MATH 304 MIDTERM EXAM 2 SAMPLE PROBLEMS

Note: problems marked with asterisque (*) have multiple questions asked. An actual examination problem will only ask one of these questions.

Level: EASY

1. Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Find $A^2 - 3A + 2I_2$.

2. Is the following statement true? Justify your answer with clear explanation.
"If A and B are $m \times n$ matrices with $m \neq n$, then AB^T and A^TB are both well-defined matrix products."

3. Let A be a 2×2 matrix. Solve for A from the following matrix equation.

$$5A - 4I_{2\times2} = 0_{2\times2}.$$

4. Is $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, 5z - 1 = x + 2y \right\}$ a vector space, under the usual addition and scalar multiplication operations?

5. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear map satisfying $T(\vec{u}_1) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $T(\vec{u}_2) = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$. Find $T(2\vec{u_1} - \vec{u_2})$.

6. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear map satisfying $T(\vec{u}_1) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $T(\vec{u}_2) = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$. Is there a vector $\vec{w} \in \mathbb{R}^2$ such that $T(\vec{w}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

7. Find the standard matrix A for the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates every vector counterclockwise by $\frac{\pi}{2}$ radians, then doubles the magnitude.

8. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Determine if the vector $\mathbf{w} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is in the span of the set of vectors $\{\mathbf{u}, \mathbf{v}\}$.

9. Do the vectors $\mathbf{v}_1 = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{v}_2 = \mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{v}_3 = \mathbf{e}_2 + \mathbf{e}_3$ span all of $\mathbb{R}^3$?

10. Let V be the set of vectors $\mathbf{v} \in \mathbb{R}^2$ where the sum of the components of $\mathbf{v}$ equals 2. Show that V is not a subspace of $\mathbb{R}^2$.

11. Which of the following sets of vectors are linearly independent in $\mathbb{R}^2$?

$$A = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad B = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},$$

$$C = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad D = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$$

12. If two vectors $\mathbf{v}_1, \ \mathbf{v}_2 \in V$ are multiples of one another, can they be linearly independent? Explain.

13. Are the vectors

$$\begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 5 \\ 0 \end{bmatrix}$$

linearly independent in $\mathbb{R}^4$?

14. What is the dimension of the space P_3 of polynomials of degree up to 3?

15. Find the coordinates of the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in the basis $\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$.

16. Construct a linearly independent collection of vectors in $\mathbb{R}^3$ that is not a basis of $\mathbb{R}^3$.

17. Consider an augmented matrix A of a homogeneous linear system in a row reduced echelon form. Suppose that there are 3 leading variables. Then, what is the dimension of the column space of A?

18. Find the dimension of the range space of a linear map T_A given by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

19. Determine whether the linear map T_A defined by a matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$ is a monomorphism (injective linear map).

20*. Given $\vec{u}, \vec{v} \in \mathbb{R}^3$ that span the subspace $S = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mid x,\ y \in \mathbb{R} \right\}$ of $\mathbb{R}^3$. Is the following set a basis of $\mathbb{R}^3$?

(a) $\left\{ \vec{u},\ \vec{v},\ \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \right\}$

(b) $\left\{ \vec{u}+\vec{v},\ \vec{v},\ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\}$

(c) $\left\{ 2\vec{u},\ 3\vec{v},\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$

(d) $\{\vec{u}-\vec{v},\ \vec{u}+\vec{v}\}$

Level: MEDIUM

21. Find $(A+B)^t A$ if $A^t A = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$ and $A^t B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

22. Give an example of two matrices A and B such that AB is a zero matrix and BA is undefined.

23. Find $(3A + I_2)B$ if $AB = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$, where A is a 2×2 matrix and $B = \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}$.

24. Is $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid xy = 0 \right\}$ a subspace of $\mathbb{R}^2$?

25. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{ and } T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

What is the matrix A so that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$?

26. Is the map $f : M_{2\times 2} \to \mathbb{R}^2$ with $f\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ linear?

27. Determine if the set H of all matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is a subspace of $\mathcal{M}_{2\times 2}$, the vector space of all 2×2 matrices with real number entries.

28. Determine if the set of all polynomials $p(t)$ of degree at most n satisfying $p(1) = 0$ is a subspace of $\mathbb{P}_n$.

29. Give an example of a vector in $\mathbb{R}^3$ that is not in Span$\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$.

30. Let $S = \{v_1, v_2, v_3\}$ be the set of three non-zero vectors from $\mathbb{R}^3$. Suppose that S is linearly dependent. Then, what are all possible geometric shapes of $span(S)$?

31. Find a vector v in $\mathbb{R}^3$ such that $\{(1, 0, -1)^t, (0, -1, 1)^t, v\}$ is linearly independent.

32. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is such that

$$T\begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}, \; T\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$$

Find the standard matrix of T.

33. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each vector counterclockwise by $\frac{\pi}{4}$ radians. Let A be its standard matrix. Show that the columns of A form a basis of $\mathbb{R}^2$.

34. Let A be the coefficient matrix for a system of 3 homogeneous equations in 3 variables. If the system has infinitely many solutions, what are the possible dimensions of the space spanned by the columns of A?

35. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Find a basis for the column space of A.

36. Let $A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$. Find a basis for the null space of A.

37. Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation with the following action on the standard basis vectors:

$$\begin{array}{lll} \mathbf{e}_1 & \mapsto & \mathbf{e}_2 + 2\mathbf{e}_3 - 3\mathbf{e}_4 \\ \mathbf{e}_2 & \mapsto & 3\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3 \end{array}.$$

Find the kernel of T.

38*. Let $M_{2\times 2}$ be the vector space of all 2×2 matrices. Define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^T$.

(a) Show that T is a linear transformation.
(b) Let B be in $M_{2\times 2}$ such that $B = B^T$. Find an A such that $T(A) = B$.
(c) Show that the range of T is the set of B such that $B = B^T$.
(d) Describe the kernel of T.

Level: HARD

39. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$. Find a matrix B such that $AB = I_2$ or show that such a B does not exist.

40. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$. Find a matrix B such that $BA = I_3$ or show that such a B does not exist.

41. Let A and B be two 2×2 matrix such that $AB = 0$. Explain why there is a vector $\mathbf{x} \neq \mathbf{0}$ such that $BA\mathbf{x} = \mathbf{0}$

42. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which rotates a vector counter-clockwise by the angle θ with respect to the origin. And let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which reflects a vector with respect to the line $x_1 = x_2$. Find the standard matrix A of $S \circ R_\theta$.

43. Find all subspaces of $\mathbb{R}^3$ that contain vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

44. If the columns of a 3×3 matrix A span $\mathbb{R}^3$, do the rows of A have to span $\mathbb{R}^3$? Justify your answer.

45. Consider the set of all matrices $A \in M_{2\times 2}$ such that $A^2 = 0$. Is this a subspace of $M_{2\times 2}$? Justify your answer.

46. Recall that for every homogeneous linear system given by a row

reduced echelon form, the solution set is

$$\{x_1 v_1 + x_2 v_2 + \cdots x_k v_k : \alpha_1, \ldots, \alpha_k \in \mathbb{R}\}$$

where $x_1, \ldots, x_k$ are free variables. Show that $S = \{v_1, \ldots, v_k\}$ is linearly independent.

47. Consider the subspace S of $M_{2\times 2}(\mathbb{R})$ spanned by the set of all 2×2 matrices in RREF. Construct a basis of $M_{2\times 2}(\mathbb{R})$ that extends a basis of S.

48. Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ is a linearly independent set in $\mathcal{P}_4$. Each v_i is a polynomial in t. We know that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ belong to $\mathcal{P}_2$. What are all the possibilities for the degrees of $\vec{v}_4$ and $\vec{v_5}$ if $\deg(\vec{v}_4) \leq \deg(\vec{v_5})$?

49. Suppose $\mathbf{v} \in \mathbb{R}^2$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbb{R}^2$. Suppose the coordinate vector for $\mathbf{v}$ with respect to $\mathcal{B}$ is $\mathbf{v}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ What is the coordinate vector for $\mathbf{v}$ with respect to a new basis $\mathcal{C} = \{\mathbf{b}_1, \mathbf{b}_1 + 2\mathbf{b}_2\}$?

50. Find a basis of the subpace $V = \{p(t) \mid p(2) = 0\}$ of $\mathcal{P}_2$.

51. Consider a linear map $F : M_{2\times 2} \to M_{2\times 2}$ defined as $F(X) = X - X^t$ Find a basis of the Kernel of F.

52. Consider a linear map $F : M_{2\times 2} \to M_{2\times 2}$ defined as $F(X) = X - X^t$ Find a basis of the Range of F.

53. For every $a \in \mathbb{R}$ consider a linear function $F_a : P_2 \to P_2$ given by the formula

$$(F_a(p))(x) = xp'(x) - a \cdot p(x)$$

(the polynomials in P_2 are in the variable x and the derivative is with respect to x).

Find all values of a for which F_a is not injective.

54. Suppose A is a 3×3 matrix such that $A^2 = 0$. What are all possible values of $rk(A)$?