MATH 304 FINAL EXAMINATION - Sample 3

Problem 1. a) Use **Gauss-Jordan elimination** (reduced row echelon form) to solve the system of linear equations

$$\begin{cases} \quad\;\; y - 6z + 5w = -7 \\ x \quad\;\; + 2z - w = 5 \\ 3x + y \quad\;\; + 2w = 8 \end{cases}$$

or explain why the system is inconsistent. If the system is consistent, write down the solution in a vector form. NO CREDIT will be given, if **any other method** is used.

b) For the matrix $A = \begin{pmatrix} 0 & 1 & -6 & 5 \\ 1 & 0 & 2 & -1 \\ 3 & 1 & 0 & 2 \end{pmatrix}$,

i) find a basis of the column space of A;

ii) find a basis of the row space of A;

iii) Determine rank and nullity of A;

iv) Determine if the vector $\mathbf{v} = \begin{bmatrix} -7 \\ 5 \\ 8 \end{bmatrix}$ is in $col(A)$.

Give detailed explanation.

Problem 2. a) Determine if the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 4 & 5 \\ 1 & 1 & 1 \end{pmatrix}$

is invertible. If A is invertible, find the inverse matrix A^{-1}. If A is not invertible, explain why.

b) Determine if the vectors

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}, \text{ and } \mathbf{v_3} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix},$$

form a basis of $\mathbb{R}^3$ or not. Explain your conclusion in details.

Problem 3. Find the standard matrix of the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$, that stretches a vector by a factor of 8 in the $x-$coordinate, then reflects it about the line $y = x$, and then rotates a vector clockwise by $30°$ about the origin.

Problem 4. The matrix A is given by

$$A = \begin{pmatrix} 0 & 3 & 6 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}$$

a) Find all eigenvalues of A.

b) Find a basis for each eigenspace of A.

c) Find the algebraic and geometric multiplicities for each eigenvalue of A.

d) Determine if A is diagonalizable. If it is, find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.
(You DO NOT have to find P^{-1})

Problem 5. If $W = span\,\{[1,0,-1,1,0],\ [3,1,0,0,1],\ [-1,2,3,1,-2]\}$,

a) Find a basis for the orthogonal complement $W^{\perp}$ of W.

b) Use Gram-Schmidt process to construct an orthogonal basis of W.

Problem 6. Determine if each of the statements below is TRUE or FALSE. Circle your choice and give the explanation for your answer.

a) If columns of a square $n \times n$ matrix Q form an orthogonal basis of $\mathbb{R}^n$, then Q must be an orthogonal matrix

TRUE FALSE

Explanation:

b) If n vectors in $\mathbb{R}^n$ are linearly independent, then they form a basis of $\mathbb{R}^n$

TRUE FALSE

Explanation:

c) For any matrix A, any vector in $null(A)$ is orthogonal to any vector in $row(A)$

TRUE FALSE

Explanation:

d) The product of two elementary matrices is an elementary matrix

TRUE FALSE

Explanation:

e) For any matrix A, if $\mathbf{v}$ is in $null(A)$ then the system $(A|\mathbf{v})$ must be consistent

TRUE FALSE

Explanation:

f) For any square matrix A

$$\det(A^T) = \frac{1}{\det A}$$

TRUE FALSE

Explanation:

g) For any square $n \times n$ matrix A, if $det(A) \neq 0$, then A must be diagonalizable

TRUE FALSE

Explanation: