MATH 304 Midterm Examination 3, Sample 1 - ANSWERS

Problem 1. One possible solution:

$$\left[\begin{array}{ccc|ccc} 0 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & 1 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

a)

$$A^{-1} = \begin{bmatrix} -\frac{2}{3} & 1 & \frac{2}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2.
a) $\det(AB) = \det(A)\det(B) = 15$
b) $\det(A^{-1}B^t) = \frac{1}{\det(A)} \cdot \det(B) = \frac{5}{3}$
c) $\det(2A) = 2^3 \cdot \det(A) = 24$
d) $\det(A+B)$ cannot be determined: if $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ the $\det(A+B)$ is different.
e) $\det\Big(B^{-1}(A+B) - I_3\Big) = \det(B^{-1}A + I_3 - I_3) = \det(B^{-1}A) = \frac{3}{5}$

Problem 3. a)

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

b) $-\lambda(2-\lambda)^2$
c) $0,\ 2$
d) $\lambda = 0 : \{1\}$; $\lambda = 2 : \{(2x-1)\}$; $\lambda = 4 : \{(2x-1)^2\}$

Problem 4. (Many different counterexamples are possible in a,b,c)

a) No. For example, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible but not diagonalizable.

b) No. For example, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable but not invertible.

c) No. For example, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is elementary but not diagonalizable.

d) Yes. The inverse is the elementary matrix that corresponds to the elementary transformation that reverses the original one.

e) Yes. $(AB)^{-1} = B^{-1}A^{-1}$

Problem 5. 12

Problem 6.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 6 \\ 4 & 4 \\ 4 & 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|cc} 1 & 0 & 0 & 5 & 6 \\ 2 & 1 & 0 & 4 & 4 \\ 0 & 2 & 2 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 5 & 6 \\ 0 & 1 & 0 & -6 & -8 \\ 0 & 2 & 2 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 5 & 6 \\ 0 & 1 & 0 & -6 & -8 \\ 0 & 1 & 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 5 & 6 \\ 0 & 1 & 0 & -6 & -8 \\ 0 & 0 & 1 & 8 & 11 \end{array}\right]$$

Answer:

$$\begin{bmatrix} 5 & 6 \\ -6 & -8 \\ 8 & 11 \end{bmatrix}$$

Problem 7.

$char(A) = \det(A - \lambda I) = (3 - \lambda)(-\lambda) - 4 = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4)$

For $\lambda = -1$, we find an eigenvector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

For $\lambda = 4$, we find an eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

So

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$$