MATH 304 Midterm 2, Sample 1 - ANSWERS

Problem 1. (One possible explanation). Ignoring the zero in the left corner, we can identify the space of the upper-triangular 2×2 matrices with $\mathbb{R}^3$, so that the given three matrices are identified with vectors

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

Then one can show by row reduction (details are omitted here but would need to be provided on the actual test) that the matrix made of these vectors is non-singular, so they are linearly independent and span the whole $\mathbb{R}^3$.

Problem 2.

a) $\left\{ \begin{bmatrix} 2 \\ 4 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 11 \\ 5 \end{bmatrix} \right\}$ (This is the "standard" choice; other correct answers are possible, but would have to be justified).

b) $\{[2\ 1\ 0\ 0\ -8],\ [0\ 0\ 3\ 1\ 1],\ [0\ 0\ 0\ 1\ 2]\}$ (These are non-zero columns of some row echelon form. Your answer may be a bit different if you used a different row echelon form, for example the reduced row echelon form. It would be a mistake to just blindly use the first three rows of the orginal matrix, in analogy with the basis for the column space. For this particular matrix this would work, but would have to be justified).

c) $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ \frac{1}{3} \\ -2 \\ 1 \end{bmatrix} \right\}$ (Or $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 12 \\ 0 \\ 1 \\ -6 \\ 3 \end{bmatrix} \right\}$ if you don't like fractions. Other answers are possible, but would have to be justified).

d) Rank is 3, nullity is 2.

Problem 3. a) Yes. For justification, one can check that it is non-empty (contains 0 polynomial), is closed under multiplication by the scalars and addition. Alternatively, one can identify P_2 with $\mathbb{R}^3$ and express this set as a nullspace of some matrix. A third way of proving this is to note that this is a kernel of a **linear** map from P_2 to $\mathbb{R}$ that sends every polynomial $f(x)$ to $f(1)$, its value at $x = 1$.

b) No. This set does not contain the zero polynomial. (It is also not closed by scalar multiplication and by addition, which is best shown by providing a concrete counterexample).

Problem 4. No. One of the simplest counterexamples is the 1×2 matrix $[1\ 0]$.

Problem 5. Yes. One possible explanation is by the rank-nullity theorem. Suppose that A is $n \times n$. Since the columns are linearly dependent, the nullity of A is positive, so the rank of A is strictly less than n. But then the rank of A^t, which equals to rank of A, is also less than n. So the nullity of A^t is positive, so the columds of A^t, are linearly dependent, and so are the rows of A.

Alernatively, one can argue that the RREF of A must have less than n leading entries, and thus less than n non-zero columns. So the dimension of the row space of A is less than n, and therefore the rows of A are linearly dependent.

Problem 6. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (The columns are $f(\vec{e}_1)$ and $f(\vec{e}_2)$).

Problem 7. Yes. Consider an arbitrary polynomial of degree 2, $f(x) = ax^2 + bx + c$. Then $f'(x) = 2ax + b$ has degree 1 and $f''(x) = 2a$ has degree 0. If we identify P_2 with $\mathbb{R}^3$ by listing the coefficients in the increasing order of degree of x, we can write that $f'' = \begin{bmatrix} 2a \\ 0 \\ 0 \end{bmatrix}$, $f' = \begin{bmatrix} b \\ 2a \\ 0 \end{bmatrix}$, $f = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$. Because $a \neq 0$ (the degree of f is **2**), putting these vectors together gives a row echelon form with three non-zero rows. Thus these vectors form a basis of $\mathbb{R}^3$, and so f'', f', and f form a basis of P_2.