MATH 304 MIDTERM EXAM 2 SAMPLE PROBLEMS

Note: problems marked with asterisque (*) have multiple questions asked. An actual examination problem will only ask one of these questions.

Level: EASY

1. Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Find $A^2 - 3A + 2I_2$.

Solution. By direct calculation, the result is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

2. Is the following statement true? Justify your answer with clear explanation.

"If A and B are $m \times n$ matrices with $m \neq n$, then AB^T and $A^T B$ are both well-defined matrix products."

Solution. Yes. For the product of matrices to be well-defined the number of columns of the first factor must equal the number of rows of the second factor.

3. Let A be a 2×2 matrix. Solve for A from the following matrix equation.

$$5A - 4I_{2\times2} = 0_{2\times2}.$$

Solution. $A = \frac{1}{5}(4I_{2\times2} + 0_{2\times2}) = \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & \frac{4}{5} \end{bmatrix}$.

4. Is $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, 5z - 1 = x + 2y \right\}$ a vector space, under the usual addition and scalar multiplication operations?

Solution. No. Any of these explanations will suffice:

- Not closed under the addition: $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \in S$, but

$$\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \notin S$$

• Not closed under the scalar multiplication:: $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \in S$, but

$(-1)\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin S$

• Does not contain the zero vector, which would have to be $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

5. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear map satisfying $T(\vec{u_1}) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $T(\vec{u_2}) = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$. Find $T(2\vec{u_1} - \vec{u_2})$.

Solution. $T(2\vec{u_1} - \vec{u_2}) = 2\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$

6. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear map satisfying $T(\vec{u_1}) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $T(\vec{u_2}) = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$. Is there a vector $\vec{w} \in \mathbb{R}^2$ such that $T(\vec{w}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

Solution. No. $T(\vec{w}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin \mathbb{R}^3$.

7. Find the standard matrix A for the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates every vector counterclockwise by $\frac{\pi}{2}$ radians, then doubles the magnitude.

Solution. $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$. So the matrix is $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$.

8. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Determine if the vector $\mathbf{w} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

is in the span of the set of vectors $\{\mathbf{u}, \mathbf{v}\}$.

Solution. The vector $\mathbf{w} = \mathbf{v} - \mathbf{u}$, is a linear combination of the vectors in the set $\{\mathbf{u}, \mathbf{v}\}$, so $\mathbf{w}$ is in $\text{Span}(\{\mathbf{u}, \mathbf{v}\})$.

9. Do the vectors $\mathbf{v}_1 = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{v}_2 = \mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{v}_3 = \mathbf{e}_2 + \mathbf{e}_3$ span all of $\mathbb{R}^3$?

Solution.

Performing row reduction, one gets $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. So the rank of the matrix is 3, thus the column space must equal the entire $\mathbb{R}^3$. (**Note:** There are many other ways to finish the argument, once the RREF is obtained. Talk to your instructor to check if your reasonaing is correct.)

10. Let V be the set of vectors $\mathbf{v} \in \mathbb{R}^2$ where the sum of the components of $\mathbf{v}$ equals 2. Show that V is not a subspace of $\mathbb{R}^2$.

Solution. The set V can not be a subspace of $\mathbb{R}^2$ for many different reasons. One such reason is because the zero vector of $\mathbb{R}^2 \notin V$. **Note:** Come to office hours to see if you can explain some of the other reasons!

11. Which of the following sets of vectors are linearly independent in $\mathbb{R}^2$?

$$A = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad B = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},$$

$$C = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad D = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$$

Solution. Not linearly independent: A is not linearly independent because it contains the $\mathbf{0}$ vectors. D is not linearly independent, because the dimension of $\mathbb{R}^2$ is 2, so 3 vectors can never be linearly independent.

Linearly independent: B is linearly independent because a single non-zero vector is always linearly independent. C is linearly independent because the two vectors are not multiples of one another.

12. If two vectors $\mathbf{v}_1$, $\mathbf{v}_2 \in V$ are multiples of one another, can they be linearly independent? Explain.

Solution. No. If $\mathbf{v}_1 = c\mathbf{v}_2$, then $\mathbf{v}_1 - c\mathbf{v}_2$ is a dependence relation.

13. Are the vectors

$$\begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 5 \\ 0 \end{bmatrix}$$

linearly independent in $\mathbb{R}^4$?

Solution. Yes. When we put these vectors into a matrix: $\begin{bmatrix} 3 & 3 & 1 \\ 0 & 1 & 7 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ we get a pivot in each column. Therefore these vectors are linearly independent. (The row of zeros does not affect the answer to this problem. What it tells us is that these 3 vectors cannot span $\mathbb{R}^4$.)

14. What is the dimension of the space P_3 of polynomials of degree up to 3?

Solution. The dimension is 4: one possible basis is $\{1, x, x^2, x^3\}$.

15. Find the coordinates of the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in the basis $\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$.

Solution.

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

So $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, therefore the coordinates are $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

16. Construct a linearly independent collection of vectors in $\mathbb{R}^3$ that is not a basis of $\mathbb{R}^3$.

Solution. Any two vectors in $\mathbb{R}^3$ that are not multiples of each other will be linearly independent, but not a basis: any basis of $\mathbb{R}^3$ must have three elements.

17. Consider an augmented matrix A of a homogeneous linear system in a row reduced echelon form. Suppose that there are 3 leading variables. Then, what is the dimension of the column space of A?

Solution. The dimension of the column space of A should be 3. In fact, one can get a basis of the column space by taking the columns of A that corrsepond to the pivot columns in its $RREF$.

18. Find the dimension of the range space of a linear map T_A given by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Solution. Since the range space of T_A is equal to the column space of A, it is enough to find a basis of the column space and its size. One possible basis $\mathcal{B}$ consists of 1st, 2nd and 3rd column vectors of A. Note that $\mathcal{B}$ is linearly independent, since every vector in $\mathcal{B}$ cannot be represented by a linear combination of the other two vectors. We also have that $\mathcal{B}$ spans the column space of A, since $span(\mathcal{B})$ contains the 4th column vector of A. Therefore, $\mathcal{B}$ is a basis of the column space, and we can conclude that the column rank is 3.

(Alternatively, one can row-reduce A to an echelon form).

19. Determine whether the linear map T_A defined by a matrix

$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is a monomorphism (injective linear map).

Solution. By row reduction, we can check that the nullity of A is 1. This means that $\ker(T_A) \neq \{\vec{0}\}$, which implies that T_A is not a monomorphism.

20*. Given $\vec{u}, \vec{v} \in \mathbb{R}^3$ that span the subspace $S = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mid x,\ y \in \mathbb{R} \right\}$

of $\mathbb{R}^3$. Is the following set a basis of $\mathbb{R}^3$?

(a) $\left\{ \vec{u},\ \vec{v},\ \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \right\}$

(b) $\left\{ \vec{u}+\vec{v},\ \vec{v},\ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right\}$

(c) $\left\{ 2\vec{u},\ 3\vec{v},\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$

(d) $\{\vec{u}-\vec{v},\ \vec{u}+\vec{v}\}$

Solution.
(a) Yes. The third vector is not in S, so the vectors must span the whole $\mathbb{R}^3$. There are three of them, so it is a basis.
(b) No. All three vectors are in S, so the do not span $\mathbb{R}^3$.
(c) No. A basis of $\mathbb{R}^3$ must have three, vectors, not four.
(d) No. A basis of $\mathbb{R}^3$ must have three vectors, not two.

Level: MEDIUM

21. Find $(A+B)^tA$ if $A^tA = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$ and $A^tB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Hint. What is the formula for the transpose of the product of two matrices?

Solution. $(A+B)^tA = (A^t+B^t)A = A^tA + B^tA = A^tA + (A^tB)^t = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 3 & 9 \end{pmatrix}$.

22. Give an example of two matrices A and B such that AB is a zero

matrix and BA is undefined.

Hint. What is an easiest way to get all zeros in the product of matrices?

Solution. Take, for example, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

23. Find $(3A + I_2)B$ if $AB = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$, where A is a 2×2 matrix and $B = \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}$.

Hint. What is I_2?

Solution. $(3A + I_2)B = 3(AB) + I_2B = 3(AB) + B$. Plugging in the given matrices, we get $3\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 10 & 11 \end{pmatrix}$.

24. Is $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid xy = 0 \right\}$ a subspace of $\mathbb{R}^2$?

Hint. Check the conditions for a subset to be a subspace.

Solution. No. For example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are both in S, but their sum, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, is not in S.

25. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{ and } T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

What is the matrix A so that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$?

Hint. What is $T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

Solution. Since T is linear, we have that $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = T\begin{bmatrix} 1 \\ 1 \end{bmatrix} - T\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Thus, $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. The matrix A has columns $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$, so we have $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$.

26. Is the map $f : M_{2\times 2} \to \mathbb{R}^2$ with $f\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ linear?

Hint. What is $f(A+B)$ for two **matrices** A and B?

Solution. Yes, it is, because matrix multiplication is distributive. Suppose A and B are two matrices, and t is a constant.

First we check that $f(A+B) = f(A)+f(B)$. $f(A+B) = (A+B)\begin{bmatrix} 2 \\ 3 \end{bmatrix} = A\begin{bmatrix} 2 \\ 3 \end{bmatrix} + B\begin{bmatrix} 2 \\ 3 \end{bmatrix} = f(A) + f(B)$.

Then we check that $f(cA) = cf(A)$: $f(cA) = (cA)\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c\left(A\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = cf(A)$.

Therefore, f is a linear map.

27. Determine if the set H of all matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is a subspace of $\mathcal{M}_{2\times 2}$, the vector space of all 2×2 matrices with real number entries.

Hint. Express H as a span of some set of matrices in $\mathcal{M}_{2\times 2}$.

Solution. $H = span(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$.
Therefore H is a subspace of $\mathcal{M}_{2\times 2}$.
(Alternatively, one can check the conditions for a subset to be a subspace).

28. Determine if the set of all polynomials $p(t)$ of degree at most n satisfying $p(1) = 0$ is a subspace of $\mathbb{P}_n$.

Hint. Check the conditions for a subset to be a subspace.

Solution. Yes, it is a subspace.
1) Zero polynomial is in this subset.
2) If a polynomial $p(t)$ is in this subset, then any multiple of it also is: $(cp)(1) = cp(1) = c \cdot 0 = 0$.
3) If $p(t)$ and $q(t)$ are in this subset, then so is their sum: $(p+q)(1) = p(1) + q(1) = 0 + 0 = 0$.

29. Give an example of a vector in $\mathbb{R}^3$ that is not in $\text{Span}\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\}$.

Hint. You can try to express an arbitrary vector in $\mathbb{R}^3$ as a linear combination of the given two vectors.

Solution.

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -1 & b-a \\ 0 & 1 & c \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -1 & b-a \\ 0 & 0 & c+b-a \end{array}\right]$$

So if $c+b-a \neq 0$ the system is inconsistent, and the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is not in the span. In particular, we can take $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(Alternatively, one can choose a vector more or less randomly and check that it is not in the span).

30. Let $S = \{v_1, v_2, v_3\}$ be the set of three non-zero vectors from $\mathbb{R}^3$. Suppose that S is linearly dependent. Then, what are all possible geometric shapes of $span(S)$?

Hint. The dimension of $\mathbb{R}^3$ is three.

Solution. The span $span(S)$ cannot be a point, since all vectors in S are non-zero. It cannot be $\mathbb{R}^3$ either: Suppose otherwise. Then the span should be 3-dimensional. We can always find a basis $\mathcal{B}$ of $span(S) = \mathbb{R}^3$ as a subset of S, which must be S. But this leads to a contradiction, since S is linearly independent.

The span $span(S)$ can be a line (by setting $v_1 = v_2 = v_3$) or a plane (for example, by choosing $v_1 = (1,0,0)^t$, $v_2 = (0,1,0)^t$ and $v_3 = (1,1,0)^t$).

31. Find a vector v in $\mathbb{R}^3$ such that $\{(1,0,-1)^t, (0,-1,1)^t, v\}$ is linearly independent.

Hint. It might be helpful to use a dot product.

Solution. We denote the set $\{(1,0,-1)^t, (0,-1,1)^t\}$ by S. We claim that $v = (1,1,1)^t$ makes the desired result. Note that for every vector $w \in span(S)$, we have $w \cdot v = 0$ by the linearity of dot product. By this, we can conclude that $v \notin span(S)$ since v is a non-zero vector. We also have that S is linearly independent, since the two vectors are not multiples of each other. By these two conclusions, we can derive that the set $S \cup \{v\}$ is linearly independent.

(Alternatively, one can realize that this problem is similar to Problem 29 and solve it using the methods from the solution to Problem 29).

32. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is such that

$$T\begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}, \; T\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$$

Find the standard matrix of T.

Hint. The columns of the standard matrix are $T\vec{e}_1$ and $T\vec{e}_2$.

Solution. $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}T\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = T\begin{pmatrix} 1 \\ 1 \end{pmatrix} - T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

So $A = \begin{pmatrix} 3 & 0 \\ 5 & -3 \end{pmatrix}$.

33. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates each vector counterclockwise by $\frac{\pi}{4}$ radians. Let A be its standard matrix. Show that the columns of A form a basis of $\mathbb{R}^2$.

Hint. Find A and solve $A\vec{x} = \vec{0}$

Solution. $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

$$\left[\begin{array}{cc|c} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \end{array}\right]$$

So the columns are linearly independent, and since there are two of them they form a basis of $\mathbb{R}^2$.

(Alternatively, one can note that the columns are not multiples of each other because they are obtained by rotation of two vectors that are not multiples of each other.)

34. Let A be the coefficient matrix for a system of 3 homogeneous equations in 3 variables. If the system has infinitely many solutions, what are the possible dimensions of the space spanned by the columns of A?

Hint. Recall the Rank-Nullity Theorem.

Solution. Rank-Nullity Theorem for the matrix A implies that $rk(A) + nullity(A) = 3$. Because the system has infinitely many solutions, the nullity of the matrix is at least 1. By the Rank-Nullity Theorem, $rk(A) \leq 3 - 1 = 2$. So $rk(A)$, which is the dimension of the Column Space of A can only be 0, 1, or 2. It isnot hard to construct example for each of these possibilities:

$$rk(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}) = 0, \; rk(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}) = 1, \; rk(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}) = 2$$

(Instead of formally using the Rank-Nullity Theorem, one can also consider the number of pivot and free variables in the RREF of the system).

35. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Find a basis for the column space of A.

Hint. Find a set of linearly independent columns of A.

Solution. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. The pivots indicate the location of the independent columns of A. The vectors $\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\}$ form a basis for $\mathrm{Col}(A)$.

36. Let $A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$. Find a basis for the null space of A.

Hint. Find the general solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

Solution. $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The parametric form of the general solution is $\mathbf{x} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$, for $s, t \in \mathbb{R}$. The vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} \right\}$ form a basis for $\mathrm{Nul}(A)$.

37. Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation with the following action on the standard basis vectors:

$$\begin{array}{lcl} \mathbf{e}_1 & \mapsto & \mathbf{e}_2 + 2\mathbf{e}_3 - 3\mathbf{e}_4 \\ \mathbf{e}_2 & \mapsto & 3\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3 \end{array}.$$

Find the kernel of T.

Hint. The kernel of T is also the null space of the standard matrix for T.

Solution. $\begin{bmatrix} 0 & 3 \\ 1 & -2 \\ 2 & 1 \\ -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Since there is a pivot in every column, T is injective. The only vector in $\text{Ker}(T)$ is $\mathbf{0}$.

38*. Let $M_{2\times 2}$ be the vector space of all 2×2 matrices. Define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^T$.
(a) Show that T is a linear transformation.
(b) Let B be in $M_{2\times 2}$ such that $B = B^T$. Find an A such that $T(A) = B$.
(c) Show that the range of T is the set of B such that $B = B^T$.
(d) Describe the kernel of T.

Hints. a) Check the conditions for a transformation to be linear.
b,c,d) Write down T explicitly for an arbitray matrix A.

Solution. a) $T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) = T\left(\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} + \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix} = \ldots = T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) + T\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right)$.

The scalar multiple portion works as expected as well.

b) If $B = B^T$, then $T(B) = B + B^T = B + B = 2B$. Choose $A = 0.5B$ so that $T(A) = 0.5T(B) = (0.5)(2)B = B$.

c) Let B be in the range of T. Then $B = T(A)$ for some $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in $M_{2\times 2}$. We can calculate $B = T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} \\ a_{12}+a_{21} & a_{22}+a_{22} \end{bmatrix}$. Now it is easy to see that B^T is equal to B.

d) $T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} \\ a_{12}+a_{21} & a_{22}+a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ means, $a_{11} = 0 = a_{22}$ and $a_{12} = -a_{21}$. So the kernel is the set of matrices of the from $\begin{bmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{bmatrix}$.

Level: HARD

39. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$. Find a matrix B such that $AB = I_2$ or show that such a B does not exist.

Hint. Because A is 2×3, B must a 3×2 for the product to be 2×2. Use arbitrary values for the entries in B and then solve for them.

Solution. Let $B = \begin{bmatrix} x_1 & x_4 \\ x_2 & x_5 \\ x_3 & x_6 \end{bmatrix}$ so that $AB = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_4 \\ x_2 & x_5 \\ x_3 & x_6 \end{bmatrix} =$ $\begin{bmatrix} x_1 + 2x_2 + 3x_3 & x_4 + 2x_5 + 3x_6 \\ 3x_1 + 2x_2 + x_3 & 3x_4 + 2x_5 + x_6 \end{bmatrix}$.

If $AB = I_2$, then $\begin{bmatrix} x_1 + 2x_2 + 3x_3 & x_4 + 2x_5 + 3x_6 \\ 3x_1 + 2x_2 + x_3 & 3x_4 + 2x_5 + x_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ can be thought of as a system of 4 equations in 6 unknowns. The augmented matrix for this system is $\left[\begin{array}{cccccc|c} 1 & 2 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & -0.5 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0.75 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & 2 & -0.25 \end{array}\right]$.

There are 4 pivots and 2 free variables, $x_3 = s$ and $x_6 = t$. The parametric form of the solution space is $\begin{bmatrix} -0.5 \\ 0.75 \\ 0 \\ 0.5 \\ -0.25 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$. Since we only need to provide one such matrix, we can choose the parameters $s = t = 0$. Then $B = \begin{bmatrix} -0.5 & 0.5 \\ 0.75 & -0.25 \\ 0 & 0 \end{bmatrix}$.

40. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$. Find a matrix B such that $BA = I_3$ or show that such a B does not exist.

Hint. Because A is 2×3, B must a 3×2 for the product to be 3×3. Use arbitrary values for the entries in B and then solve for them.

Solution. Again we need B to be $3{\times}2$. This time $BA = \begin{bmatrix} x_1 & x_4 \\ x_2 & x_5 \\ x_3 & x_6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} =$ $\begin{bmatrix} x_1 + 3x_4 & 2x_1 + 2x_4 & 3x_1 + x_4 \\ x_2 + 3x_5 & 2x_2 + 2x_5 & 3x_2 + x_5 \\ x_3 + 3x_6 & 2x_3 + 2x_6 & 3x_3 + x_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. After it is set up, this will lead to an inconsistent system with 9 equations and 6 unknowns. Hence there is no such matrix B that has the property $BA = I_3$.

Note: This is not the only solution. Ask your instructor for a "better", more conceptual, solution!

41. Let A and B be two 2×2 matrix such that $AB = 0$. Explain why there is a vector $\mathbf{x} \neq \mathbf{0}$ such that $BA\mathbf{x} = \mathbf{0}$

Hint. When B is not the zero matrix try to find a nonzero vector in the column space of B that A can be applied to.

Solution. If B is the zero matrix, then for every $\mathbf{x}$ we have $BAx = \mathbf{0}$. It is also possible that B is not the zero matrix. In that case then we can think of B as the standard matrix for some transformation. At least one of the standard basis vectors will be mapped to a non-zero vector, $B\mathbf{e} = \mathbf{x} \neq 0$. Now consider $BA\mathbf{x} = BA(B\mathbf{e}) = B(AB)\mathbf{e} = B\mathbf{0}\mathbf{e} = \mathbf{0}$.

Note: This is not the only solution. If you think that you found a different one, check with your instructor to see if your argument is correct.

42. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which rotates a vector counter-clockwise by the angle θ with respect to the origin. And let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which reflects a vector with respect to the line $x_1 = x_2$. Find the standard matrix A of $S \circ R_\theta$.

Hint. Composition of linear maps corresponds to multiplication of standard matrices.

Solution. The columns of the standard matrix of a linear map R_θ are

$R_\theta(\vec{e}_1)$ and $R_\theta(\vec{e}_2)$. So the standard matrix of R_θ is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Likewise, the matrix of S is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. So the matrix of $S \circ R_\theta$ is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

43. Find all subspaces of $\mathbb{R}^3$ that contain vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Hint. Span of the vectors is the smallest subspace that contains the vectors.

Solution. The two vectors are linearly independent, so their span is a plane. One can easily see that it is a plane $x_1 + x_2 + x_3 = 0$. Any strictly bigger subspace that contains these vectors must have dimension strictly greater than 2. So it must be the whole space $\mathbb{R}^3$.

44. If the columns of a 3×3 matrix A span $\mathbb{R}^3$, do the rows of A have to span $\mathbb{R}^3$? Justify your answer.

Hint. What is the relation between $rk(A)$ and $rk(A^t)$?

Solution. Recall that $rk(A) = rk(A^t)$, that is the dimension of the row space of A must equal the dimension of the column space of A, so it is 3. Thus the rows of A must span $\mathbb{R}^3$ because any smaller span would have smaller dimension.

(Alternatively, one can argue that the RREF of A is the identity matrix).

Note. The difficulty in this problem is in the correct justification, not in finding the answer. Depending on what was covered in your section and and what your instructor accepts as a correct argument, this problem may be Medium or even Easy difficulty level for your section.

45. Consider the set of all matrices $A \in M_{2\times2}$ such that $A^2 = 0$. Is this a subspace of $M_{2\times2}$? Justify your answer.

Hint. Find a non-zero matrix A such that $A^2 = 0$.

Solution. This is not a subspace: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ are in the set, but their sum is not.

46. Recall that for every homogeneous linear system given by a row reduced echelon form, the solution set is

$$\{x_1v_1 + x_2v_2 + \cdots x_kv_k : \alpha_1, \ldots, \alpha_k \in \mathbb{R}\}$$

where $x_1, \ldots, x_k$ are free variables. Show that $S = \{v_1, \ldots, v_k\}$ is linearly independent.

Hint. Just consider the coordinates corresponding to free variables.

Solution. Suppose that $t_1v_1 + t_2v_2 + \cdots + t_kv_k = 0$. We need to show that $t_1 = \cdots = t_k = 0$. Denote $t_1v_1 + t_2v_2 + \cdots t_kv_k$ by w. Also, let us assume that a free variable x_i appears at j_ith coordinate.

Fix an arbitrary $i \in \{1, \ldots, k\}$ and consider the j_ith coordinate of w. It must be

$$t_1 \cdot 0 + \cdots + t_i \cdot 1 + \cdots + t_k \cdot 0$$

which is equal to t_i. By the assumption, we have $t_i = 0$. Since this holds for every $i \in \{1, \ldots, k\}$ we can conclude that S is linearly independent.

47. Consider the subspace S of $M_{2\times 2}(\mathbb{R})$ spanned by the set of all 2×2 matrices in RREF. Construct a basis of $M_{2\times 2}(\mathbb{R})$ that extends a basis of S.

Hint. Find all possible RREF 2×2 matrices.

Solution. Based on which columns are pivot, all possible RREF are the following:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

One can show that these matrices span the space of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$.

So one possible basis of $M_{2\times 2}(\mathbb{R})$ that extends a basis of S is the following:

$$\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\}$$

Here the first three matrices form a basis of S.

48. Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ is a linearly independent set in $\mathcal{P}_4$. Each v_i is a polynomial in t. We know that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ belong to $\mathcal{P}_2$. What are all the possibilities for the degrees of $\vec{v}_4$ and $\vec{v_5}$ if $\deg(\vec{v}_4) \leq \deg(\vec{v_5})$?

Hint. Can all five linearly independent polynomials lie in $\mathcal{P}_3$?

Solution. Because $\dim(\mathcal{P}_3) = 4$, it cannot contain all five linearly independent polynomials. So the largest degree, which is $\deg(\vec{v_5})$ must be 4. Also, $\deg(\vec{v}_4)$ cannot be less than 3, because this would give four linearly independent polynomials $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ in a three-dimensional space $\mathcal{P}_2$. So $\deg(\vec{v}_4)$ can only be 3 or 4. Both of these are possible: we can have, for example, the following polynomials: $\{1, t, t^2, t^3, t^4\}$ or $\{1, t, t^2, t^3 + t^4, t^4\}$.

49. Suppose $\mathbf{v} \in \mathbb{R}^2$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbb{R}^2$. Suppose the coordinate vector for $\mathbf{v}$ with respect to $\mathcal{B}$ is $\mathbf{v}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ What is the coordinate vector for $\mathbf{v}$ with respect to a new basis $\mathcal{C} = \{\mathbf{b}_1, \mathbf{b}_1 + 2\mathbf{b}_2\}$?

Hint. Recall the definition of the coordinate vector in the language of linear combinations.

Solution. We need to find constants x and y so that $\mathbf{v} = x\mathbf{b}_1 + y(\mathbf{b}_1 + 2\mathbf{b}_2)$ Distributing and rearranging, we get $\mathbf{v} = (x + y)\mathbf{b}_1 + 2y\mathbf{b}_2$ Since $\mathbf{v}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, we know that $\mathbf{v} = 2\mathbf{b}_1 + 4\mathbf{b}_2$ Because $\mathcal{B}$ is a basis, these are the only coefficients that work, so $x + y = 2, 2y = 4$ We solve this system to see that $x = 0$ and $y = 2$. Thus, $\mathbf{v}_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Alternatively, one can express the vector $\mathbf{b}_2$ in terms of the new basis: $\mathbf{b}_2 = \frac{1}{2}((\mathbf{b}_1 + 2\mathbf{b}_2) - \mathbf{b}_1)$. Then

$$\mathbf{v} = 2\mathbf{b}_1 + 4\mathbf{b}_2 = 2\mathbf{b}_1 + 2((\mathbf{b}_1 + 2\mathbf{b}_2) - \mathbf{b}_1) = 2(\mathbf{b}_1 + 2\mathbf{b}_2)$$

50. Find a basis of the subpace $V = \{p(t) \mid p(2) = 0\}$ of $\mathcal{P}_2$.

Hint. Can you find a pair of linearly independent polynomials in this set?

Solution. We know that $p(t) = t - 2$ and $q(t) = (t-2)^2$ are both in V. Moreover, they are linearly independent, since if $a(t-2) + b(t-2)^2 = 0$, then $a = b = 0$. Because V is not the whole three-dimensional space $\mathcal{P}_2$, $dim(V) \leq 2$. Since these two polynomials are linearly independent, they must form a basis. So we can take $\mathcal{B} = \{t-2, (t-2)^2\}$.

Alternatively, one can identify $\mathcal{P}_2$ with the $\mathbb{R}^3$ by looking at the coefficients of the polynomials and find a basis of V as a plane in $\mathbb{R}^3$. Make sure to rewrite your answer in terms of polynomials.

51. Consider a linear map $F : M_{2\times 2} \to M_{2\times 2}$ defined as $F(X) = X - X^t$ Find a basis of the Kernel of F.

Hint. Describe the Kernel of F.

Solution. $Ker(F) = \{X \in M_{2\times 2} \mid X - X^t = 0\} = \{X \in M_{2\times 2} \mid X^t = X\}$. So it is the space of symmetric matrices, that is matrices of the form $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$, where $a, b, d \in \mathbb{R}$. Clearly, the following is a basis of this space.

$$\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

52. Consider a linear map $F : M_{2\times 2} \to M_{2\times 2}$ defined as $F(X) = X - X^t$ Find a basis of the Range of F.

Hint. Write down $F(X)$ for an arbitray matrix X.

Solution. $F(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}$. So $\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}$ is a basis of Range(F).

53. For every $a \in \mathbb{R}$ consider a linear function $F_a : P_2 \to P_2$ given by the

formula

$$(F_a(p))(x) = xp'(x) - a \cdot p(x)$$

(the polynomials in P_2 are in the variable x and the derivative is with respect to x).

Find all values of a for which F_a is not injective.

Hint. Write down $(F_a(p))(x)$ for an arbitrary quadratic polynomial $p(x)$.

Solution. For $p(x) = c_0 + c_1x + c_2x^2$,

$$(F_a(p))(x) = -ac_0 + (1-a)c_1x + (2-a)c_2x^2$$

For $a \neq 0, 1, 2$ the Kernel of F_a is clearly $\{0\}$, so the map is injective. For $a = 0$, the Kernel contains $\{1\}$, for $a = 1$ it contains x, and for $a = 2$ it contains x^2. So F_a is not injective for a equal to 0, 1, or 2.

54. Suppose A is a 3×3 matrix such that $A^2 = 0$. What are all possible values of $rk(A)$?

Hint. Consider the Range and the Kernel of the linear transformation defined by A.

Solution. Because $A^2 = 0$, for every $\vec{x} \in \mathbb{R}^3$ we have

$$A(A\vec{x}) = (A \cdot A)\vec{x} = \vec{0}$$

So $A\vec{y} = \vec{0}$ for every $\vec{y}$ of the form $A\vec{x}$. In other words, Column Space of A is a subspace of the Null Space of A. This implies that $rk(A) \leq nullity(A)$. By the Rank-Nullity Theorem, $nullity(A) = 3 - rk(A)$, so $rk(A)$ can only be 0 or 1. Both values of rank are possible: A can be one of the following two matrices:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$