MATH 304 FINAL EXAMINATION - Sample 5

Problem 1. a) Use **Gauss-Jordan elimination** (reduced row echelon form) to solve the system of linear equations

$$\begin{cases} x & +y & +2z & -w & = & 2 \\ x & -y & & +3w & = & -4 \\ x & +2y & +3z & -3w & = & 5 \\ 4x & +3y & +7z & -2w & = & 5 \end{cases}$$

or explain why the system is inconsistent. If the system is consistent, write down the solution in a vector form. NO CREDIT will be given, if **any other method** is used.

b) Determine if the vectors

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 4 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 7 \end{bmatrix}, \text{ and } \mathbf{v_4} = \begin{bmatrix} -1 \\ 3 \\ -3 \\ -2 \end{bmatrix}$$

are linearly independent or not. Explain your conclusion in details.

Problem 2. a) Use Cramer's Rule to solve the system of linear equations

$$\begin{cases} x + 2y + z & = & 2 \\ 3x - y & = & 1 \\ 9x + y + 2z & = & 5 \end{cases}$$

b) Determine if the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 0 \\ 9 & 1 & 2 \end{pmatrix}$

is invertible. If A is invertible, find the inverse matrix A^{-1}. If A is not invertible, explain why.

Problem 3. Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, that rotates a vector clockwise by 60° about the origin, then reflects it about the line $y = x$, and then reflects it about the x-axis.

a) Find the standard matrix of the linear transformation T.

b) Determine if the transformation T is invertible.
Give detailed explanation. If T is invertible, find the standard matrix of the inverse transformation T^{-1}.

Problem 4. A matrix A is given below:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -2 & -1 & 1 \\ -2 & -3 & 3 \end{pmatrix}$$

a) Find all eigenvalues of A.

b) Find a basis for each eigenspace of A.

c) Determine if A is diagonalizable. If it is, find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.
(You DO NOT have to find P^{-1})

d) Determine if A is orthogonally diagonalizable.
Give detailed explanation.
(You do not have to find an orthogonal matrix Q such that $D = Q^T AQ$.)

Problem 5. Given the matrix $A = \begin{pmatrix} 1 & 0 & -1 & 1 & 3 \\ 2 & 6 & -1 & 0 & 7 \\ -1 & 6 & 2 & -3 & -2 \end{pmatrix}$,

a) Find a basis and dimension for each of the following:
i) the row space of A;
ii) the column space of A.
iii) the null space of A.

b) If $W = span\,\{[1, 0, -1, 1, 3],\ [2, 6, -1, 0, 7],\ [-1, 6, 2, -3, -2]\}$,
find a basis for the orthogonal complement $W^{\perp}$ of W.

c) Construct an orthogonal basis for $col(A)$ containing vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

d) Find the projection of the vector $\mathbf{v} = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$ onto $col(A)$.

Problem 6. Determine if each of the statements below is TRUE or FALSE. Circle your choice and give the explanation for your answer.

a) For any square $n \times n$ matrix A, $\det(kA) = k \det A$

TRUE FALSE

Explanation:

b) For any square $n \times n$ matrices A and B, $\det(AB) = \det A \cdot \det B$

TRUE FALSE

Explanation:

c) Any diagonalizable matrix is orthogonally diagonalizable

TRUE FALSE

Explanation:

d) Any orthogonally diagonalizable matrix is diagonalizable

TRUE FALSE

Explanation:

e) Any three non-zero orthogonal vectors in $\mathbb{R}^4$ are linearly independent

TRUE FALSE

Explanation:

f) Any three linearly independent vectors in $\mathbb{R}^4$ are orthogonal

TRUE FALSE

Explanation:

g) If W is a subspace of $\mathbb{R}^n$, then a basis of W contains exactly n vectors

TRUE FALSE

Explanation: