Upstate New York Online Number Theory Colloquium

Time and Date: 12:00 pm EST September 14, 2020

Speaker: Wadim Zudilin

Title: Creative microscoping

Abstract: Let $A_k = 2^{-6k}\binom{2k}{k}^3$ for $k = 0, 1, \dots$. Though traditional techniques of establishing the hypergeometric evaluation

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)A_k = \frac{2}{\pi}$$

and (super)congruences

$$\sum_{k=0}^{p-1}(-1)^k(4k+1)A_k \equiv p(-1)^{(p-1)/2} \pmod{p^3} \quad \text{for primes } p > 2$$

share certain similarities, they do not display intrinsic reasons for the two to be related. In my talk I will outline basic ingredients of a method developed in joint works with Victor Guo, which does the missing part, also for many other instances of such arithmetic duality. The main idea is constructing suitable q-deformations of the infinite sum (and many such sums are already recorded in the q-literature), and then look at the asymptotics of that at roots of unity. Interestingly enough, the q-deformations may offer more. For example, the q-deformation of the above infinite sum also implies

$$\sum_{k=0}^{\infty} A_k = \frac{\Gamma(1/4)^4}{4\pi^3} = \frac{8L(f,1)}{\pi} \quad \text{and} \quad \sum_{k=0}^{p-1} A_k \equiv a(p) \pmod{p^2}$$

(in fact, the latter congruences in their stronger modulo p^3 form proven by Long and Ramakrishna), where $a(p)$ is the p-th Fourier coefficient of (the weight 3 modular form) $f = q\prod_{m=1}^{\infty}(1-q^{4m})^6$. (*NB*: The variable q in the last definition is related to the modular parameter τ through $q = e^{2\pi i\tau}$ and has nothing to do with the q in the q-deformation.)