MATH 304 Midterm Examination 3, Sample 3 -ANSWERS

There are **seven (7)** problems on **two** pages in this examination. All work must be shown. NO CALCULATORS allowed.

Problem 1.

a) The matrix of T in the standard basis is

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

b) The matrix of T in the basis $\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$ can be found as

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

This calculates to

$$\begin{bmatrix} -1 & -2 & -1 \\ 4 & 3 & 3 \\ -1 & 1 & 0 \end{bmatrix}$$

Problem 2. (Many solutions exist)

a)

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b)

$$A^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Problem 3. 25

Problem 4.

a)

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^{-1}$$

b)

$$A^n = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5^n & 0 \\ 0 & 2^n \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{5^n+2\cdot 2^n}{3} & \frac{2\cdot 5^n-2\cdot 2^n}{3} \\ \frac{5^n-2^n}{3} & \frac{2\cdot 5^n+2^n}{3} \end{bmatrix}$$

Problem 5.

a) No. A is invertible. It may or may not be diagonalizable.

b) Yes. This means that $A - 3I_n$ has determinant 0, so it has a non-zero vector in the null space, which gives an eigenvector of T with eigenvalue 3.

c) Yes. Composition of linear transformations corresponds to multiplication of matrices.

d) Yes. A is invertible, so T is invertible.

e) Yes. If $A = PBP^{-1}$ then the matrix of T in the basis of columns of P is B.

Problem 6.

a) Yes. If $\vec{v} \neq \vec{0}$ and $A\vec{v} = 3\vec{v}$, then $A^2\vec{v} = A(3\vec{v}) = 3A\vec{v} = 9\vec{v}$.

b) No, for example for $A = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$.

c) Yes. If $\vec{v} \neq \vec{0}$ and $A\vec{v} = \vec{0}$, then $A^2\vec{v} = A\vec{0} = \vec{0}$.

d) Yes. We will prove the contrapositive. Since A does not have eigenvalue 0, it is invertible, so A^2 is also invertible, so A^2 does not have eigenvalhue 0.

e) Yes. Any $n \times n$ with n distinct eigenvalues is diagonalizable.

Problem 7.

$$A^{-1} = \begin{bmatrix} -1 & -3 & 2 & 4 \\ -1 & -4 & 2 & 6 \\ 1 & 4 & -2 & -5 \\ 1 & 2 & -1 & -3 \end{bmatrix}$$