MATH 304 Midterm Examination 2, Sample 2

There are **seven (7)** problems on **two** pages in this examination. All work must be shown. NO CALCULATORS allowed.

Problem 1. Consider the function $T : P_2 \to \mathbb{R}^2$ defined as follows: $T(f) = \begin{bmatrix} f(1) \\ f'(1) \end{bmatrix}$.

a) Show that T is linear.
b) Find a basis of the Range of T.
c) Find a basis of the Kernel of T.

Problem 2. Consider the set V of all 2×2 matrices that commute with the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$.

a) Show that V is a subspace of the space $M_{2,2}$ of all 2×2 matrices.
b) Find a basis of V.

Problem 3. Suppose $\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$ is a linearly independent collection of vectors in $\mathbb{R}^4$. For each of the following statements tell if it must be true, must be false, or may be true or false depending on the given vectors. Justify your answers.

a) $\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$ is a basis of $\mathbb{R}^4$.
b) $\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$ is a basis of $span(\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3)$.
c) $\vec{v}_1$ and $\vec{v}_2$ are not multiples of each other.
d) $\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3,\ \vec{e}_4\}$ is a basis of $\mathbb{R}^4$.
e) $\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$ is a basis of $\mathbb{R}^3$.
f) Dimension of $span(\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3)$ is 3.

Problem 4. Given two matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & k \end{bmatrix}$$

a) Find AB.
b) Find all values of k for which $rk(AB) = 1$.
c) For each value of k from (b), find a basis of the column space of AB.
d) For each value of k from (b), find a basis of the row space of AB.
e) For each value of k from (b), find a basis of the null space of AB.

Problem 5. Determine if the set of vectors $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ such that } y = xz \right\}$ in $\mathbb{R}^3$ forms a subspace of $\mathbb{R}^3$. Either show that S is a subspace, or give a counterexample to show that S is not a subspace.

Problem 6.
a) Give an example of a 2×2 matrix A such that $A^2 = 0$, but $A \neq 0$.
b) Give an example of a 2×2 matrix A such that $A^2 = 0$, but all entries of A are non-zero.

Problem 7. Suppose A is a 2×2 matrix with real entries, such that $A^t A = 0$. Show that $A = 0$.