MATH 304 FINAL EXAMINATION - Sample 2

Problem 1. Find the distance from the point $(1,1,1)$ to the line

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Problem 2. Determine if the following vectors are linearly dependent or independent. If they are linearly dependent, find a linear dependence relation among the vectors. Justify your answer.

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}.$$

Problem 3. Consider the following linear system:

$$\begin{cases} x + y + kz &= 1 \\ x + ky + z &= 1 \\ kx + y + z &= -2 \end{cases}$$

Find all values of k such that the above linear system has
(i) no solution, (ii) a unique solution, (iii) infinitely many solutions.

Problem 4. For the matrix

$$A = \begin{pmatrix} 2 & -4 & 0 & 2 & 1 \\ 1 & -2 & -1 & -2 & -2 \\ -1 & 2 & 1 & 2 & 2 \\ 1 & -2 & 1 & 4 & 4 \end{pmatrix}$$

Find each of the following. Show all work.
(i) a basis for $\text{row}(A)$; (ii) a basis for $\text{col}(A)$; (iii) a basis for $\text{null}(A)$; (iv) $\text{rank}(A)$; (v) $\text{nullity}(A)$.

Problem 5. Denote by U_2 the space of upper-triangular real 2×2 matrices and by $M_{2,2}$ the space of all real 2×2 matrices. Consider the linear transformation $T : U_2 \to M_{2,2}$, given by the formula

$$T(A) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} A$$

Find the matrix of T with respect to the bases

$$\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\}$$

and

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$$

Problem 6. Use **Cramer's Rule** to solve the following system. NO CREDIT will be given if **any other method** is used.

$$\begin{cases} x + y &= 3 \\ 2x - 3y &= 1 \end{cases}$$

Problem 7. The matrix A is given by

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

Determine if A is diagonalizable. If it is, find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$. (You DO NOT have to find P^{-1}). If it is not diagonalizable, explain why.

Problem 8. Suppose W is a subspace of $\mathbb{R}^4$, such that

$$W = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

a) Use the Gram-Schmidt process to construct an orthogonal basis of W.

b) Find the orthogonal projection $\text{proj}_W(\vec{v})$ of vector $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \end{bmatrix}$ onto W.

Problem 9. Determine if each of the statements below is TRUE or FALSE. Circle your choice and give the explanation for your answer.

a) Any 3 vectors in $\mathbb{R}^3$, such that no two are parallel, form a basis of $\mathbb{R}^3$

TRUE FALSE

Explanation:

b) For any vectors $\mathbf{u}$ and $\mathbf{v}$ from $\mathbb{R}^n$, if $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$

TRUE FALSE

Explanation:

c) Every elementary matrix has determinant 1

TRUE FALSE

Explanation:

d) If $\mathbf{v}$ is an eigenvector of a square matrix A, then $\mathbf{v}$ must be in $null(A)$

TRUE FALSE

Explanation:

e) A linear system can have exactly 2 solutions

TRUE FALSE

Explanation:

f) If row 3 of a square matrix is equal to the sum of row 1 and row 2, then the columns of A form a spaning set of $\mathbb{R}^n$

TRUE FALSE

Explanation:

g) For any square $n \times n$ matrices A and B , $(AB)^T = (A^T)(B^T)$

TRUE FALSE

Explanation: