MATH 304 MIDTERM EXAM I SAMPLE - SOLUTIONS

Level: EASY

1. What is the graph of a (nontrivial) linear equation in two variables? Describe the geometric (graphical) meaning of the solution set of a linear system of two equations in two variables.

Solution

The graph is a line. The solution set of a linear system of two non-trivial equations is the intersection of two lines. It can be a line (if the lines are the same), a point, or an empty set (if the lines are parallel).

2. Describe the relationship between the consistency of a linear system, its number of free variables and its number of solutions.

Solution

"The system is consistent" and ''The system has one or more free variables" are logically independent statements: each one of them can be true or false regardless of the other. If both are true, then the system has infinitely many solutions. If the system is consistent but has no free variables, then it has exactly one solution. If the system is inconsistent, then it has no solutions (by definition).

3. Below is the Echelon Form of a linear system of 4 equations in 4 variables. How many leading variables and how many free variables are there in the system? Explain how you determined your answer.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Solution: This system has 2 pivot entries, **bold-faced** in the matrix below,

$$\left[\begin{array}{cccc|c} \mathbf{1} & 0 & 0 & -1 & 0 \\ 0 & \mathbf{1} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

so the system has 2 leading variables. The total number of variables is 4, so the system has $4 - 2 = 2$ free variables.

4. The first three columns describe a function. Check all the boxes that corespond to a true statement; justify your answers.

$f(x)$	Domain	Codomain	One-to-one	Onto	Bijective
x^2	$[0, \infty)$	$[0, \infty)$			
x^2	$(-1, 1)$	$(-1, 1)$			
$2x - 1$	$\mathbb{R}$	$\mathbb{R}$			
$2x - 1$	$[0, 1]$	$[-1, 1]$			
x^3	$\mathbb{N}$	$\mathbb{Z}$			
$2x^2$	$\mathbb{N}$	$\mathbb{N}$			

Answers

$f(x)$	Domain	Codomain	One-to-one	Onto	Bijective
x^2	$[0, \infty)$	$[0, \infty)$	yes	yes	yes
x^2	$(-1, 1)$	$(-1, 1)$	no	no	no
$2x - 1$	$\mathbb{R}$	$\mathbb{R}$	yes	yes	yes
$2x - 1$	$[0, 1]$	$[-1, 1]$	yes	yes	yes
x^3	$\mathbb{N}$	$\mathbb{Z}$	yes	no	no
$2x^2$	$\mathbb{N}$	$\mathbb{N}$	yes	no	no

5. Suppose we are given that every student in this section is currently enrolled in at least 3 and at most 4 courses. A, B, and C are three students in this section. Additionally, we know:

(i) A and B are both taking a computer science course together;

(ii) A is taking one course more than B and C each;

(iii) Two people from $\{A, B, C\}$ are taking a course in macroeconomics together;

(iv) At least one of A, B, C is taking a course in psychology.

Based on this information, can you deduce who is taking which courses? If yes, list the courses for each student. If no, explain why.

Solution

There is not enough information to conclude who is taking what courses. From (ii) we can conclude that A is taking 4 courses, while B and C are taking 3 courses, but nothing else can be deduced.

6. Consider the following statement and a sketch of its proof. Is this proof mathematically correct? Justify your answer.

Stetement: "$\forall n \in \mathbb{N}$, $1 + 2 + \ldots + n = \frac{n(n+1)}{2} + 10$"

"Sketch of the proof":
Assume that this formula is true for some $n \in \mathbb{N}$. We will show that it is true for $n + 1$ as follows:

$$\begin{aligned} 1 + 2 + \ldots + n + (n+1) &= (1 + 2 + \ldots + n) + (n+1) \\ &= \frac{n(n+1)}{2} + 10 + (n+1) \\ &= (n+1)\left(\frac{n}{2} + 1\right) + 10 \\ &= \frac{(n+1)(n+2)}{2} + 10 \\ &= \frac{(n+1)((n+1)+1)}{2} + 10 \end{aligned}$$

By the Principle of Mathematical Induction, this proves the statement for all natural numbers.

Solution: Note: Saying that the statement is false is not enough to justify the answer "No".

The proof includes the induction step, however, the base case ($n = 1$) must be verified for it to work.

$P(1): \quad 1 = \frac{1 \cdot 2}{2} + 10$ is not true.

7. a) Create A, an augmented matrix for a homogeneous system of two equations in three unknowns. The choice for entries a_{ij} is up to you.

Solution: $\left[\begin{array}{ccc|c} \star & \star & \star & 0 \\ \star & \star & \star & 0 \end{array}\right]$

b) Describe the augmented column (the right-hand-side column) of RREF(A) and explain your reasoning.

Solution:
Since each entry of the augmented column is 0, the augmented column will not change when we apply elementary row operations and will stay all-zero in the RREF(A) .

8. Let A be a 2×5 augmented matrix with two pivots.

a) If A is the matrix of a homogeneous system, is it possible for the system to have the trivial solution? why?

Solution: Yes. Homogeneous systems always have the trivial solution.

b) If A is the matrix of a homogeneous system, is it possible for the system to have a non-trivial solution? why?

Solution: Yes. The matrix A has 5 columns and 2 pivots. Hence there are 2 free variables. (Note that one free variable is sufficient for an affirmative answer.)

c) If A is the matrix of a homogeneous system, is it possible for the system to have no solutions at all? i.e. can the system be inconsistent? why?

Solution: No. Homogeneous systems always have the trivial solution.

9. Determine if the following homogeneous systems have non-trivial solutions.

a) $\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$

Solution: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

There are three pivots, hence no free variables. So there are no non-trivial solutions.

b) $\begin{cases} x_1 + x_2 = 0 \\ x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$

Solution: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

There are two pivots, hence one free variable. So there exist non-trivial solutions.

10. Which of the following matrices are in Reduced Row Echelon Form? Why or why not?

$$\begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

Solution: $\begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 1 \end{pmatrix}$ is in RREF. It is in Echelon form because the pivots form a staircase pattern, all the entries below the pivots are zero, and there are no rows of zeros. It is in RREF because furthermore, the pivots are all 1, and they're the only non-zero entries in their columns.

$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \end{pmatrix}$ is not in RREF. The pivots are 2 and 3, which are not 1.

$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{pmatrix}$ is in RREF for the same reasons as the first one. The leading column of zeros does not change the conclusion.

11. Solve $A\mathbf{x} = 0$ for $A = \begin{pmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & 6 \end{pmatrix}$.

Solution: We row reduce:

$$\begin{pmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & 6 \end{pmatrix} \rightarrow \begin{matrix} \\ R_2 - R_1 \\ R_3 + 3R_1 \end{matrix} \begin{pmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 18 \end{pmatrix} \rightarrow \begin{matrix} \\ \\ R_3 - 2R_2 \end{matrix} \begin{pmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 12 \end{pmatrix}$$

$$\rightarrow \begin{matrix} \\ \\ \frac{1}{12}R_3 \end{matrix} \begin{pmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{matrix} R_1 - 4R_3 \\ R_2 - 3R_3 \\ \\ \end{matrix} \begin{pmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{matrix} R_1 - 3R_2 \\ \\ \\ \end{matrix} \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, x_3 is a free variable, and we have:

$$\begin{aligned} x_1 &= -4x_3 \\ x_2 &= 3x_3 \\ x_3 & \quad \text{free} \\ x_4 &= 0 \end{aligned}$$

12. Consider the system

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -3 & -3 \end{array}\right)$$

Find two different vectors that are the solution to this system. Are there only two solutions?

Solution: This system is already in RREF, so we can read off the solutions. The free variable is x_3, and the solutions are

$$\begin{aligned} x_1 &= -2x_3 + 2 \\ x_2 &= 3x_3 - 3 \\ x_3 &- \quad \text{free} \end{aligned}$$

Since x_3 is free, we can plug in any values for x_3 to get a solution. Plugging in $x_3 = 0$ and $x_3 = 1$, we see that

$$\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are two solutions. In fact, since we can plug in any number we like for x_3, there are infinitely many solutions.

13. Find the points of intersection of the coordinate axes and the plane $2x + 3y + 4z = 10$.

Solution

To intersect with the x-axis, set $y = z = 0$. So we get $2x = 10$ and the point is $(5, 0, 0)$ (in horizontal notation). Likewise, the intersection with the y-axis is $(0, \frac{10}{3}, 0)$ and the intersection with the z-axis is $(0, 0, \frac{5}{2})$.

14. Given points $A = (1, 1, 2, 3)$ and $B = (3, 0, 2, 5)$, find the vector $\vec{AB}$ and its length.

Solution

$$\vec{AB} = \begin{bmatrix} 3-1 \\ 0-1 \\ 2-2 \\ 5-3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}; \ |\vec{AB}| = \sqrt{2^2 + (-1)^2 + 0^2 + 2^2} = 3$$

15. Describe how to add vectors in $\mathbb{R}^n$ algebraically and geometrically.

Solution

Algebraically, one must add vectors coordinate-wise. Geometrically, one can use the Triangle Rule: $\vec{AB} + \vec{BC} = \vec{AC}$. Alternatively, one can use the Parallelogram Rule: $\vec{AB} + \vec{AD} = \vec{AC}$, where $ABCD$ is a parallelogram. (**Note:** You should be able to draw it).

16. Find the angle between two vectors $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -2 \\ 7 \\ 5 \end{bmatrix}$ (Simplify by evaluating the arccos function, if possible.)

Solution: The angle θ between $\vec{v}$ and $\vec{w}$ is determined by

$$\theta = \arccos\left(\frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}\right) = \arccos\left(\frac{-2+21+10}{\sqrt{1^2+3^2+2^2}\sqrt{(-2)^2+7^2+5^2}}\right)$$

$$= \arccos\left(\frac{29}{\sqrt{14}\sqrt{78}}\right)$$

17. Is it true that every matrix has exactly one Echelon form? Why or why not?

Solution: No, this is not true. While reduced row echelon form is unique for a matrix, simply Echelon form is not. For example, if U is in Echelon form, you can scale any row of U to get a new matrix that is also in Echelon form.

Level: MEDIUM

18. Let P_1 be the plane through the origin, orthogonal to $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and P_2 be the plane orthogonal to $\begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$. What is the line of intersection between P_1 and P_2?

Solution: The plane **through the origin** orthogonal to $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ is given by the equation $x+3y+2z=0$. And the plane through the origin orthogonal to $\begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}$ is given by $2x + 7y + 3 = 0$. These planes intersect when both equations hold. In other words, we must solve the system

$$\begin{cases} x + 3y + 2z &= 0 \\ 2x + 7y + 3z &= 0 \end{cases}$$

We put this system into a matrix and solve. Since the system is homogeneous, we do not need to augment by a column of 0s.

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 7 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \end{pmatrix}$$

So the solution is that x_3 is free, and $x_1 = -5x_3$ and $x_2 = x_3$. Thus, the line of intersection is given by

$$\left\{ \begin{pmatrix} -5t \\ t \\ t \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

19. For what values of h is the system $\begin{cases} x_1 + x_2 &= 0 \\ x_2 + x_3 &= h \\ x_1 - x_3 &= 1 \end{cases}$ consistent?

Solution: We write the augmented matrix, and row reduce just until Echelon form.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & h \\ 1 & 0 & -1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & h \\ 0 & -1 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & h \\ 0 & 0 & 0 & 1+h \end{array}\right)$$

This system is consistent if and only if $1 + h = 0$. That is, whenever $h = -1$.

20. Suppose that A is a 3×3 matrix, $\mathbf{b}$ is a vector, and the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. What is the maximum number of

pivots A can have? Why?

Solution: The number of pivot variables is at most 2.

Since $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, then A has at least one free variable. Thus, there can not be more than 3-1=2 pivot variables. Two pivot variables are possible, for example if the augmented matrix $(A|\mathbf{b})$ is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

21. For a given vector $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, determine the set of all vectors w such that v and w satisfy the Cauchy-Schwarz Inequality with equality (without a proof).

Solution: The Cauchy-Schwarz inequality for a non-zero vector $\vec{v}$ and any vector $\vec{w}$ holds with equality if and only if $\vec{w}$ is a scalar multiple of $\vec{v}$. So the set of all such vectors $\vec{w}$ is $\{t \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \mid t \in \mathbb{R}\}$.

22. Let $f : X \to Y$ and $g : Y \to Z$ be functions.
a) Show that if both f and g are one-to-one, then $g \circ f$ is also one-to-one.

Solution: Given distinct $x_1, x_2 \in X$, we show that $g \circ f(x_1) \neq g \circ f(x_2)$. Since x_1 and x_2 are distinct, we have $f(x_1) \neq f(x_2)$ by the condition that f is one-to-one. From $f(x_1) \neq f(x_2)$, we have that $g \circ f(x_1) = g(f(x_1)) \neq g(f(x_2)) = g \circ f(x_2)$.

b) Show that if both f and g are onto, then $g \circ f$ is also onto.

Solution: We show that for any $z \in Z$, there is x such that $g \circ f(x) = z$. For any $z \in Z$, we have $y \in Y$ such that $g(y) = z$. And for y, there is $x \in X$ such that $f(x) = y$. So we have

$$g \circ f(x) = g(f(x)) = g(y) = z.$$

Therefore, there is $x \in X$ such that $g \circ f(x) = z$.

c) Show that if both f and g are bijective, then $g \circ f$ is also bijective.

Solution: Suppose that f and g are bijective. Then, f and g are both one-to-one and onto. By (a), we have that $g \circ f$ is one-to-one, and by (b), we have that $g \circ f$ is onto. Therefore, $g \circ f$ is bijective.

23. Find all values of the parameter a such that the homogeneous system with the following coefficient matrix has a non-trivial solution.

$$\begin{bmatrix} 1 & 2 & 3 \\ a & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution

We use Gaussian Elimination (row reduction):

$$\begin{bmatrix} 1 & 2 & 3 \\ a & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{matrix} \\ R_2 - aR_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2-2a & 1-3a \\ 0 & -1 & -2 \end{bmatrix} \to \begin{matrix} \\ R_3 \\ R_2 \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 2-2a & 1-3a \end{bmatrix}$$

$$\to \begin{matrix} \\ \\ R_3 + (2-2a)R_2 \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & (1-3a) - 2(2-2a) \end{bmatrix}$$

For every value of a this matrix is in row echelon form. Simplifying, $(1-3a) - 2(2-2a) = -3 + a$. So when $a = 3$ this entry is 0 and we have a free variable, and thus a non-trivial solution. If $a \neq 3$ we have no free variables, and the system has only the trivial solution.

24. Suppose $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ are both solutions of some system of equations. Find the following:

a) a non-trivial solution of the corresponding homogeneous system;

b) two more solutions of that homogeneous system

c) two more solutions of the original system.

Solution

a) The difference of any two solutions of a system is always a solution of the corresponding homogeneous system: $\begin{bmatrix} 2-1 \\ 4-2 \\ 1-3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

b) Multiplying any solution of a homogeneous system by any real number, we get another solution of the system. So, for example, we get $\begin{bmatrix} 10 \\ 20 \\ -20 \end{bmatrix}$ and $\begin{bmatrix} 20 \\ 40 \\ -40 \end{bmatrix}$. We also have a trivial solution $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

c) We get more solutions of the original system by adding solutions of the homogeneous system to a solution of the orginial system. For example, $\begin{bmatrix} 2+10 \\ 4+20 \\ 1-20 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \\ -19 \end{bmatrix}$ and $\begin{bmatrix} 2+20 \\ 4+40 \\ 1-40 \end{bmatrix} = \begin{bmatrix} 22 \\ 44 \\ -39 \end{bmatrix}$.

25. Suppose some system of linear equations has exactly one solution. How many solutions does the corresponding homogeneous system have? Can you describe these solutions?

Solution

The corresponding homogeneous system has exactly one solution: the trivial one, that is when all variables are equal to zero. Here is the justification. Every homogeneous system has the trivial solution. If our system had a nontrivial solution, we would be able to add it to the given solution (our system is consistent!) and get another solution of our system. Since our system has exactly one solution, this is impossible, therefore the corresponding homogeneous system has only the trivial solution.

26. Find the Row Reduced Echelon Form of the following augmented matrix. Then describe the solution set in parametric vector form.

$$\left[\begin{array}{rrrr|r} -2 & 0 & 0 & 2 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 2 & 0 & 0 & -2 & 0 \end{array}\right].$$

Solution:

$$\left[\begin{array}{rrrr|r} -2 & 0 & 0 & 2 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 2 & 0 & 0 & -2 & 0 \end{array}\right] \underset{R_4 := R_4 + R_1}{\longrightarrow} \left[\begin{array}{rrrr|r} -2 & 0 & 0 & 2 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to$$

$$\underset{R_3 := R_3 + R_2}{\longrightarrow} \left[\begin{array}{rrrr|r} -2 & 0 & 0 & 2 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \underset{\substack{R_1 := R_1/(-2) \\ R_2 := R_2/(-2)}}{\longrightarrow} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Leading variables: $x_1, \quad x_2$; free variables: $x_3, \quad x_4$.
Set $x_3 = t, \quad x_4 = s$, and solve for x_1 and x_2 in terms of t and s :
$x_1 = s$
$x_2 = t$
$x_3 = t$
$x_4 = s$
Solution in a vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} s$$

27. Reduce the system to Echelon Form to determine if it is consistent. You do not need to find the solutions.

$$\begin{cases} x + y + z &= 0 \\ x - y + z &= 2 \\ x - y - z &= 4 \end{cases}$$

Solution: Write down the augmented matrix of the system and reduce it to REF:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 2 \\ 1 & -1 & -1 & 4 \end{array}\right] \underset{\substack{R_2 := R_2 - R_1 \\ R_3 := R_3 - R_1}}{\longrightarrow} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & -2 & -2 & 4 \end{array}\right] \to$$

$$\xrightarrow[R_3 := R_3 - R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & 2 \end{array}\right], \text{ so the system is consistent.}$$

28. Write down all possible Reduced Row Echelon Forms of the augmented matrix of a homogeneous linear system of 2 equations in 3 variables. Use stars to represent entries that are arbitrary real numbers

Solution: All possible RREF are the following:

$$\left[\begin{array}{ccc|c} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \end{array}\right]; \left[\begin{array}{ccc|c} 1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]; \left[\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 0 & 0 & 0 \end{array}\right];$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]; \left[\begin{array}{ccc|c} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]; \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]; \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

29. Consider the line L in $\mathbb{R}^3$ that passes through the points $(1, 0, -1)$ and $(-1, 0, 1)$.
a) Write parametric equations of this line.
b) Describe the intersection of this line with the $xy-$plane and the $xz-$plane.
c) Which of these points are on the line L?
$P = (10, 0, -10)$, $Q = (10, -10, 0)$, $R = (-10, 0, 10)$

Solution:

a) The direction vector of the line is $\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$, and the equations are

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

b) The $xy-$plane has equation $z = 0$, so the line intersects it at such t that $-1 + 2t = 0$. This is $t = \frac{1}{2}$, so the point is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The $xz-$plane has equation $y = 0$ which is satisfied for every value of t, so the line lies entirely on this plane: the intersection is the line itself.

c) P and R are on the line, which one can see by solving for t. Q is not on the line, because it is not on the $xz-$plane.

30. Consider the planes described below:

$\pi_1 :\ x + y + z = 0$
$\pi_2 :\ 2x + 2y + 2z = 5$
$\pi_3 :\ -x + y + z = 0$

Describe $\pi_1 \cap \pi_2,\ \pi_1 \cap \pi_3,\ \pi_1 \setminus \pi_3$ (set difference) and $\pi_1 \cup \pi_2$.

Solution:

$\pi_1 \cap \pi_2$ is an empty set, because $x + y + z$ cannot be 0 and $\frac{5}{2}$ at the same time.

$\pi_1 \cap \pi_3$ can be found by solving the system of two equations. The answer:

$$\{t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \mid t \in \mathbb{R}\}$$

π_1 can be described as $\{s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \mid s, t \in \mathbb{R}\}$. Its intersection with π_3 is determined by the condition $-(-s-t)+s+t = 0$ which simplifies to $s + t = 0$. So

$\pi_1 \setminus \pi_3 = \{s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \mid s, t \in \mathbb{R} \ and \ s + t \neq 0\}$

$\pi_1 \cup \pi_2$ is a disjoint union of the two planes, since they do not intersect.

31. Write down an equation of the plane through $(1, 1, 1),\ (1, 2, -1)$ and $(0, 1, -1)$.
Is the point $(0, 2, 1)$ on this plane?

Solution:

The vectors that connect $(1, 1, 1)$ to $(1, 2, -1)$ and $(0, 1, -1)$ are $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

and $\begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$. So the parametric equation of the plane is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$$

Setting this equal to $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, we get a system of three equations in the variables s and t. It is easy to see that the system is inconsistent. So the point is not on the plane. (**Note:** Alternatively, one can find the (non-parametric) equation of the plane by finding its normal vector by the property of being orthogonal to the vectors $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$).

32. Prove the symmetry property for the dot product of vectors $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^3$. In other words, explain why it is always true that $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$, $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

Solution: For arbitrary $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$ we have $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$ which is simply a sum of products of real numbers. Since multiplication is commutative in $\mathbb{R}$ this expression equals $v_1u_1 + v_2u_2 + v_3u_3 = \mathbf{v} \cdot \mathbf{u}$.

33. Let $\mathbf{u} = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}$.

a) Find the distance between $\mathbf{u}$ and $\mathbf{v}$ (when the vectors are identified with the points).

Solution: For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$. Compute $\mathbf{u}-\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix}$ so then $\|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22}$.

b) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

Solution: Since $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$, we can compute

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) = \cos^{-1}\left(\frac{4+18}{(\sqrt{16+9})(\sqrt{4+1+36})}\right) = \cos^{-1}\left(\frac{22}{5\sqrt{41}}\right)$$

.

34. Let $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

a) Which pairs of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are orthogonal?

Solution: Vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are pairwise orthogonal:
$\mathbf{u} \cdot \mathbf{v} = 0, \quad \mathbf{u} \cdot \mathbf{w} = 0, \quad \mathbf{w} \cdot \mathbf{v} = 0.$

b) Are any of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ unit vectors?

Solution: The only unit vector is $\mathbf{u}$ because
$\|\mathbf{u}\| = 1, \quad \|\mathbf{v}\| = \sqrt{2}, \quad \|\mathbf{w}\| = \sqrt{3}.$

c) Normalize $\mathbf{v}$ (find the vector of length 1 in the direction of $\mathbf{v}$).

Solution: The unit vector in the direction of $\mathbf{v}$ is

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

Level: HARD

35. Choose h and k such that the system

$$\begin{cases} x_1 & +hx_2 & = 2 \\ 5x_1 & +10x_2 & = k \end{cases}$$

has

a) no solution

Solution: After rewriting this system as a matrix we can row reduce to the following:

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & 10-5h & k-10 \end{array}\right]$$

The system will be inconsistent if the **bold-faced** entry is a pivot.

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & 10-5h & \mathbf{k-10} \end{array}\right]$$

This occurs when $k - 10 \neq 0$ and $10 - 5h = 0 \quad \Rightarrow \quad k \neq 10$ and $h = 2$.

b) a unique solution

Solution: The system has a unique solution when this **bold-faced** entry is a pivot.

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & \mathbf{10-5h} & k-10 \end{array}\right]$$

This occurs when $h \neq 2$ and we note that there is no restriction on values for k.

c) many solutions

Solution: Since there are two unknowns and at least one pivot, to ensure that there is a free variable, the bottom row must be a zero row.

$$\left[\begin{array}{cc|c} \mathbf{1} & h & 2 \\ 0 & 10-5h & k-10 \end{array}\right]$$

There will be infinitely many solutions when $h = 2$ and $k = 10$ simultaneously.

36. Let A be the augmented matrix for a system for which the number of variables equals the number of equations.

a) Does A have to be reduced to RREF in order to determine if the system is inconsistent?

Solution: A system is inconsistent if the matrix for the system has a pivot in the augmented column. We can determine the location of the pivot columns once A is in EF. There is no need to continue to RREF.

b) If A has 4 pivots, what conditions would need to be satisfied in order to conclude that the system has exactly one non-trivial solution?

Solution: The system would need to have 4 rows, 4 basic variables. and 0 free variables. The system needs to be non-homogeneous.

37. Suppose f and g are two finctions and $f \circ g$ is their composition. Suppose $f \circ g$ is injective.

a) Can we conclude that f is injective? Explain or construct a counterexample.

b) Can we conclude that g is injective? Explain or construct a counterexample.

Solution

a) The function f does not have to be injective. For example, if $f(x) = |x|$ and $g(x) = e^x$ are functions from $\mathbb{R}$ to $\mathbb{R}$, then $f \circ g$ is injective, but f is not injective.

b) The function g is injective, otherwise we would get $x_1 \neq x_2$ such that $g(x_1) = g(x_2)$ which would imply that $f(g(x_1)) = f(g(x_2))$, contradicting the injectivity of $f \circ g$.

38. Suppose f and g are two finctions and $f \circ g$ is their composition. Suppose $f \circ g$ is surjective.

a) Can we conclude that f is surjective? Explain or construct a counterexample.

b) Can we conclude that g is surjective? Explain or construct a counterexample.

Solution

a) The function f is surjective. Indeed, if it were not surjective, then some values in the codomain of f, which is also the codomain of $f \circ g$, could never be obtained as outputs of f, and thus of $f \circ g$.

b) The function g does not have to be surjective. Suppose $A = \{1\}, B = \{1, 2\}, C = \{1\}$. We define the functions $f : B \to C$ and $g : A \to B$ as follows: $f(1) = f(2) = 1$ and $g(1) = 1$. Then $f \circ g$ is surjective but g is not surjective. (**Note:** This is also an example for the part (a) of the previous problem).

39. Suppose $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ are both solutions of some system of equations. Show that this system is homogeneous.

Solution

The difference of two solutions of the system is a solution of the corresponding homogeneous system. So $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a solution of the corresponding homogeneous system. Multiplying this by (-1), we get another solution of the homogeneous system: $\begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$. Adding this to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, we get a trivial solution of the original system. And this implies that the original system is homogeneous. (**Note:** One can also argue that since $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a solution both of the original and the corresponding homogeneous system, the two systems must have the same right hand side, so the original system is homogeneous).

40. Let L be a linear system and let vectors $\vec{w_1}$ and $\vec{w_2}$ be solutions of L.

Suppose that the corresponding homogeneous system of L has the solution set

$$\{t_1\vec{v_1} + \cdots + t_k\vec{v_k} : t_1, \ldots, t_k \in \mathbb{R}\}.$$

Show that

$$\{\vec{w_1}+t_1\vec{v_1}+\cdots+t_k\vec{v_k} : t_1, \ldots, t_k \in \mathbb{R}\} = \{\vec{w_2}+t_1\vec{v_1}+\cdots+t_k\vec{v_k} : t_1, \ldots, t_k \in \mathbb{R}\}.$$

Solution: Without loss of generality, it is enough to show that

$$\{\vec{w_1}+t_1\vec{v_1}+\cdots+t_k\vec{v_k} : t_1, \ldots, t_k \in \mathbb{R}\} \subseteq \{\vec{w_2}+t_1\vec{v_1}+\cdots+t_k\vec{v_k} : t_1, \ldots, t_k \in \mathbb{R}\}.$$

Let us denote the set on the left and on the right by W_1 and W_2, respectively. We need to show that $\vec{w_1} + \sum_{i=1}^k t_i\vec{v_i} \in W_2$ for every $t_1, \ldots, t_k \in \mathbb{R}$.

Note that $\vec{w_1}-\vec{w_2}$ is a solution for the corresponding homogeneous system of L, since for every equation $a_1x_1 + \cdots + a_nx_n = d$ of L, we have

$$\sum_{i=1}^{n} a_i(w_{1,i} - w_{2,i}) = \sum_{i=1}^{n} a_iw_{1,i} - \sum_{i=1}^{n} a_iw_{2,i} = d - d = 0,$$

when $\vec{w_j} = [w_{j,1}, w_{j,2}, \ldots, w_{j,n}]$ for $j \in \{1, 2\}$. Hence, by the assumption, there exist $s_1, \ldots, s_k \in \mathbb{R}$ such that

$$\vec{w_1} - \vec{w_2} = \sum_{i=1}^{k} s_i\vec{v_i}.$$

Therefore, we have

$$\vec{w_1} + \sum_{i=1}^{k} t_i\vec{v_i} = \vec{w_2} + (\vec{w_1} - \vec{w_2}) + \sum_{i=1}^{k} t_i\vec{v_i} = \vec{w_2} + \sum_{i=1}^{k}(s_i + t_i)\vec{v_i} \in W_2.$$

41. Let L be a homogeneous linear system. Using an echelon form of L, show that if there are strictly more variables than equations, then there is a non-zero (or non-trivial) solution of L.

Solution: Suppose that L has strictly more variables than equations. Then an echelon form E of L has strictly more variables than non-zero rows there, since the number of non-zero rows of the echelon form E is at most the number of equations of L. Since the number of non-zero rows in E is the same as the number of leading variables of E, we have at least one free variable for E. So, the solution set of E, which is same as the solution set of L, has the form

$$\left\{ \sum_{i=1}^{k} t_i \vec{v_i} : t_1, \ldots, t_k \in \mathbb{R} \right\},$$

where $k \geq 1$, and each $\vec{v_i}$ is non-zero. Therefore, there are infinitely many solutions, and in particular there is a non-zero (non-trivial) solution of L.

42. a) Suppose that there are two distinct solutions $\vec{v}$ and $\vec{w}$ of a linear system L. Show that the set

$$\{t\vec{v} + (1-t)\vec{w} : t \in \mathbb{R}\}$$

is contained in the solution set of L.

Solution: We need to show that $t\vec{v} + (1-t)\vec{w}$ is a solution of L for any $t \in \mathbb{R}$. That is, we need to show that for any given $t \in \mathbb{R}$, $t\vec{v} + (1-t)\vec{w}$ satisfies each equation

$$\sum_{i=1}^{n} a_i x_i = d.$$

Let $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, \ldots, w_n]$. Then we have,

$$\sum_{i=1}^{n} a_i(tv_i + (1-t)w_i) = \sum_{i=1}^{n} (ta_i v_i + (1-t)a_i w_i) = t\sum_{i=1}^{n} a_i v_i + (1-t)\sum_{i=1}^{n} a_i w_i$$

$$= td + (1-t)d = d.$$

Therefore, $t\vec{v} - (1-t)\vec{w}$ is a solution of L.

b) Using (a), prove that for a linear system L, the size of the solution set of L is either 0, 1, or infinite.

Solution: It is enough to show that if we have two distinct solutions $\vec{v}$ and $\vec{w}$ for L, then we have infinitely many solutions. By (a), we have that all elements from

$$\{t\vec{v} + (1-t)\vec{w} : t \in \mathbb{R}\}$$

are solutions of L. This set can also be rewritten as

$$\{t(\vec{v} - \vec{w}) + \vec{w} : t \in \mathbb{R}\}$$

Because $\vec{v}$ and $\vec{w}$ are distinct, $\vec{v} - \vec{w} \neq \vec{0}$. Therefore, for any two distinct $t \in \mathbb{R}$ we get distinct solutions. Thus L has infinitely many solutions.

43. Using the Triangle Inequality and mathematical induction, prove

$$||\vec{u_1} + \vec{u_2} + \cdots + \vec{u_n}|| \leq ||\vec{u_1}|| + ||\vec{u_2}|| + \cdots + ||\vec{u_n}||$$

for every n given vectors $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}$ and every positive integer n.

Solution: Use mathematicial induction on the number n of the given vectors. When $n = 1$, we have $||\vec{u_1}|| = ||\vec{u_1}||$, so the inequality holds clearly. Now, we assume that the inequality holds for any given n vectors, and show that it holds for any $n+1$ vectors, say $\vec{u_1}, \ldots, \vec{u_{n+1}}$. But, we have

$$||\vec{u_1} + \vec{u_2} + \cdots + \vec{u_n} + \vec{u_{n+1}}|| = ||(\vec{u_1} + \vec{u_2} + \cdots + \vec{u_n}) + \vec{u_{n+1}}||$$

$$\leq ||\vec{u_1} + \vec{u_2} + \cdots + \vec{u_n}|| + ||\vec{u_{n+1}}|| \leq ||\vec{u_1}|| + ||\vec{u_2}|| + \cdots + ||\vec{u_n}|| + ||\vec{u_{n+1}}||,$$

where the last inequality uses induction hypothesis, and the second last inequality uses the triangle inequality. This completes the proof.

44. Let a function $f : X \to Y$ be given. Also, recall that $f(X')$ is the set

$$\{f(x) : x \in X'\}$$

where X' is a subset of X.

a) Let $X_1, X_2 \subseteq X$. Show that $f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$.

Solution: We need to show two inclusions: $f(X_1 \cup X_2) \subseteq f(X_1) \cup f(X_2)$ and $f(X_1 \cup X_2) \supseteq f(X_1) \cup f(X_2)$.

Let us consider the first inclusion. Suppose that $y \in f(X_1 \cup X_2)$. Then, there exists $x \in X_1 \cup X_2$ such that $f(x) = y$. Since x is either in X_1 or in X_2, y is either in $f(X_1)$ or $f(X_2)$. Therefore, we have $y \in f(X_1) \cup f(X_2)$.

Let us consider the other inclusion. Suppose that $y \in f(X_1) \cup f(X_2)$. Then y is either in $f(X_1)$ or in $f(X_2)$. In the first case, there exists $x_1 \in X_1 \subseteq X_1 \cup X_2$ such that $f(x_1) = y$, and in the second case, similarly, there exists $x_2 \in X_2 \subseteq X_1 \cup X_2$ such that $f(x_2) = y$. In both cases, we have $y \in f(X_1 \cup X_2)$. This completes the proof.

b) Using mathematical induction and (a), show that

$$f(\bigcup_{i=1}^{n} X_i) = \bigcup_{i=1}^{n} f(X_i)$$

for all given n subsets $X_1, \ldots, X_n$ and for every positive integer n.

Solution: Use mathematical induction on the number n of sets at the union. When $n = 1$, it obviously hods. Now assume that the equality holds for any n sets, and we show the equality for $n+1$ sets, say $X_1, \ldots, X_{n+1}$.

But then,

$$\begin{aligned} f(\bigcup_{i=1}^{n+1} X_i) &= f(\left(\bigcup_{i=1}^{n} X_i\right) \cup X_{n+1}) = f\left(\bigcup_{i=1}^{n} X_i\right) \cup f(X_{n+1}) \\ &= \left(\bigcup_{i=1}^{n} f(X_i)\right) \cup f(X_{n+1}) = \bigcup_{i=1}^{n+1} f(X_i), \end{aligned}$$

where at the second equaltiy we used (a), and at the third equality we used the induction hypothesis. This completes the proof.

45. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. We consider a 2×2 array as in Table 1, where each entry is either 0 or 1. The proposition $p(x_i, y_j)$ for $i \in \{1, 2\}$ and $j \in \{1, 2\}$ means that the corresponding entry - which is in ith row and jth column - is 1.

Table 1.

	y_1	y_2
x_1	0	0
x_2	1	0

For each quantified statement below, list all possible arrays that the given statement is true.

a) $\forall x_i \in X \, \exists y_j \in Y \, p(x_i, y_j)$.

Solution: The condition means that in each row we have at least one 1. So we have the following possibilities:

	y_1	y_2
x_1	1	0
x_2	1	0

	y_1	y_2
x_1	1	0
x_2	0	1

	y_1	y_2
x_1	0	1
x_2	1	0

	y_1	y_2
x_1	0	1
x_2	0	1

	y_1	y_2
x_1	1	1
x_2	1	0

	y_1	y_2
x_1	1	1
x_2	0	1

	y_1	y_2
x_1	0	1
x_2	1	1

	y_1	y_2
x_1	1	0
x_2	1	1

	y_1	y_2
x_1	1	1
x_2	1	1

b) $\exists y_j \in Y \, \forall x_i \in X \, p(x_i, y_j)$.

Solution: The condition means that there at least one column that consists entirely of 1s. The list of the arrays is:

	y_1	y_2
x_1	1	0
x_2	1	0

	y_1	y_2
x_1	0	1
x_2	0	1

	y_1	y_2
x_1	1	1
x_2	1	0

	y_1	y_2
x_1	1	1
x_2	0	1

	y_1	y_2
x_1	0	1
x_2	1	1

	y_1	y_2
x_1	1	0
x_2	1	1

	y_1	y_2
x_1	1	1
x_2	1	1

46. For which values of the parameter k does the matrix A have the given RREF?

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 2 & -4 & 6 \\ -1 & k & -3 \end{bmatrix} \text{ and RREF: } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Scale the second tow by $\frac{1}{2}$ and then add to the third row the second row to obtain $\begin{bmatrix} 5 & 3 & 2 \\ 1 & -2 & 3 \\ 0 & (k-2) & 0 \end{bmatrix}$.

If $k-2=0$ then we get a matrix with a zero row on the bottom and, clearly, its further row reduction will lead to a RREF that is different from the one given.

If $k-2 \neq 0$, we can divide the third row by it and obtain a matrix $\begin{bmatrix} 5 & 3 & 2 \\ 1 & -2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$. It is easy to show that it will reduce to the required RREF.

So A has the given RREF if and only if $k \neq 2$.

47. Consider the linear system below; $a_i,\ b_i,\ c_i \in \mathbb{R}$ are fixed:

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 &= c_1 \\ b_1x_1 + b_2x_2 + b_3x_3 &= c_2 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{cases}$$

List all possible RREF for this system, assuming that the system is consistent.

Solution

We have at most two non-zero rows, so at most two pivot columns. Because the system is consistent, the last column cannot be a pivot column.

If we have two pivot columns, we have three posibilities: the pivots could be in columns 1 and 2, 1 and 3, 2 and 3. These give the following RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right]$$

If we have one pivot column, then the only non-zero row is the first one, and the pivot can be in the any of the first three columns:

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Finally, we can have no pivots at all, and the zero matrix as RREF:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

48. Consider the linear system

$$\begin{cases} 3x + 5y - 4z &= 0 \\ -3x - 2y + 4z &= 0 \\ 6x + y - 8z &= 0 \end{cases}$$

a) Find the RREF of the augmented matrix of the given system.
b) Identify the basic and the free variables.
c) Describe the solutions sets of the following systems:

$$\text{(i)} \begin{cases} 3x + 5y - 4z &= 1 \\ -3x - 2y + 4z &= 0 \\ 6x + y - 8z &= 0 \end{cases} \qquad \text{(ii)} \begin{cases} 3x + 5y - 4z &= 5 \\ -3x - 2y + 4z &= -2 \\ 6x + y - 8z &= 1 \end{cases}$$

Solution

Since all three systems have the same left hand side, we first apply Gaussian Elimination (row reduction) to the augmented matrix for a system with this left hand side and arbitrary right hand side:

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & a \\ -3 & -2 & 4 & b \\ 6 & 1 & -8 & c \end{array}\right] \rightarrow \begin{array}{r} \\ R_2 + R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|c} 3 & 5 & -4 & a \\ 0 & 3 & 0 & a+b \\ 0 & -9 & 0 & c-2a \end{array}\right]$$

$$\rightarrow \begin{array}{r} \\ \\ R_3 + 3R_1 \end{array} \left[\begin{array}{ccc|c} 3 & 5 & -4 & a \\ 0 & 3 & 0 & a+b \\ 0 & 0 & 0 & a+3b+c \end{array}\right] \rightarrow \tfrac{1}{3}R_3 \left[\begin{array}{ccc|c} 3 & 5 & -4 & a \\ 0 & 1 & 0 & \tfrac{1}{3}(a+b) \\ 0 & 0 & 0 & a+3b+c \end{array}\right]$$

a) When $a = b = c = 0$, this further reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & -\tfrac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

b) Here z is free, x and y are basic variables, and the soultions are $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \tfrac{4}{3}z \\ 0 \\ z \end{bmatrix}$, where z is any real number.

c) For (i) we get a pivot in the last column, so the system is inconsistent. For (ii) the matrix reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which gives $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as a particular solution. So the solution set is

$$\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \}$$

49. Suppose the solution set to the system $A\mathbf{x} = \mathbf{0}$ is a line. Can we say anything about the shape of the solution set to $A\mathbf{x} = \mathbf{b}$ for some vector $\mathbf{b} \neq \mathbf{0}$? Can it be a point, a line, a plane, or some other shape?
Construct an example or explain why it is impossible.

Solution: A solution can be a line. For example, if $A = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$ and a vector $\mathbf{b} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ the set of solutions is the line $x_1 = 1$ which is a shift of the line $x_1 = 0$, the solution of the corresponding homogeneous system.

A solution cannot be a single point, because we wouldbe able to get more solutions by adding to this particular solution other solutions on the homogneous system.

A solution set cannot be a plane, because a plane needs to parameters to describe it, and we only have one free variable.

A solution set can also be empty. For example, take $A = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

50. Can the solution set of a system with 2 equations in 3 unknowns ever be a point? A line? A plane?
Construct an example or explain why it is impossible.

Solution: If we have two equations with 3 unknowns, then we have at most 2 pivot variables. Since there are 3 variables in total (3 columns) we have at least 1 free variable. That means that the solution set can never be just a single point. In fact, we can have 1, 2 or 3 free variables. In these cases, if the system is consistent, the solution set is a line, a plane, and all of $\mathbb{R}^3$, respectively. We can also have no solutions at all.

Here are matrices that show all 4 situations:

1. No solutions:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

2. 1 free variable, a line:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

3. 2 free variables, a plane:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

4. 3 free variables, whole space:

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

51. Let U and V be vectors in $\mathbb{R}^n$. Use the properties of the dot product and other results to show that

$$||U+V||^2 + ||U-V||^2 = 2||U||^2 + 2||V||^2.$$

Solution

By the definition of the length of the vector as the square root of the dot product of the vector with itself,

$$||U+V||^2 + ||U-V||^2 = (U+V)\cdot(U+V) + (U-V)\cdot(U-V)$$

By the distributivity property of the dot product, applied twice to each of the products $(U+V)\cdot(U+V)$ and $(U-V)\cdot(U-V)$ this can be rewritten as

$$(U\cdot U + U\cdot V + V\cdot U + V\cdot V) + (U\cdot U - U\cdot V - V\cdot U + V\cdot V)$$

Simplifying, we get $2(U \cdot U) + 2(V \cdot V)$, which equals $2||U||^2 + 2||V||^2$.

52. Describe each side of the unit square in $\mathbb{R}^2$ as a set consisting of a linear combination of vectors with parameters. Find the angle between the diagonal of the unit square and one of the axes. Do the same for the unit cube in $\mathbb{R}^3$. Generalize to any dimension n. Find the angle between the diagonal of the unit cube in $\mathbb{R}^n$ and one of the axes.

Solution

The unit square is the set $\{\begin{bmatrix} s \\ t \end{bmatrix} \mid 0 \leq s \leq 1 \; and \; 0 \leq t \leq 1\}$.

This can be written as $\{s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid 0 \leq s \leq 1 \; and \; 0 \leq t \leq 1\}$

The four sides are:

$\{s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid 0 \leq s \leq 1 \; and \; t = 0\}$,

$\{s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid 0 \leq s \leq 1 \; and \; t = 1\}$,

$\{s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid s = 0 \; and \; 0 \leq t \leq 1\}$

$\{s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid s = 1 \; and \; 0 \leq t \leq 1\}$

The diagonal of the unit cube in $\mathbb{R}^n$ is the vector $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}$. To find its angle with the vector $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$ we use the formula $\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$.

We get $\theta = \arccos(\frac{1}{\sqrt{n}})$. In particular for the square we get $\arccos(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$ and for the three-dimensional cube we get $\theta = \arccos(\frac{1}{\sqrt{3}})$.