MATH 304 Midterm Examination 3, Sample 1

There are **seven (7)** problems on **two** pages in this examination. All work must be shown. NO CALCULATORS allowed.

Problem 1. Given a matrix

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Show that A is invertible and find A^{-1}.
b) Express A as a product of elementary matrices.

Problem 2. Suppose A and B are 3×3 matrices, $\det(A) = 3$, and $\det(B) = 5$. Find the following, if possible, and justify your answer. If the answer cannot be determined with the given data, explain why.
a) $\det(AB)$
b) $\det(A^{-1}B^t)$
c) $\det(2A)$
d) $\det(A + B)$
e) $\det\Big(B^{-1}(A + B) - I_3\Big)$

Problem 3. Consider the space P_2 of real polynomials of degree up to 2 in the variable x. Define $T : P^2 \to P^2$ as follows:

$$T(f)\ (x) = (2x - 1) \cdot f'(x)$$

a) Find the matrix of T with respect to the basis $\{1, x, x^2\}$.
b) Find the characteristic polynomial of the matrix from part (a).
c) Find all eigenvalues of T.
d) For each eigenvalue of T find a basis of the corresponding eigenspace (as polynomials in P_2).

Problem 4. Justify or provide a counterexample.
a) Every invertible matrix is diagonalizable.
b) Every diagonalizable matrix is invertible.
c) Every elementary matrix is diagonalizable.
d) Every elementary matrix is invertible.
e) The product of two invertible matrices is always invertible.

Problem 5. Find the following determinant.

$$\begin{vmatrix} 0 & 1 & 0 & 2 & 3 \\ 2 & 1 & 0 & 2 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 1 & 1 \end{vmatrix}$$

Problem 6. Linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ is given by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$$

Find the matrix of T with respect to the basis $\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}\}$ of $\mathbb{R}^2$ and the basis $\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\}$ of $\mathbb{R}^3$.

Problem 7. Given the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$$

Find an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$. If such P and D do not exist, explain why.