MATH 304 Midterm Examination 3, Sample 2 - ANSWERS

There are **seven (7)** problems on **two** pages in this examination. All work must be shown. NO CALCULATORS allowed.

Problem 1.
From the Change of Basis formula, the standard matrix equals

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1}$$

Calculating the inverse matrix and multiplying (steps skipped), we get

$$\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}$$

Problem 2.
By calculation (steps skipped) $\det(C) = 22$. So $\det(F) = (\frac{1}{2})^4 \cdot 22 = \frac{11}{8}$.

Problem 3.
a) T is a matrix transformation with the matrix

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ 3 & 0 & 1 & 0 \end{pmatrix}$$

Since it is matrix transformation, it is linear.

b) Invertible. The matrix of the inverse is:

$$\begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} & -1 & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{3}{4} & \frac{3}{4} & 3 & -\frac{5}{4} \\ 1 & 0 & 2 & -1 \end{pmatrix}$$

Problem 4. (Many correct answers exist)

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 2 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 5. (Many counterexamples for (a) and (d) exist)

a) No. $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ is diagonalizable, because it has two distinct eigenvalues (1 and 3), but it is not diagonal.

b) Yes. If the determinant of the matrix is 0, then it has non-trivial nullspace, so it has an eigenvector with the eigenvalue 0.

c) Yes. This is part of the Fundamental Theorem of Invertible Matrices.

d) No. $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ is a product of elementary matrices, but it is not similar to the identity matrix.

e) Yes. If A is similar to I, then $A = PIP^{-1} = PP^{-1} = I$.

Problem 6.

Eigenvalue 1: $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ (direction of the line of reflection)

Eigenvalue -1: $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (perpendicular direction)

Problem 7.

a) $char(A) = -\lambda(\lambda - 4)^2$

b) $\lambda = 0$: Eigenspace is $span\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \}$

$\lambda = 4$: Eigenspace is $span\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}\}$

c)

$$P = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \; D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$