MATH 304 MIDTERM EXAM 3 SAMPLE PROBLEMS

Group 1

Level: EASY

1. Given points $A = (2, 0, 0, -1)$ and $B = (1, 0, 1, 1)$, find the length of the vector $\vec{AB}$.

2. Let $A = \begin{bmatrix} 1 & -3 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$
and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ via $\mathbf{x} \mapsto A\mathbf{x}$.
Find a particular vector $\mathbf{x}$ whose image under T is $\mathbf{b}$.

3. Can one 2×3 matrix have both of these REFs?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

4. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that $L((1,1)^t) = (1,0,2)^t$ and $L((2,3)^t) = (1,-1,4)^t$. What is $L((8,11)^t)$?

5. Find the rank of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix}$.

Level: MEDIUM

6. Are the following two linear systems equivalent?

$$\begin{cases} x_1 + x_2 &= 0 \\ x_2 - x_3 &= 0 \\ x_1 + x_3 &= 0 \end{cases}$$

and

$$\begin{cases} x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \\ x_1 + x_3 &= 0. \end{cases}$$

7. Let V be a finite dimensional vector space with subspaces H and K. Define $H + K = \{\mathbf{v} \in V | \mathbf{v} = \mathbf{h} + \mathbf{k}\}$, the set of vectors in V that can be written as a sum of two vectors, one in H and the other in K.

Show $H + K$ is a subspace of V.

8. Can a 3×3 matrix have exactly three different REFs?

9. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$. Choose vectors among column vectors of A which form a basis of the column space of A.

10. Suppose $\{\vec{v}_1, \vec{v}_2\}$ is a basis of some vector space V. Suppose $\vec{v}_3 = 2\vec{v}_1 + 3\vec{v}_2$ and $\vec{v}_4 = 3\vec{v}_1 - \vec{v}_2$. Find the coordinates of $\vec{v}_4$ in the basis $\{\vec{v}_2, \vec{v}_3\}$.

Level: HARD

11. Is there a non-zero 3×3 matrix A such that for all natural n $rank(A^n) = (rank(A))^n$?

12. Let A be a 2×4 matrix such that $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has no solution.

List all possible dimensions for the column space of A and all possible dimensions for the null space of A.

13. Show that for every square matrix A we have $rk(A^2) \leq rk(A)$.

14. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with $L((1,0)^t) = (1,4)^t$ and $L((1,1)^t) = (2,5)^t$. Is L an injection?

15. Is the following set a subspace of $\mathbb{M}_{3\times 3}$?

$$S = \{A \in M_{3\times 3} \mid A^2 = 0\}$$

Group 2

Level: EASY

16. Use the determinant to decide if the list of vectors below is linearly independent:

$$\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

17. Let $\mathbf{u} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

Compute the area of the parallelogram determined by $\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}$ and $\mathbf{0}$.

18. Find the inverse of $A = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$.

19. Compute $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix}$.

20. Find the matrix of the linear transformation $T : P_1 \to P_1$ that sends $p(t)$ to $p'(t)$ in the basis $\mathcal{B} = \{t, 1\}$.

Level: MEDIUM

21. Are the following two matrices similar?

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

22. Let A, B, C be square matrices of the same size such that

$$\det(A) = \frac{1}{2}, \quad \det(B) = 2, \quad \det(C) = \sqrt{2}.$$

Compute $\det\left(A^T B^2 C^{-1}\right)$.

23. Consider the linear transformation $F : M_{2\times 2} \to M_{2\times 2}$ defined as follows.

$$F(X) = 2X - Tr(X)I_2$$

Find the matrix of F with respect to the input basis

$$\mathcal{B}_1 = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

and the output basis

$$\mathcal{B}_2 = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

24. Find all possible values of $\det(C)$ if $C^2 = \begin{pmatrix} 5 & 6 & 5 \\ 5 & 6 & 6 \\ 6 & 4 & 6 \end{pmatrix}$.

25. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Find an ordered basis $\mathfrak{B}$ such that $A = [id_{\mathbb{R}^2}]_{\mathfrak{B}}^{\mathcal{E}}$ where $\mathcal{E}$ is the standard basis of $\mathbb{R}^2$, and $id_{\mathbb{R}^2}$ is an identity map on $\mathbb{R}^2$. (Here the notation means that $\mathcal{E}$ is the basis of the domain and $\mathfrak{B}$ is the basis of the codomain).

Level: HARD

26. Let A be the matrix below:

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Write A^{-1} as a product of elementary matrices.

27. Let A be a square matrix such that the Gram matrix for A is I_n. What are all of the possibilities for $\det(A)$?

28. Show that the inverse of a square matrix, if exists, is unique.

29. Suppose that $A \sim B$ and $C \sim D$. Then does $AC \sim BD$ always hold? Justify your reason.

30. Find the matrix of a linear transformation $F : P^2 \to P^2$ that sends $f(t)$ to $(t+2)f'(t)$ with respect to the input basis $\mathcal{B}_1 = \{1, t+2, t^2\}$ and the output basis $\mathcal{B}_2 = \{t, t+1, t^2+4\}$.

Group 3

Level: EASY

31. Let A and B be $n \times n$ orthogonal matrices. Is $A + B$ orthogonal?

32. Let $A = \begin{bmatrix} 3 & -2 & -2 \\ -4 & 1 & 2 \\ 8 & -4 & -5 \end{bmatrix}$.

Check that $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is an eigenvector of A.

33. Consider the matrices P and D are given below and let $A = PDP^{-1}$.

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

What are eigenvalues of A?

34. Let $v = (1, 2, 3)$ and $w = (1, -3, 2)$. Compute $proj_w(v)$.

35. Is the following matrix orthogonally diagonalizable? $A = \begin{bmatrix} 3 & 4 & 3 \\ 4 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$.

Level: MEDIUM

36. Let $A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. Find all the eigenvalues λ_i, of A, and the corresponding algebraic multiplicities.

37. Let $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$.

Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

38. Consider the matrices P and D as given below and let $A = PDP^{-1}$.

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Calculate, with justification, eigenvalues of A^{-1}.

39. Consider $A = B^t B$, where $B = \begin{bmatrix} 1 & 2 & 1 \\ 5 & 3 & 0 \end{bmatrix}$. Is A orthogonally diagonalizable?

40. Can we find a basis of $\mathbb{R}^2$ which consists of eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix}$?

Level: HARD

41. Let $U_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $U_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $Y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$.

Write Y as the sum of two vectors, one in $\mathcal{W} = \mathrm{Span}\{\mathrm{U}_1, \mathrm{U}_2\}$ and the other orthogonal to $\mathcal{W}$.

42. Let $A = \begin{bmatrix} 3 & -2 & -2 \\ -4 & 1 & 2 \\ 8 & -4 & -5 \end{bmatrix}$, and let $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ be an (ordered) eigenbasis for $\mathbb{R}^3$.

Compute A^n for all $n \geq 0$.

43. Consider the matrices P and D given below and let $A = PDP^{-1}$.

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Calculate A^n for all natural n.

44. Show that no $n+1$ nonzero vectors in $\mathbb{R}^n$ can be mutually orthogonal to one another.

45. Is the following matrix A diagonalizable? $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

46. Orthogonally diagonalize $A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ or explain why it is impossible.