MATH 304 MIDTERM EXAM 3 SAMPLE PROBLEMS

Group 1
Level: EASY

1. Given points $A = (2, 0, 0, -1)$ and $B = (1, 0, 1, 1)$, find the length of the vector $\vec{AB}$.

Solution. $|\vec{AB}| = \sqrt{(1-2)^2 + (0-0)^2 + (1-0)^2 + (1-(-1))^2} = \sqrt{6}$.

2. Let $A = \begin{bmatrix} 1 & -3 & -1 \\ -2 & 1 & 2 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$ and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ via $\mathbf{x} \mapsto A\mathbf{x}$.

Find a particular vector $\mathbf{x}$ whose image under T is $\mathbf{b}$.

Solution.
The general solution can be found by row reducing $\left[\begin{array}{ccc|c} 1 & -3 & -1 & -4 \\ -2 & 1 & 2 & -7 \end{array}\right]$ to $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 0 & 3 \end{array}\right]$. Use $\begin{cases} x_1 = 5 + t \\ x_2 = 3 \\ x_3 = t \end{cases}$ and pick any $t \in \mathbb{R}$.
One possible vector is $\mathbf{x} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$.

3. Can one 2×3 matrix have both of these REFs?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution. No. These REFs lead to different RREF, because their pivot column numbers are different (1st and 2nd in the first matrix and 1st and 3rd in the second). Alternatively, the first and second columns of the second matrix are multiples of each other, and the columns of the first matrix are not.

4. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that $L((1,1)^t) = (1,0,2)^t$ and $L((2,3)^t) = (1,-1,4)^t$. What is $L((8,11)^t)$?

Soliution. $L((8,11)^t) = L(2\cdot(1,1)^t + 3\cdot(2,3)^t) = 2\cdot L((1,1)^t) + 3\cdot L((2,3)^t)$ $= 2 \cdot (1,0,2)^t + 3 \cdot (1,-1,4)^t = (2,0,4)^t + (3,-3,12)^t = (5,-3,16)^t$.

5. Find the rank of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix}$.

Solution.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the rank of A is 2.

Level: MEDIUM

6. Are the following two linear systems equivalent?

$$\begin{cases} x_1 + x_2 & = & 0 \\ x_2 - x_3 & = & 0 \\ x_1 + x_3 & = & 0 \end{cases}$$

and

$$\begin{cases} x_1 + 2x_2 - x_3 & = & 0 \\ 2x_1 + x_2 + x_3 & = & 0 \\ x_1 + x_3 & = & 0. \end{cases}$$

Hint: Write the augmented matrix of both matrices and row reduce until RREF.

Solution. The augmented matrices of both linear systems row-reduce to the same RREF: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Since row operations are reversible, we can row reduce the first linear system to the RREF and then apply the inverse of the row operations applied to the second linear system to get the second linear system back from the RREF. Therefore, the two linear systems are row-equivalent.

7. Let V be a finite dimensional vector space with subspaces H and K. Define $H+K=\{\mathbf{v}\in V|\mathbf{v}=\mathbf{h}+\mathbf{k}\}$, the set of vectors in V that can be written as a sum of two vectors, one in H and the other in K.

Show $H+K$ is a subspace of V.

Hint: Check the conditions for a subset to be a subspace.

Solution.

$H+K$ is nonempty. Consider $\mathbf{0}_H+\mathbf{0}_K=\mathbf{0}_V$.

$H+K$ is closed under vector addition. Let $\mathbf{v}_1=\mathbf{h}_1+\mathbf{k}_1$ and $\mathbf{v}_2=\mathbf{h}_2+\mathbf{k}_2$. Then $\mathbf{v}_1+\mathbf{v}_2=(\mathbf{h}_1+\mathbf{k}_1)+(\mathbf{h}_2+\mathbf{k}_2)=(\mathbf{h}_1+\mathbf{h}_2)+(\mathbf{k}_1+\mathbf{k}_2)$, a sum of a vector in H with a vector in K.

$H+K$ is closed under scalar multiplication. For $\mathbf{v}=\mathbf{h}+\mathbf{k}$ and $c\in\mathbb{R}$, $c\mathbf{v}=c(\mathbf{h}+\mathbf{k})=c\mathbf{h}+c\mathbf{k}$, a sum of a vector in H with a vector in K.

8. Can a 3×3 matrix have exactly three different REFs?

Hint: If you have one REF, how can you generate more?

Solution. No. If a matrix has a non-zero REF, then it has infinitely many more, by multiplying one of its non-zero rows (or the whole matrix) by a non-zero number. And if it has a zero REF, then it is a zero matrix, and it has only one REF.

9. Let $A=\begin{bmatrix}1&-1&0\\0&1&-1\\-1&0&1\end{bmatrix}$. Choose vectors among column vectors of A which form a basis of the column space of A.

Hint: Use row reduction.

Solution. $A=\begin{bmatrix}1&-1&0\\0&1&-1\\-1&0&1\end{bmatrix}\to\begin{bmatrix}1&-1&0\\0&1&-1\\0&-1&1\end{bmatrix}\to\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&0\end{bmatrix}$

So $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis of the column space of A.

10. Suppose $\{\vec{v}_1, \vec{v}_2\}$ is a basis of some vector space V. Suppose $\vec{v}_3 = 2\vec{v}_1 + 3\vec{v}_2$ and $\vec{v}_4 = 3\vec{v}_1 - \vec{v}_2$. Find the coordinates of $\vec{v}_4$ in the basis $\{\vec{v}_2, \vec{v}_3\}$.

Hint: Start by solving the first equation for $\vec{v}_1$.

Solution. From the first equation, $\vec{v}_1 = \frac{1}{2}\vec{v}_3 - \frac{3}{2}\vec{v}_2$. Plugging this into the second equation, we get $\vec{v}_4 = 3(\frac{1}{2}\vec{v}_3 - \frac{3}{2}\vec{v}_2) - \vec{v}_2 = -\frac{11}{2}\vec{v}_2 + \frac{3}{2}\vec{v}_3$. So the coordinates of $\vec{v}_4$ in the basis $\{\vec{v}_2, \vec{v}_3\}$ are $\begin{bmatrix} -11/2 \\ 3/2 \end{bmatrix}$.

Level: HARD

11. Is there a non-zero 3×3 matrix A such that for all natural n $rank(A^n) = (rank(A))^n$?

Hint: What are the possibilities for $rank(A)$ that can make this possible?

Solution. Yes. Since the rank of a 3×3 matrix is at most 3, we need $(rank(A))^n$ to always be at most 3. So the only possibility is for $rank(A) = 1$ or $rank(A) = 0$. In fact, since A is non-zero, we must have $rank(A) = 1$. This helps to find an example. If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then $A^n = A$ for each n, and the rank of A^n is always $1 = (rank(A))^n$.

12. Let A be a 2×4 matrix such that $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has no solution.

List all possible dimensions for the column space of A and all possible dimensions for the null space of A.

Hint: $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has no solution implies that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin \text{Col}(A)$.

Solution. Since A has 2 rows, the largest possible rank of A could be 2. Since the columns of A do not span $\mathbb{R}^2$, the only possible dimensions of Col(A) are 0 and 1. This means that A could have either 3 or 4 non-pivot columns, which are the only possible dimensions on Nul(A).

13. Show that for every square matrix A we have $rk(A^2) \leq rk(A)$.

Hint. What is the relation between the range spaces of the transformations defined by A^2 and A?

Solution. If $\vec{v} = A^2\vec{x}$, then $\vec{v} = A\vec{y}$, where $y = A\vec{x}$. So the Range of the transformation defined by A^2 is contained in that for A. The rank of a matrix is the dimension of the Range of the corresponding linear transformation, so $rk(A^2) \leq rk(A)$.

14. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with $L((1,0)^t) = (1,4)^t$ and $L((1,1)^t) = (2,5)^t$. Is L an injection?

Hint: Use the dimension theorem.

Solution. Because $(1,4)^t$ and $(2,5)^t$ are not multiples of each other, they span $\mathbb{R}^2$. Thus $Range(L) = \mathbb{R}^2$. So $dim(Ker(L)) = 2 - 2 = 0$, and so L is an injection.

15. Is the following set a subspace of $\mathbb{M}_{3\times 3}$?

$$S = \{A \in M_{3\times 3} \mid A^2 = 0\}$$

Hint: Look for non-trivial examples of such A.

Solution. It is not a subspace: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in S, and so is A^t. But $A + A^t = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin S$.

Group 2

Level: EASY

16. Use the determinant to decide if the list of vectors below is linearly independent:

$$\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solution.

$$\begin{vmatrix} 3 & -2 & 1 \\ 2 & 4 & 0 \\ 4 & 0 & 0 \end{vmatrix} = -16 \neq 0.$$

Therefore, the set of vectors is linearly independent.

17. Let $\mathbf{u} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

Compute the area of the parallelogram determined by $\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}$ and $\mathbf{0}$.

Solution. The matrix $A = [\mathbf{uv}]$ produces a shear transformation of the unit square into a parallelogram. The determinant of A, $\begin{vmatrix} 4 & 2 \\ 0 & 5 \end{vmatrix} = 20$, gives the area of the parallelogram.

18. Find the inverse of $A = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$.

Solution. We use that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible, then its inverse is

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Then we see that

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 4 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/4 \end{pmatrix}$$

19. Compute $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix}$.

Solution.

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 5 + 0 = 5.$$

20. Find the matrix of the linear transformation $T : P_1 \to P_1$ that sends $p(t)$ to $p'(t)$ in the basis $\mathcal{B} = \{t, 1\}$.

Solution. $T(t) = 1 = 0 \cdot t + 1 \cdot 1$; $T(1) = 0 = 0 \cdot t + 0 \cdot 1$. Putting the coordinates of $T(t)$ and $T(1)$ as columns, we get the matrix of T in the basis $\mathcal{B}$ as $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Level: MEDIUM

21. Are the following two matrices similar?

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hint: Similar matrices have the same rank.

Solution. A has rank 1 while B has rank 2. Therefore, the two matrices are not similar.

22. Let A, B, C be square matrices of the same size such that

$$\det(A) = \frac{1}{2}, \quad \det(B) = 2, \quad \det(C) = \sqrt{2}.$$

Compute $\det\left(A^T B^2 C^{-1}\right)$.

Hint: The determinant of a product equals the product of the determinants.

Solution. $\det\left(A^T B^2 C^{-1}\right) = \det\left(A^T\right)\det\left(B^2\right)\det\left(C^{-1}\right) = \left(\frac{1}{2}\right)(4)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$.

23. Consider the linear transformation $F : M_{2\times 2} \to M_{2\times 2}$ defined as follows.

$$F(X) = 2X - Tr(X)I_2$$

Find the matrix of F with respect to the input basis

$$\mathcal{B}_1 = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

and the output basis

$$\mathcal{B}_2 = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

Hint: Find where F sends the matrices from $\mathcal{B}_1$ and express the outputs in the basis $\mathcal{B}_2$.

Solution.

$$F(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix}$$

24. Find all possible values of $\det(C)$ if $C^2 = \begin{pmatrix} 5 & 6 & 6 \\ 5 & 6 & 5 \\ 6 & 4 & 6 \end{pmatrix}$.

Hint: Find determinant of C^2 first.

Solution. We use that $det(C^2) = (det(C))^2$. So first we'll compute $det(C^2)$: Since we can do a replacement row operation to get

$$\begin{pmatrix} 5 & 6 & 5 \\ 5 & 6 & 6 \\ 6 & 4 & 6 \end{pmatrix} \to \begin{pmatrix} 5 & 6 & 5 \\ 0 & 0 & 1 \\ 6 & 4 & 6 \end{pmatrix}$$

we see that the determinants of these two matrices are the same. Now, using cofactor expansion, we see that

$$det \begin{pmatrix} 5 & 6 & 6 \\ 0 & 0 & 1 \\ 6 & 4 & 6 \end{pmatrix} = (-1) det \begin{pmatrix} 5 & 6 \\ 6 & 4 \end{pmatrix} = 16$$

Thus, $(det(C))^2 = 16$. Therefore, $det(C) = 4$ or $det(C) = -4$.

25. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Find an ordered basis $\mathfrak{B}$ such that $A = [id_{\mathbb{R}^2}]_{\mathfrak{B}}^{\mathcal{E}}$ where $\mathcal{E}$ is the standard basis of $\mathbb{R}^2$, and $id_{\mathbb{R}^2}$ is an identity map on $\mathbb{R}^2$. (Here the notation means that $\mathcal{E}$ is the basis of the domain and $\mathfrak{B}$ is the basis of the codomain).

Hint: Express A^{-1} as a change of basis matrix.

Solution. Using the formula for an inverse of a 2×2 matrix, we obtain $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. Note that it is enough to find a basis $\mathfrak{B}$ such that $A^{-1} = [id_{\mathbb{R}^2}]_{\mathcal{E}}^{\mathfrak{B}}$ since this implies

$$A = (A^{-1})^{-1} = ([id_{\mathbb{R}^2}]_{\mathcal{E}}^{\mathfrak{B}})^{-1} = [id_{\mathbb{R}^2}]_{\mathfrak{B}}^{\mathcal{E}}.$$

We can take the desired ordered basis as $\mathfrak{B} = \{(2, -1)^t, (-3, 2)^t\}$ by choosing column vectors of A^{-1}.

Level: HARD

26. Let A be the matrix below:

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Write A^{-1} as a product of elementary matrices.

Hint: Use the elementary matrices corresponding to the row operations for reducing A to I_3.

Solution: Here is one possible sequence of row operations where we keep track of the row operations by labeling them.

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \underset{E_1}{[R_2 \rightarrow -2R_1 + R_2]} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \underset{E_2}{[R_2 \rightarrow -R_2]} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \underset{E_3}{[R_1 \rightarrow -3R_2 + R_1]} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The elementary matrices corresponding to the row operations above are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary row operations correspond to multiplying the matrix on the left by the elementary matirces. Our calculations show that $E_3E_2E_1A = I_3$. Therefore, one possible answer is $A^{-1} = E_3E_2E_1$.

27. Let A be a square matrix such that the Gram matrix for A is I_n. What are all of the possibilities for $\det(A)$?

Hint: The Gram matrix is A^TA.

Solution. If $A^TA = I_n$ then $\det(A^TA) = 1$. Since $\det(A^T) = \det A$, we know $\det(A)\det(A) = (\det(A))^2 = 1$. Therefore $\det(A) = \pm 1$.

28. Show that the inverse of a square matrix, if exists, is unique.

Hint: Suppose B and C are inverses of A. Compute BAC in two different ways.

Solution. Suppose A is an invertible square matrix, and B and C are two inverses. Then we consider the product BAC. First, we note that

$$BAC = B(AC) = BI = B$$

since C is an inverse of A, and so $AC = I$. On the other hand,

$$BAC = (BA)C = IC = C$$

since B is an inverse of A, and so $BA = I$.

Thus, $BAC = B = C$, an so $B = C$. So if A is invertible, then its inverse is unique.

29. Suppose that $A \sim B$ and $C \sim D$. Then does $AC \sim BD$ always hold? Justify your reason.

Hint: Use invariants of similar matrices to find a counterexample.

Solution. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \text{ and } C = D = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}.$$

Note that A and B are similar since

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

And C and D are similar obviously. But we have

$$AC = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and}$$

$$BD = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}.$$

Note that $tr(AC) = 3 \neq 2 = tr(BD)$. Therefore AC cannot be similar to BD.

30. Find the matrix of a linear transformation $F : P^2 \to P^2$ that sends $f(t)$ to $(t+2)f'(t)$ with respect to the input basis $\mathcal{B}_1 = \{1, t+2, t^2\}$ and the output basis $\mathcal{B}_2 = \{t, t+1, t^2+4\}$.

Hint: Find where F sends the polynomials from $\mathcal{B}_1$ and express the outputs in the basis $\mathcal{B}_2$.

Solution. $F(1) = 0$, $F(t+2) = t+2$, $F(t^2) = (t+2)\cdot(2t) = 4t+2t^2$. To express these in the basis $\mathcal{B}_2$, we simultaneously solve three systems by row reduction, writing everything in the basis $\{1, t, t^2\}$:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 4 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 & 2 & -8 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 12 \\ 0 & 1 & 0 & 0 & 2 & -8 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right]$$

So the answer is $\begin{bmatrix} 0 & -1 & 12 \\ 0 & 2 & -8 \\ 0 & 0 & 2 \end{bmatrix}$.

Group 3

Level: EASY

31. Let A and B be $n \times n$ orthogonal matrices. Is $A + B$ orthogonal?

Solution. Not necessarily. For instance, the identity matrix I_n is orthogonal while $I_n + I_n$ is not.

32. Let $A = \begin{bmatrix} 3 & -2 & -2 \\ -4 & 1 & 2 \\ 8 & -4 & -5 \end{bmatrix}$.

Check that $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is an eigenvector of A.

Solution. $A\mathbf{v} = \begin{bmatrix} 3 & -2 & -2 \\ -4 & 1 & 2 \\ 8 & -4 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ so $\mathbf{v}$ is an eigenvector with eigenvalue $\lambda = 1$.

33. Consider the matrices P and D are given below and let $A = PDP^{-1}$.

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

What are eigenvalues of A?

Solution. The eigenvalues of A are the entries of the diagonal matrix D. So they are -1 and 2.

34. Let $v = (1, 2, 3)$ and $w = (1, -3, 2)$. Compute $proj_w(v)$.

Solution. Using the formula for orthogonal projection to a line, we get $proj_w(v) = \frac{v \cdot w}{w \cdot w} w = \frac{(1,2,3)\cdot(1,-3,2)}{(1,-3,2)\cdot(1,-3,2)}(1, -3, 2) = \frac{1}{14}(1, -3, 2) = (1/14, -3/14, 1/7)$.

35. Is the following matrix orthogonally diagonalizable? $A = \begin{bmatrix} 3 & 4 & 3 \\ 4 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$.

Solution. Yes, because every symmetric matrix is orthogonally diagonalizable.

Level: MEDIUM

36. Let $A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$. Find all the eigenvalues λ_i, of A, and the corresponding algebraic multiplicities.

Hint: Find the zeros of det(A − xI).

Solution. The characteristic polynomial of A is

$$det(A-\lambda I) = \begin{vmatrix} -\lambda & 0 & 0 & 2 \\ 0 & -\lambda & 2 & 0 \\ 0 & 2 & -\lambda & 0 \\ 2 & 0 & 0 & -\lambda \end{vmatrix} =$$

$$-\lambda \begin{vmatrix} -\lambda & 2 & 0 \\ 2 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} - 0 + 0 - 2 \begin{vmatrix} 0 & -\lambda & 2 \\ 0 & 2 & -\lambda \\ 2 & 0 & 0 \end{vmatrix} =$$

$$\lambda^4 - 8\lambda^2 + 16 = (\lambda^2-4)^2 = ((\lambda-2)(\lambda+2))^2 = (\lambda-2)^2(\lambda+2)^2.$$

The zeros of this equation are $\lambda = 2$ (with multiplicity 2) and $\lambda = -2$ (with multiplicity 2). So these are the eigenvalues of A with their algebraic multiplicites.

37. Let $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$.

Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Hint: A is upper triangular, so its eigenvalues lie on the diagonal.

Solution. $\lambda = \pm 1$. Each will have a one dimensional eigenspace.
$E_{-1} = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$ and $E_1 = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$.
Construct $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.
$PDP^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = ... = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$.

38. Consider the matrices P and D as given below and let $A = PDP^{-1}$.

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Calculate, with justification, eigenvalues of A^{-1}.

Hint: Show that A^{-1} is similar to D^{-1}

Solution. Since $A = PDP^{-1}$, then we can compute its inverse using the fact that $(CD)^{-1} = D^{-1}C^{-1}$, so we have to reverse the order:

$$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$$

The inverse of D is also a diagonal matrix. In fact,

$$D^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Thus, $A^{-1} = PD^{-1}P^{-1}$ gives a diagonalization of A^{-1}, and its eigenvalues are the entries of the diagonal matrix. In particular, its eigenvalues are -1 and 1/2.

Another way to see this is as follows. First, $A = PDP^{-1}$ is the product of invertible matrices. So, A is invertible. From the diagonalization of A, we see that its eigenvalues are -1 and 2. That is, there are vectors $\mathbf{v}$ and $\mathbf{w}$ so that

$$A\mathbf{v} = -\mathbf{v}, \ A\mathbf{w} = 2\mathbf{w}$$

Multiplying both of these equations on both sides by A^{-1}, we see that

$$\mathbf{v} = -A^{-1}\mathbf{v} \text{ implies } A^{-1}\mathbf{v} = -\mathbf{v}$$

so -1 is an eigenvalue of A^{-1} and

$$\mathbf{w} = 2A^{-1}\mathbf{w} \text{ implies } A^{-1}\mathbf{w} = \frac{1}{2}\mathbf{w}$$

so 1/2 is another eigenvalue.

Since A is a 2×2 matrix, it can only have 2 eigenvalues, so they must be -1 and 1/2.

39. Consider $A = B^tB$, where $B = \begin{bmatrix} 1 & 2 & 1 \\ 5 & 3 & 0 \end{bmatrix}$. Is A orthogonally diagonalizable?

Hint: Recall the necessary and sufficient condition for a real matrix to be orthogonally diagonalizable.

Solution. A real matrix A is orthogonally diagonalizable if and only if $A^t = A$. From the properties of the transpose,

$$A^t = (B^tB)^t = B^t(B^t)^t = B^tB = A$$

So A is orthogonally diagonalizable.
(**Note:** One can also, of course, just calculate A and note that it is symmetric).

40. Can we find a basis of $\mathbb{R}^2$ which consists of eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix}$?

Hint: Find all eigenvalues and eigenspaces of A.

Solution. The characteristic polynomial $\chi_A(t)$ of A is $(t-1)(t-5)+4 = t^2 - 6t + 9 = (t-3)^2$. So 3 is the only eigenvalue of A. To form a basis of $\mathbb{R}^2$ which only consists of eigenvectors of A, we need to find the basis inside V_3, which is the eigenspace of A associated with the eigenvalue 3. So it is necessary to have $\dim_{\mathbb{R}}(V_3) = 2$.

By definition, the eigenspace V_3 is $\ker(T_{A-3I_2})$. Since $rank(A - 3I_2) = 1$, by the dimension theorem, we have $nullity(A - 3I_2) = 1$ which is the dimension of V_3. Therefore, we cannot find a basis which only consists of eigenvectors of A.

Level: HARD

41. Let $U_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $U_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $Y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$.

Write Y as the sum of two vectors, one in $\mathcal{W} = \mathrm{Span}\{\mathrm{U}_1, \mathrm{U}_2\}$ and the other orthogonal to $\mathcal{W}$.

Hint: Change U_2 to be orthogonal to U_1 without changing $\mathcal{W}$.

Solution. Let $v_1 = U_1$ and $v_2 = U_2 - \mathrm{proj}_{[v_1]} U_2$. We have

$$v_2 = U_2 - \frac{U_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{(1)(1) + (1)(1) + (1)(0)}{(1)(1) + (1)(1) + (0)(0)} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then $\langle v_1, v_2 \rangle$ is an orthogonal basis for $\mathcal{W}$.

$$\text{proj}_{\mathcal{W}}Y = \frac{Y \cdot v_1}{v_1 \cdot v_1}v_1 + \frac{Y \cdot v_2}{v_2 \cdot v_2}v_2 =$$

$$\frac{(-1)(1)+(4)(1)+(3)(0)}{(1)(1)+(1)(1)+(0)(0)}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{(-1)(0)+(4)(0)+(3)(1)}{(0)(0)+(0)(0)+(1)(1)}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 3 \end{bmatrix}$$

Therefore,

$$Y = \text{proj}_{\mathcal{W}}Y + (Y - \text{proj}_{\mathcal{W}}Y) = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 3 \end{bmatrix} + \left(\begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 3 \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 3 \end{bmatrix} + \begin{bmatrix} -\frac{5}{2} \\ \frac{5}{2} \\ 0 \end{bmatrix}.$$

42. Let $A = \begin{bmatrix} 3 & -2 & -2 \\ -4 & 1 & 2 \\ 8 & -4 & -5 \end{bmatrix}$, and let $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ be an (ordered) eigenbasis for $\mathbb{R}^3$.

Compute A^n for all $n \geq 0$.

Hint1: Diagonalize A.
Hint2: Those are eigenvectors!

Solution. Use the (ordered) eigenvectors to find eigenvalues $\lambda = 1, -1, -1$. Construct $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

A is diagonalizable so $A = PDP^{-1}$ and hence $A^n = PD^nP^{-1}$. Notice that $D^n = \begin{bmatrix} 1^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{bmatrix}$ which means that we have to consider two cases:

When n is even, $D^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$, so $A^n = PI_3P^{-1} = PP^{-1} = I_3$.

When n is odd, $D^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D$, so $A^n = PDP^{-1} = A$.

43. Consider the matrices P and D given below and let $A = PDP^{-1}$.

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Calculate A^n for all natural n.
Hint 1: Write out and simplify $(PDP^{-1})^n$ first.
Hint 2: A^n is similar to D^n.

We use that

$$A = PDP^{-1}$$

Thus,

$$A^n = (PDP^{-1})^n = (PDP^{-1})(PDP^{-1})\dots(PDP^{-1})$$

All of the terms of the form PP^{-1} cancel, and we are left with

$$A^n = PDD\dots DP^{-1} = PD^nP^{-1}$$

Using the matrices for P and D we are given, we get

$$A^n = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^n \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

We compute that

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and

$$D^n = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}$$

Thus,

$$A^n = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} (1/2((-1)^n + 2^n) & 1/2((-1)^n - 2^n) \\ 1/2((-1)^n - 2^n) & 1/2((-1)^n + 2^n)) \end{bmatrix}$$

44. Show that no $n+1$ nonzero vectors in $\mathbb{R}^n$ can be mutually orthogonal to one another.

Hint: Prove that the vectors must be linearly independent.

Solution. Suppose $\vec{v}_1, ..., \vec{v}_{n+1}$ are mutually orthogonal. We will show that they are linearly independent. Indeed, suppose $c_1\vec{v}_1 + ... + c_{n+1}\vec{v}_{n+1} = \vec{0}$. For each i, taking the dot product with $\vec{v}_i$, we get $c_i(\vec{v}_i \cdot \vec{v}_i) = 0$. So $c_i = 0$. It remains to recall that there cannot be more than n linearly independent vectors in $\mathbb{R}^n$.

45. Is the following matrix A diagonalizable? $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Hint: Compare algebraic and geometric multiplicities of the eigenvalues.

Solution. Suppose that A is diagonalizable, that is, A is similar with a diagonal matrix D. Since A has eigenvalues 1 (with algebraic multiplicity 2) and has eigenvalue 2 (with algebraic multiplicity 1), the diagonal entries of D should consist of exactly two 1's and one 2.

Since $A \sim D$, there is an invertible matrix Q such that $D = Q^{-1}AQ$. This implies

$$D - I_3 = Q^{-1}AQ - Q^{-1}I_3Q = Q^{-1}(A - I_3)Q.$$

Hence we obtain that $A - I_3$ is similar with $D - I_3$, which implies $rank(A - I_3) = rank(D - I_3)$. But this leads to a contradiction since we have $rank(D - I_3) = 1 \neq 2 = rank(A - I_3)$. Therefore, A is not diagonalizable.

46. Orthogonally diagonalize $A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ or explain why it is impossible.

Hint: One of the eigenvalues is 2.

Solution. The characteristic poylnomial of A is

$$\begin{vmatrix} -\lambda & -1 & 1 \\ -1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 2 = -(\lambda - 2)(\lambda + 1)^2$$

So the eigenvalues are $\lambda = -1, 2$.

For $\lambda = -1$, we find a basis of eigenspace by row reduction: $\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$.

Then we apply Gram-Schmidt algortihm to it and normalize to find an orthonormal basis. Specifically, we first replace the second vector by

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

Then by normalizing, we get a basis

$$\{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}\}$$

For eigenvalue $\lambda = 2$, we find get the row reduction gives as the following:

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -2 \\ -1 & -2 & -1 \\ -2 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So a normalized basis is $\{\begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\}$

And therefore $P^{-1}AP = D$, where the diagonal matrix D is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$,

and the orthogonal matrix P is $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1\sqrt{3} \end{bmatrix}$, with its inverse

$P^{-1} = P^t$ being $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1\sqrt{3} \end{bmatrix}$.