, determine the set of all vectors w such that v and w satisfy the Cauchy-Schwartz Inequality with equality (without a proof).

22. Let $f : X \to Y$ and $g : Y \to Z$ be functions.
a) Show that if both f and g are one-to-one, then $g \circ f$ is also one-to-one.

b) Show that if both f and g are onto, then $g \circ f$ is also onto.
c) Show that if both f and g are bijective, then $g \circ f$ is also bijective.

23. Find all values of the parameter a such that the homogeneous system with the following coefficient matrix has a non-trivial solution?

$$\begin{bmatrix} 1 & 2 & 3 \\ a & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

24. Suppose $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ are both solutions of some system of equations. Find the following:

a) a non-trivial solution of the corresponding homogeneous system;
b) two more solutions of that homogeneous system
c) two more solutions of the origninal system.

25. Suppose some system of linear equations has exactly one solution. How many solutions does the corresponding homogeneous system have? Can you describe these solutions?

26. Find the Row Reduced Echelon Form of the following augmented matrix. Then describe the solution set in parametric vector form.

$$\left[\begin{array}{cccc|c} -2 & 0 & 0 & 2 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 2 & 0 & 0 & -2 & 0 \end{array}\right].$$

27. Reduce the system to Echelon Form to determine if it is consistent. You do not need to find the solutions.

$$\begin{cases} x + y + z &= 0 \\ x - y + z &= 2 \\ x - y - z &= 4 \end{cases}$$

28. Write down all possible Reduced Row Echelon Forms of the augmented matrix of a homogeneous linear system of 2 equations in 3 variables. Use stars to represent entries that are arbitrary real numbers

29. Consider the line L in $\mathbb{R}^3$ that passes through the points $(1, 0, -1)$ and $(-1, 0, 1)$.
a) Write parametric equations of this line.
b) Describe the intersection of this line with the $xy-$plane and the $xz-$plane.
c) Which of these points are on the line L?
$P = (10, 0, -10),\ Q = (10, -10, 0),\ R = (-10, 0, 10)$

30. Consider the planes described below:

$\pi_1 :\ x + y + z = 0$
$\pi_2 :\ 2x + 2y + 2z = 5$
$\pi_3 :\ -x + y + z = 0$

Describe $\pi_1 \cap \pi_2,\ \pi_1 \cap \pi_3,\ \pi_1 \setminus \pi_3$ (set difference) and $\pi_1 \cup \pi_2$.

31. Write down an equation of the plane through $(1, 1, 1),\ (1, 2, -1)$ and $(0, 1, -1)$.
Is the point $(0, 2, 1)$ on this plane?

32. Prove the symmetry property for the dot product of vectors $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^3$. In other words, explain why it is always true that $\mathbf{u}{\cdot}\mathbf{v} = \mathbf{v}{\cdot}\mathbf{u},\ \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

33. Let $\mathbf{u} = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}$.

a) Find the distance between $\mathbf{u}$ and $\mathbf{v}$ (when the vectors are identified with the points).
b) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

34. Let $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

a) Which pairs of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are orthogonal?
b) Are any of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ unit vectors?
c) Normalize $\mathbf{v}$ (find the vector of length 1 in the direction of $\mathbf{v}$).

Level: HARD

35. Choose h and k such that the system

$$\begin{cases} x_1 & +hx_2 & = 2 \\ 5x_1 & +10x_2 & = k \end{cases}$$

has
a) no solution
b) a unique solution
c) many solutions

36. Let A be the augmented matrix for a system for which the number of variables equals the number of equations.
a) Does A have to be reduced to RREF in order to determine if the system is inconsistent?
b) If A has 4 pivots, what conditions would need to be satisfied in order to conclude that the system has exactly one non-trivial solution?

37. Suppose f and g are two finctions and $f \circ g$ is their composition. Suppose $f \circ g$ is injective.
a) Can we conclude that f is injective? Explain or construct a counter-example.
b) Can we conclude that g is injective? Explain or construct a counter-example.

38. Suppose f and g are two finctions and $f \circ g$ is their composition. Suppose $f \circ g$ is surjective.
a) Can we conclude that f is surjective? Explain or construct a counter-example.

b) Can we conclude that g is surjective? Explain or construct a counter-example.

39. Suppose $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ are both solutions of some system of equations. Show that this system is homogeneous.

40. Let L be a linear system and let vectors $\vec{w_1}$ and $\vec{w_2}$ be solutions of L. Suppose that the corresponding homogeneous system of L has the solution set

$$\{t_1\vec{v_1} + \cdots + t_k\vec{v_k} : t_1, \ldots, t_k \in \mathbb{R}\}.$$

Show that

$$\{\vec{w_1}+t_1\vec{v_1}+\cdots+t_k\vec{v_k} : t_1, \ldots, t_k \in \mathbb{R}\} = \{\vec{w_2}+t_1\vec{v_1}+\cdots+t_k\vec{v_k} : t_1, \ldots, t_k \in \mathbb{R}\}.$$

41. Let L be a homogeneous linear system. Using an echelon form of L, show that if there are strictly more variables than equations, then there is a non-zero (or non-trivial) solution of L.

42. a) Suppose that there are two distinct solutions $\vec{v}$ and $\vec{w}$ of a linear system L. Show that the set

$$\{t\vec{v} + (1-t)\vec{w} : t \in \mathbb{R}\}$$

is contained in the solution set of L.

b) Using (a), prove that for a linear system L, the size of the solution set of L is either 0, 1, or infinite.

43. Using the Triangle Inequality and mathematical induction, prove

$$||\vec{u_1} + \vec{u_2} + \cdots + \vec{u_n}|| \leq ||\vec{u_1}|| + ||\vec{u_2}|| + \cdots + ||\vec{u_n}||$$

for every n given vectors $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}$ and every positive integer n.

44. Let a function $f : X \to Y$ be given. Also, recall that $f(X')$ is the set

$$\{f(x) : x \in X'\}$$

where X' is a subset of X.

a) Let $X_1, X_2 \subseteq X$. Show that $f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$.

b) Using mathematical induction and (a), show that

$$f(\bigcup_{i=1}^{n} X_i) = \bigcup_{i=1}^{n} f(X_i)$$

for all given n subsets $X_1, \ldots, X_n$ and for every positive integer n.

45. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. We consider a 2×2 array as in Table 1, where each entry is either 0 or 1. The proposition $p(x_i, y_j)$ for $i \in \{1, 2\}$ and $j \in \{1, 2\}$ means that the corresponding entry - which is in ith row and jth column - is 1.

For each quantified statement below, list all possible arrays that the given statement is true.
a) $\forall x_i \in X \, \exists y_j \in Y \, p(x_i, y_j)$.
b) $\exists y_j \in Y \, \forall x_i \in X \, p(x_i, y_j)$.

	y_1	y_2
x_1	0	0
x_2	1	0

46. For which values of the parameter k does the matrix A have the given RREF?

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 2 & -4 & 6 \\ -1 & k & -3 \end{bmatrix} \text{ and RREF: } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

47. Consider the linear system below; $a_i, \; b_i, \; c_i \in \mathbb{R}$ are fixed:

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 &= c_1 \\ b_1x_1 + b_2x_2 + b_3x_3 &= c_2 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{cases}$$

List all possible RREF for this system, assuming that the system is consistent.

48. Consider the linear system

$$\begin{cases} 3x+5y-4z &= 0 \\ -3x-2y+4z &= 0 \\ 6x+y-8z &= 0 \end{cases}$$

a) Find the RREF of the augmented matrix of the given system.
b) Identify the basic and the free variables.
c) Describe the solutions sets of the following systems:

$$\text{(i)} \begin{cases} 3x+5y-4z &= 1 \\ -3x-2y+4z &= 0 \\ 6x+y-8z &= 0 \end{cases} \qquad \text{(ii)} \begin{cases} 3x+5y-4z &= 5 \\ -3x-2y+4z &= -2 \\ 6x+y-8z &= 1 \end{cases}$$

49. Suppose the solution set to the system $A\mathbf{x} = \mathbf{0}$ is a line. Can we say anything about the shape of the solution set to $A\mathbf{x} = \mathbf{b}$ for some vector $\mathbf{b} \neq \mathbf{0}$? Can it be a point, a line, a plane, or some other shape?
Construct an example or explain why it is impossible.

50. Can the solution to a system with 2 equations in 3 unknowns ever be a point? A line? A plane?
Construct an example or explain why it is impossible.

51. Let U and V be vectors in $\mathbb{R}^n$. Use the properties of the dot product and other results to show that

$$||U+V||^2 + ||U-V||^2 = 2||U||^2 + 2||V||^2.$$

52. Describe each side of the unit square in $\mathbb{R}^2$ as a set consisting of a linear combination of vectors with parameters. Find the angle between the diagonal of the unit square and one of the axes. Do the same for the unit cube in $\mathbb{R}^3$. Generalize to any dimension n. Find the angle between the diagonal of the unit cube in $\mathbb{R}^n$ and one of the axes.