MATH 304 Midterm Examination 3, Sample 3

There are **seven (7)** problems on **two** pages in this examination. All work must be shown. NO CALCULATORS allowed.

Problem 1. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a counter-clockwise rotation by $\frac{\pi}{2}$ in the (xy)-plane, followed by stretching by a factor of 2 in the direction of the z-axis.

a) Find the matrix of T in the standard basis.

b) Find the matrix of T in the basis $\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$

Problem 2. Given the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

a) Express A as a product of elementary matrices.

b) Express A^t as a product of elementary matrices.

Problem 3. Calculate the determinant.

$$\begin{vmatrix} 0 & 1 & 1 & 3 & 2 & 1 \\ 2 & 1 & 0 & 1 & 9 & 3 \\ 1 & 0 & 2 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{vmatrix}$$

Problem 4. Given a matrix

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

a) Find a diagonal matrix D and an invertible P so that $A = PDP^{-1}$.

b) Use the result from part (a) to find a formula for A^n.

Problem 5. Suppose T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, given in the standard basis by a matrix A. For each of the following statements determine if it is always true. Justify.

a) If T is invertible, then A is diagonalizable.

b) If the characteristic polynomial of A has a zero at $\lambda = 3$, then T has an eigenvector with the eigenvalue 3.

c) The matrix of $T \circ T$ in the standard basis is A^2.

d) If A is a product of elementary matrices, then T is invertible.

e) If A is similar to some matrix B, then B is the matrix of T in some basis of $\mathbb{R}^n$.

Problem 6. Suppose A is a square 2×2 matrix. For each of the following statements determine if it is always true. Justify.

a) If A has eigenvalue 3, then A^2 has eigenvalue 9.

b) If A^2 has eigenvalue 9, then A has eigenvalue 3.

c) If A has eigenvalue 0 then A^2 has eigenvalue 0.

d) If A^2 has eigenvalue 0, then A has eigenvalue 0.

e) If A has eigenvalues 3 and 0, then A is diagonalizable.

Problem 7. Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Verify your result by calculating $A \cdot A^{-1}$.