Slopes In $\mathbb{Z}_p$-Towers of Curves

Daqing Wan
University of California at Irvine

August 3, 2020
Upstate New York Online Number Theory Colloquium

- C: a smooth projective geometrically irreducible curve of genus g defined over a finite field $\mathbb{F}_q$ of characteristic p.

- C: a smooth projective geometrically irreducible curve of genus g defined over a finite field $\mathbb{F}_q$ of characteristic p.
- The zeta function of C is

$$Z(C,s) = \exp\left(\sum_{k=1}^{\infty} \frac{\#C(\mathbb{F}_{q^k})}{k} s^k\right) = \frac{P(C,s)}{(1-s)(1-qs)}.$$

- C: a smooth projective geometrically irreducible curve of genus g defined over a finite field $\mathbb{F}_q$ of characteristic p.

- The zeta function of C is

$$Z(C,s) = \exp\Big(\sum_{k=1}^{\infty} \frac{\#C(\mathbb{F}_{q^k})}{k} s^k\Big) = \frac{P(C,s)}{(1-s)(1-qs)}.$$

- The degree $2g$ zeta polynomial

$$P(C,s) = \prod_{i=1}^{2g} (1-\alpha_i s) \in \mathbb{C}[s]$$

is pure of weight 1 (Weil). That is,

$$|\alpha_i| = \sqrt{q},\ \ \alpha_i \bar{\alpha}_i = q,\ \ (1 \le i \le 2g).$$

- Write the zeta polynomial

$$P(C,s)=\prod_{i=1}^{2g}(1-\alpha_i s)\in\mathbb{C}_p[s],$$

$$0\leq v_q(\alpha_1)\leq\cdots\leq v_q(\alpha_{2g})\leq 1,\ \ v_q(q)=1.$$

The set $\{v_q(\alpha_1),\cdots,v_q(\alpha_{2g})\}\subset\mathbb{Q}\cap[0,1]$ is called the q-slope sequence of C.

- Write the zeta polynomial

$$P(C,s)=\prod_{i=1}^{2g}(1-\alpha_i s)\in \mathbb{C}_p[s],$$

$$0\le v_q(\alpha_1)\le\cdots\le v_q(\alpha_{2g})\le 1,\ \ v_q(q)=1.$$

  The set $\{v_q(\alpha_1),\cdots,v_q(\alpha_{2g})\}\subset \mathbb{Q}\cap[0,1]$ is called the q-slope sequence of C.

- The set of q^k-slopes for $C\otimes\mathbb{F}_{q^k}$ is the set of q-slopes for C. Thus, the q-slope sequence is a geometric invariant of C, simply called the slope sequence of C.

- Write the zeta polynomial

$$P(C,s)=\prod_{i=1}^{2g}(1-\alpha_i s)\in \mathbb{C}_p[s],$$

$$0\le v_q(\alpha_1)\le \cdots \le v_q(\alpha_{2g})\le 1,\ \ v_q(q)=1.$$

The set $\{v_q(\alpha_1),\cdots,v_q(\alpha_{2g})\}\subset \mathbb{Q}\cap[0,1]$ is called the q-slope sequence of C.

- The set of q^k-slopes for $C\otimes \mathbb{F}_{q^k}$ is the set of q-slopes for C. Thus, the q-slope sequence is a geometric invariant of C, simply called the slope sequence of C.

- The slope sequence can be computed from the q-adic Newton polygon of $P(C,s)$ if $P(C,s)$ is given.

- Question I: Given curve C, what is the slope sequence of C?

  (This is hard in general. Any polynomial time algorithm?)

- Question I: Given curve C, what is the slope sequence of C?

  (This is hard in general. Any polynomial time algorithm?)

- Question II: How the slope sequence of C varies when C varies in an algebraic family?

- Question I: Given curve C, what is the slope sequence of C?

  (This is hard in general. Any polynomial time algorithm?)

- Question II: How the slope sequence of C varies when C varies in an algebraic family?

- Question III: How the slope sequence of C varies when C varies in a $\mathbb{Z}_p$-tower?

- Write the zeta polynomial

$$P(C, s) = 1 + c_1 s + \cdots + q^g s^{2g}, \ c_{g+i} = q^i c_{g-i}.$$

The q-adic Newton polygon $\mathrm{NP}(C)$ is the lower convex hull in $\mathbb{R}^2$ of the points

$$(0,0), (1, v_q(c_1)), \cdots, (k, v_q(c_k)), \cdots, (2g, g).$$

- Write the zeta polynomial

$$P(C,s) = 1 + c_1 s + \cdots + q^g s^{2g}, \ c_{g+i} = q^i c_{g-i}.$$

The q-adic Newton polygon $\mathrm{NP}(C)$ is the lower convex hull in $\mathbb{R}^2$ of the points

$$(0,0), (1, v_q(c_1)), \cdots, (k, v_q(c_k)), \cdots, (2g, g).$$

- The slope sequence of C is simply the slope sequence of $\mathrm{NP}(C)$, counting multiplicity.

- Write the zeta polynomial

$$P(C,s) = 1 + c_1 s + \cdots + q^g s^{2g},\ c_{g+i} = q^i c_{g-i}.$$

The q-adic Newton polygon $\mathrm{NP}(C)$ is the lower convex hull in $\mathbb{R}^2$ of the points

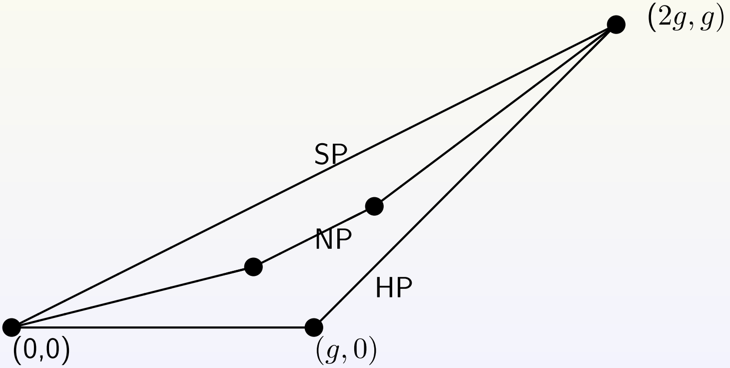
$$(0,0), (1, v_q(c_1)), \cdots, (k, v_q(c_k)), \cdots, (2g, g).$$

- The slope sequence of C is simply the slope sequence of $\mathrm{NP}(C)$, counting multiplicity.
- Hodge polygon HP: Slopes: $\{0, 1\}$ with multiplicity g.
- Supersingular polygon SP: Slopes: $1/2$ with multiplicity $2g$.

- $\mathrm{NP}(C)$ is "symmetric" (by functional equation): s is a slope if and only if $1 - s$ is a slope with the same multiplicity.

- $\mathrm{NP}(C)$ is "symmetric" (by functional equation): s is a slope if and only if $1-s$ is a slope with the same multiplicity.
- One has the trivial inequalities:

$$\mathrm{HP} \leq \mathrm{NP}(C) \leq \mathrm{SP}.$$



- Vertices of $\mathrm{NP}(C)$ are lattice points in $\mathbb{R}^2$.

  (by Dieudonne-Manin).

- Vertices of $\mathrm{NP}(C)$ are lattice points in $\mathbb{R}^2$.

  (by Dieudonne-Manin).

- If $\mathrm{NP}(C) = \mathrm{HP}$, C is called **ordinary**.

  If $\mathrm{NP}(C) = \mathrm{SP}$, C is called **supersingular**.

I. Slopes In Algebraic Families of Curves

- If $f : X \longrightarrow Y$ is a family of curves of genus g over $\bar{\mathbb{F}}_p$, the generic (lowest) Newton polygon GNP(f) of f exists by the Grothendieck specialization theorem.

- If $f : X \longrightarrow Y$ is a family of curves of genus g over $\bar{\mathbb{F}}_p$, the generic (lowest) Newton polygon GNP(f) of f exists by the Grothendieck specialization theorem.

  If GNP(f) is equal to HP, f is called **generically ordinary**.

- If $f : X \longrightarrow Y$ is a family of curves of genus g over $\bar{\mathbb{F}}_p$, the generic (lowest) Newton polygon GNP(f) of f exists by the Grothendieck specialization theorem.

  If GNP(f) is equal to HP, f is called **generically ordinary**.

- (a). The universal family $\mathcal{H}_g$ of genus g hyper-elliptic curves over $\bar{\mathbb{F}}_p$ is generically ordinary for every prime p.

- If $f : X \longrightarrow Y$ is a family of curves of genus g over $\bar{\mathbb{F}}_p$, the generic (lowest) Newton polygon GNP(f) of f exists by the Grothendieck specialization theorem.

  If GNP(f) is equal to HP, f is called **generically ordinary**.

- (a). The universal family $\mathcal{H}_g$ of genus g hyper-elliptic curves over $\bar{\mathbb{F}}_p$ is generically ordinary for every prime p.

  (b). Hence the larger family $\mathcal{M}_g$ of all genus g curves over $\bar{\mathbb{F}}_p$ is also generically ordinary for every prime p.

- If $f : X \longrightarrow Y$ is a family of curves of genus g over $\bar{\mathbb{F}}_p$, the generic (lowest) Newton polygon GNP(f) of f exists by the Grothendieck specialization theorem.

  If GNP(f) is equal to HP, f is called **generically ordinary**.

- (a). The universal family $\mathcal{H}_g$ of genus g hyper-elliptic curves over $\bar{\mathbb{F}}_p$ is generically ordinary for every prime p.

  (b). Hence the larger family $\mathcal{M}_g$ of all genus g curves over $\bar{\mathbb{F}}_p$ is also generically ordinary for every prime p.

- Big families are more likely to be generically ordinary. It is harder to decide if a small (one parameter) family of curves of genus g is generically ordinary. In this case, generically ordinary means ordinary at almost all fibers.

- Conjecture (Katz, 2018). For $n = 2g+1$ or $n = 2g+2$, the one parameter family of hyper-elliptic curves over $\bar{\mathbb{F}}_p$

$$y^2 = x^n - nx - \lambda, \ \ g \geq 1$$

is generically ordinary if $p > n^2$.

- Conjecture (Katz, 2018). For $n = 2g+1$ or $n = 2g+2$, the one parameter family of hyper-elliptic curves over $\bar{\mathbb{F}}_p$

$$y^2 = x^n - nx - \lambda,\ \ g \geq 1$$

is generically ordinary if $p > n^2$.

(For $\ell \neq p$, big ℓ-adic monodromy implies big p-adic monodromy and generic ordinariness for large p).

- Conjecture (Katz, 2018). For $n = 2g + 1$ or $n = 2g + 2$, the one parameter family of hyper-elliptic curves over $\bar{\mathbb{F}}_p$

$$y^2 = x^n - nx - \lambda, \ \ g \geq 1$$

is generically ordinary if $p > n^2$.

(For $\ell \neq p$, big ℓ-adic monodromy implies big p-adic monodromy and generic ordinariness for large p).

- Conjecture (Katz, 2018). For squarefree $f(x) \in \bar{\mathbb{F}}_p[x]$ of degree $n = 2g$ or $n = 2g + 1$, the one parameter family of hyper-elliptic curves over $\bar{\mathbb{F}}_p$

$$y^2 = f(x)(x - \lambda)$$

is generically ordinary if $p > n + 1$.

- For $f(x) = \sum_{i=0}^{d} a_i x^i \in \bar{\mathbb{F}}_p[x]$ of degree d not divisible by p, the Newton polygon of the Artin-Schreier curve $y^p - y = f(x)$ lies above the polygon with slopes

$$\{\frac{1}{d}, \frac{2}{d}, \cdots, \frac{d-1}{d}\}^{(p-1)}.$$

These two polygons coincide if and only if $p \equiv 1 \mod d$.

- For $f(x) = \sum_{i=0}^{d} a_i x^i \in \bar{\mathbb{F}}_p[x]$ of degree d not divisible by p, the Newton polygon of the Artin-Schreier curve $y^p - y = f(x)$ lies above the polygon with slopes
$$\{\frac{1}{d}, \frac{2}{d}, \cdots, \frac{d-1}{d}\}^{(p-1)}.$$
These two polygons coincide if and only if $p \equiv 1 \mod d$.

- Joe Kramer-Miller (2020, ANT etc) has extended this improved lower bound to any cyclic cover of arbitrary base curve ramified at several points, using ramification invariants.

- For $f(x) = \sum_{i=0}^{d} a_i x^i \in \bar{\mathbb{F}}_p[x]$ of degree d not divisible by p, the Newton polygon of the Artin-Schreier curve $y^p - y = f(x)$ lies above the polygon with slopes
$$\{\frac{1}{d}, \frac{2}{d}, \cdots, \frac{d-1}{d}\}^{(p-1)}.$$
These two polygons coincide if and only if $p \equiv 1 \mod d$.

- Joe Kramer-Miller (2020, ANT etc) has extended this improved lower bound to any cyclic cover of arbitrary base curve ramified at several points, using ramification invariants.

- Question. For $f(x) \in \bar{\mathbb{F}}_p[x]$ of degree d not divisible by p, the Newton polygon of the one parameter family of Artin-Schreier curves over $\bar{\mathbb{F}}_p$
$$y^p - y = \lambda f(x)$$
is fibre by fibre independent of $\lambda \in \bar{\mathbb{F}}_p^*$?

- Question (W, in Mirror Symmetry V, 2006). Is the one parameter Dwork family of Calabi-Yau hypersurfaces over $\bar{\mathbb{F}}_p$

$$x_1^n + x_2^n + \cdots + x_n^n + n\lambda x_1 \cdots x_n = 0, \ n \geq 3$$

generically ordinary if p does not divide n?

- Question (W, in Mirror Symmetry V, 2006). Is the one parameter Dwork family of Calabi-Yau hypersurfaces over $\bar{\mathbb{F}}_p$

$$x_1^n + x_2^n + \cdots + x_n^n + n\lambda x_1 \cdots x_n = 0,\ \ n \geq 3$$

  generically ordinary if p does not divide n?

- By the star decomposition theorem, the mirror family over $\bar{\mathbb{F}}_p$

$$x_1 + x_2 + \cdots + x_{n-1} + n\lambda + 1/x_1 \cdots x_{n-1} = 0$$

  is indeed generically ordinary if p does not divide n.

- Question (W, in Mirror Symmetry V, 2006). Is the one parameter Dwork family of Calabi-Yau hypersurfaces over $\bar{\mathbb{F}}_p$

$$x_1^n + x_2^n + \cdots + x_n^n + n\lambda x_1 \cdots x_n = 0,\ \ n \geq 3$$

  generically ordinary if p does not divide n?

- By the star decomposition theorem, the mirror family over $\bar{\mathbb{F}}_p$

$$x_1 + x_2 + \cdots + x_{n-1} + n\lambda + 1/x_1 \cdots x_{n-1} = 0$$

  is indeed generically ordinary if p does not divide n.

- One can ask the same question for other "invertible" one parameter family of CY hypersurfaces

$$\sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}} + n\lambda x_1 \cdots x_n = 0$$

  and its mirror family. Or determine its GNP.

II. Slopes In $\mathbb{Z}_p$-Towers of Curves

- Consider

$$C = C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_n \leftarrow \cdots \leftarrow C_\infty,$$

$$\mathrm{Gal}(C_n/C) = \mathbb{Z}_p/p^n\mathbb{Z}_p,\ p = \mathrm{char}(\mathbb{F}_q),$$

a $\mathbb{Z}_p$-tower of curves with constant field $\mathbb{F}_q$, ramified on a finite non-empty set S of closed points on C.

- Consider

$$C = C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_n \leftarrow \cdots \leftarrow C_\infty,$$

$$\mathrm{Gal}(C_n/C) = \mathbb{Z}_p/p^n\mathbb{Z}_p, \ p = \mathrm{char}(\mathbb{F}_q),$$

  a $\mathbb{Z}_p$-tower of curves with constant field $\mathbb{F}_q$, ramified on a finite non-empty set S of closed points on C.

- Such towers can be constructed explicitly using Witt vectors.

- Consider

$$C = C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_n \leftarrow \cdots \leftarrow C_\infty,$$

$$\mathrm{Gal}(C_n/C) = \mathbb{Z}_p/p^n\mathbb{Z}_p,\ p = \mathrm{char}(\mathbb{F}_q),$$

  a $\mathbb{Z}_p$-tower of curves with constant field $\mathbb{F}_q$, ramified on a finite non-empty set S of closed points on C.

- Such towers can be constructed explicitly using Witt vectors.

- $g_n = \mathrm{genus}(C_n)$ grows at least quadratically in p^n.

- Consider

$$C = C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_n \leftarrow \cdots \leftarrow C_\infty,$$

$$\mathrm{Gal}(C_n/C) = \mathbb{Z}_p/p^n\mathbb{Z}_p,\ p = \mathrm{char}(\mathbb{F}_q),$$

  a $\mathbb{Z}_p$-tower of curves with constant field $\mathbb{F}_q$, ramified on a finite non-empty set S of closed points on C.

- Such towers can be constructed explicitly using Witt vectors.

- $g_n = \mathrm{genus}(C_n)$ grows at least quadratically in p^n.

- Question: How the slope sequence of C_n varies as n varies?

  (Any stability for large n?)

- Note

$$\frac{Z(C_n, s)}{Z(C, s)} = \prod_{\chi^{p^n}=1, \chi\neq 1} L(\chi, s),$$

where χ runs over all continuous p-adic characters of $\mathrm{Gal}(C_\infty/C) = \mathbb{Z}_p$ of order dividing p^n.

- Note
$$\frac{Z(C_n,s)}{Z(C,s)} = \prod_{\chi^{p^n}=1, \chi\neq 1} L(\chi,s),$$
where χ runs over all continuous p-adic characters of $\mathrm{Gal}(C_\infty/C) = \mathbb{Z}_p$ of order dividing p^n.

- If $\mathrm{ord}(\chi) = \mathrm{ord}(\chi') = p^k > 1$, then
$$\mathrm{NP}(L(\chi,s)) = \mathrm{NP}(L(\chi',s)).$$

- Note
$$\frac{Z(C_n,s)}{Z(C,s)} = \prod_{\chi^{p^n}=1,\chi\neq 1} L(\chi,s),$$
where χ runs over all continuous p-adic characters of $\mathrm{Gal}(C_\infty/C) = \mathbb{Z}_p$ of order dividing p^n.

- If $\mathrm{ord}(\chi) = \mathrm{ord}(\chi') = p^k > 1$, then
$$\mathrm{NP}(L(\chi,s)) = \mathrm{NP}(L(\chi',s)).$$

- Let χ_n be any fixed primitive character of order p^n. Then
$$\mathrm{NP}(\frac{Z(C_n,s)}{Z(C,s)}) = \bigoplus_{k=1}^{n} \mathrm{NP}(L(\chi_k,s))^{p^{k-1}(p-1)}.$$

- It is enough to study $\mathrm{NP}(L(\chi_n,s))$, $1 \leq n < \infty$.

- Consider $\mathbb{Z}_p$-towers over $C = \mathbb{P}^1$, totally ramified at ∞ and unramified on $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$.

- Consider $\mathbb{Z}_p$-towers over $C = \mathbb{P}^1$, totally ramified at ∞ and unramified on $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$.
- Any such $\mathbb{Z}_p$-tower can be uniquely constructed (up to twist) from a primitive convergent power series

$$f(x) = \sum_{(i,p)=1} c_i x^i \in \mathbb{Z}_q[[x]],\ c_i \in \mathbb{Z}_q = W(\mathbb{F}_q),\ \lim_i c_i = 0,$$

where $f(x)$ is called primitive if not all c_i are divisible by p.

- Consider $\mathbb{Z}_p$-towers over $C = \mathbb{P}^1$, totally ramified at ∞ and unramified on $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$.
- Any such $\mathbb{Z}_p$-tower can be uniquely constructed (up to twist) from a primitive convergent power series

$$f(x) = \sum_{(i,p)=1} c_i x^i \in \mathbb{Z}_q[[x]],\ c_i \in \mathbb{Z}_q = W(\mathbb{F}_q),\ \lim_i c_i = 0,$$

where $f(x)$ is called primitive if not all c_i are divisible by p.
- The construction is explicitly given by the following Witt vector equation

$$C_\infty : [y_1^p, y_2^p, \cdots] - [y_1, y_2, \cdots] = \sum_{(i,p)=1} c_i[x^i, 0, \cdots].$$

$$C_1 : y_1^p - y_1 = \sum_{(i,p)=1} \bar{c}_i x^i.\ (\mathrm{Artin-Schreier\ curve})$$

- The genus g_n of C_n is (Kosters-W, Proc. AMS, 2018):

$$2g_n = \sum_{k=1}^{n} p^{k-1}(p-1)u_k,$$

$$u_n = -1 + p^{n-1} \max_{v_p(c_i)<n} \{\frac{i}{p^{v_p(c_i)}}\} = \deg(L(\chi_n, s)).$$

- The genus g_n of C_n is (Kosters-W, Proc. AMS, 2018):

$$2g_n = \sum_{k=1}^{n} p^{k-1}(p-1)u_k,$$

$$u_n = -1 + p^{n-1} \max_{v_p(c_i)<n} \{\frac{i}{p^{v_p(c_i)}}\} = \deg(L(\chi_n, s)).$$

- The genus sequence g_n grows at least quadratically in p^n. It can grow as fast as one wishes!

- The genus g_n of C_n is (Kosters-W, Proc. AMS, 2018):

$$2g_n = \sum_{k=1}^{n} p^{k-1}(p-1)u_k,$$

$$u_n = -1 + p^{n-1} \max_{v_p(c_i)<n} \{\frac{i}{p^{v_p(c_i)}}\} = \deg(L(\chi_n, s)).$$

- The genus sequence g_n grows at least quadratically in p^n. It can grow as fast as one wishes!

- Example. If $f(x) = x^d + a_{d-1}x^{d-1} + \cdots \in \mathbb{Z}_q[x]$ is monic of degree d not divisible by p, then the genus g_n is given by

$$u_n = -1 + dp^{n-1},\ 2g_n = (p-1)(d\frac{p^{2n}-1}{p^2-1} - \frac{p^n-1}{p-1}).$$

- The $\mathbb{Z}_p$-tower C_∞/C is called **genus stable** if

$$g_n = \alpha p^{2n} + \beta p^n + \gamma, \ n \gg 0.$$

- The $\mathbb{Z}_p$-tower C_∞/C is called **genus stable** if

$$g_n = \alpha p^{2n} + \beta p^n + \gamma, \ n \gg 0.$$

- C_∞/C is genus stable if and only if

$$a_n := \max_{v_p(c_i)<n} \{\frac{i}{p^{v_p(c_i)}}\} = \frac{1}{p-1}(\frac{a}{p^m} - \frac{b}{p^{n-1}}), \ n \gg 0$$

for $m, a, b \in \mathbb{Z}_{\geq 0}$, $(b,p) = 1$ if $b > 0$, $a \equiv b \mod (p-1)$, i.e.,

$$\deg(L(\chi_n, s)) = u_n = -1 + \frac{1}{p-1}(p^{n-1-m}a - b), \ n \gg 0.$$

- The $\mathbb{Z}_p$-tower C_∞/C is called **genus stable** if

$$g_n = \alpha p^{2n} + \beta p^n + \gamma, \ n \gg 0.$$

- C_∞/C is genus stable if and only if

$$a_n := \max_{v_p(c_i)<n}\{\frac{i}{p^{v_p(c_i)}}\} = \frac{1}{p-1}(\frac{a}{p^m} - \frac{b}{p^{n-1}}), \ n \gg 0$$

for $m, a, b \in \mathbb{Z}_{\geq 0}$, $(b, p) = 1$ if $b > 0$, $a \equiv b \mod (p-1)$, i.e.,

$$\deg(L(\chi_n, s)) = u_n = -1 + \frac{1}{p-1}(p^{n-1-m}a - b), \ n \gg 0.$$

- The $\mathbb{Z}_p$-tower C_∞/C is called **strongly genus stable** if $b = 0$, i.e., if $\delta := \max_{(i,p)=1}\{\frac{i}{p^{v_p(c_i)}}\}$ exists. In this case,

$$\deg(L(\chi_n, s)) = u_n = p^{n-1}\delta - 1, \ n \gg 0.$$

- Let $\bar{f}(x) = \sum_{i=1}^{d} \bar{a}_i x^i \in \mathbb{F}_q[x]$, $\deg(\bar{f}) = d$ not divisible by p. Let $f(x) = \sum_{i=1}^{d} a_i x^i \in \mathbb{Z}_q[x]$ be its Teichmüller lifting.

- Let $\bar{f}(x) = \sum_{i=1}^{d} \bar{a}_i x^i \in \mathbb{F}_q[x]$, $\deg(\bar{f}) = d$ not divisible by p. Let $f(x) = \sum_{i=1}^{d} a_i x^i \in \mathbb{Z}_q[x]$ be its Teichmüller lifting.

- C. Liu and W (2008, ANT) introduced the T-adic L-function

$$\mathcal{L}(T, s) = L(\chi_{\text{univ}}, s) \in 1 + s\mathbb{Z}_p[[T]][[s]],$$

where

$$\chi_{\text{univ}} : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p[[T]]^{\times}, \ \chi_{\text{univ}}(b) = (1+T)^b$$

parameterizes all continuous $\mathbb{C}_p$ valued characters χ of $\mathrm{Gal}(C_\infty/C) = \mathbb{Z}_p$, possibly infinite order. The parameter $t = \chi(1) - 1$ varies in the open unit disc $|t|_p < 1$.

- Let $\bar{f}(x) = \sum_{i=1}^{d} \bar{a}_i x^i \in \mathbb{F}_q[x]$, $\deg(\bar{f}) = d$ not divisible by p. Let $f(x) = \sum_{i=1}^{d} a_i x^i \in \mathbb{Z}_q[x]$ be its Teichmüller lifting.

- C. Liu and W (2008, ANT) introduced the T-adic L-function

$$\mathcal{L}(T, s) = L(\chi_{\text{univ}}, s) \in 1 + s\mathbb{Z}_p[[T]][[s]],$$

where

$$\chi_{\text{univ}} : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p[[T]]^{\times}, \ \chi_{\text{univ}}(b) = (1+T)^b$$

parameterizes all continuous $\mathbb{C}_p$ valued characters χ of $\mathrm{Gal}(C_\infty/C) = \mathbb{Z}_p$, possibly infinite order. The parameter $t = \chi(1) - 1$ varies in the open unit disc $|t|_p < 1$.

- Non-trivial characters are parameterized by the points in the punctured unit disk $0 < |t|_p < 1$.

If $\chi_n(1) = \zeta_{p^n}$, then $t_n = \chi_n(1) - 1 = \zeta_{p^n} - 1$ is a classical point and $L(\chi_n, s) = \mathcal{L}(t_n, s)$. Note that $v_p(t_n) = 1/p^{n-1}(p-1) \to 0$. For large n, t_n is in the annulus $\epsilon < |t|_p < 1$.

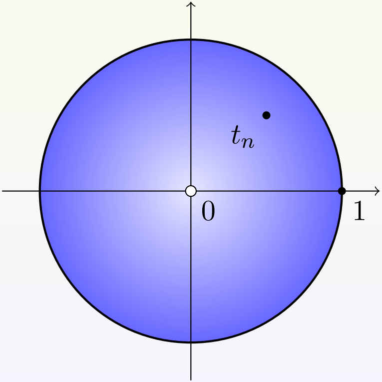


Figure: Pic 1

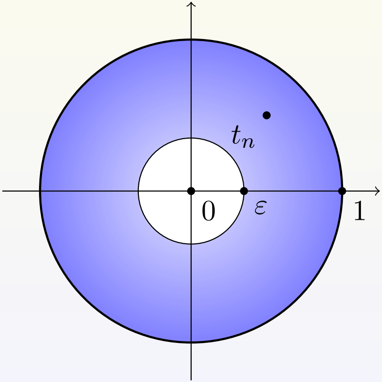


Figure: Pic 2

- Over $\mathbb{G}_m$, the T-adic Dwork trace formula gives

$$\mathcal{L}^*(T,s) = (1-s)\mathcal{L}(T,s) = \frac{D(T,s)}{D(T,qs)},$$

$$\mathcal{L}(T,s) = \frac{D(T,s)}{(1-s)D(T,qs)},$$

where $D(T,s)$ is a T-adic entire function in s.

- Over $\mathbb{G}_m$, the T-adic Dwork trace formula gives

$$\mathcal{L}^*(T,s) = (1-s)\mathcal{L}(T,s) = \frac{D(T,s)}{D(T,qs)},$$

$$\mathcal{L}(T,s) = \frac{D(T,s)}{(1-s)D(T,qs)},$$

where $D(T,s)$ is a T-adic entire function in s.
Note $\mathcal{L}^*(T,s)$ is T-adic meromorphic in s, but neither rational nor entire! (no cohomological formula here)

- Over $\mathbb{G}_m$, the T-adic Dwork trace formula gives

$$\mathcal{L}^*(T,s) = (1-s)\mathcal{L}(T,s) = \frac{D(T,s)}{D(T,qs)},$$

$$\mathcal{L}(T,s) = \frac{D(T,s)}{(1-s)D(T,qs)},$$

where $D(T,s)$ is a T-adic entire function in s.
Note $\mathcal{L}^*(T,s)$ is T-adic meromorphic in s, but neither rational nor entire! (no cohomological formula here)

- For $q = p^a$, $T^{a(p-1)}$-adic $\mathrm{NP}(D(T,s)) \geq \mathrm{HP}(D)$, where $\mathrm{HP}(D)$ is the polygon with slopes

$$\{\frac{0}{d}, \frac{1}{d}, \frac{2}{d}, \cdots, \frac{k}{d}, \cdots\}.$$

- Over $\mathbb{G}_m$, the T-adic Dwork trace formula gives

$$\mathcal{L}^*(T,s) = (1-s)\mathcal{L}(T,s) = \frac{D(T,s)}{D(T,qs)},$$

$$\mathcal{L}(T,s) = \frac{D(T,s)}{(1-s)D(T,qs)},$$

where $D(T,s)$ is a T-adic entire function in s.
Note $\mathcal{L}^*(T,s)$ is T-adic meromorphic in s, but neither rational nor entire! (no cohomological formula here)

- For $q = p^a$, $T^{a(p-1)}$-adic NP$(D(T,s)) \geq \mathrm{HP}(D)$, where $\mathrm{HP}(D)$ is the polygon with slopes

$$\{\frac{0}{d}, \frac{1}{d}, \frac{2}{d}, \cdots, \frac{k}{d}, \cdots\}.$$

- Equality "$t^{a(p-1)}$-adic NP$(D(t,s)) = \mathrm{HP}(D)$" holds for **all** $0 < |t|_p < 1$ if and only if it holds if one **single** $0 < |t|_p < 1$.

- If $p \equiv 1 \mod d$, equality holds at classical point $t_1 = \zeta_p - 1$, thus equality holds for all $0 < |t|_p < 1$.

- At classical point $t_n = \zeta_{p^n} - 1$, $v_q(t_n^{a(p-1)}) = 1/p^{n-1}$, thus the q-slope sequence for $D(t_n, s)$ is

$$\{\frac{0}{dp^{n-1}}, \frac{1}{dp^{n-1}}, \frac{2}{dp^{n-1}}, \cdots,\}.$$

  This is an arithmetic progression!

- The slope sequence for the polynomial

$$L(\chi_n, s) = \frac{D(t_n, s)}{(1-s)D(t_n, qs)}, \ n \geq 1$$

  is just the truncation

$$\{\frac{1}{dp^{n-1}}, \frac{2}{dp^{n-1}}, \cdots, \frac{dp^{n-1}-1}{dp^{n-1}}\}.$$

- If $p \not\equiv 1 \mod d$, the slope sequence is much more complicated, even for $n = 1$.

- If $p \not\equiv 1 \mod d$, the slope sequence is much more complicated, even for $n = 1$. But we have
- Davis-Xiao-Wan (Math Ann., 2016). For $q = p^a$ and all

$$n \geq n_0 := \lceil 1 + \log_p \frac{(d-1)^2 a}{8d} \rceil, \ (n_0 = 2, \text{if } p \geq \frac{da}{8})$$

the slope sequence for $L(\chi_n, s)$ is

$$\bigcup_{i=0}^{p^{n-n_0}-1} \{\frac{i}{p^{n-n_0}}, \frac{i+s_1}{p^{n-n_0}}, \frac{i+s_2}{p^{n-n_0}}, \cdots, \frac{i+s_{dp^{n_0-1}-1}}{p^{n-n_0}}\} - \{0\},$$

a truncation of dp^{n_0-1} arithmetic progressions!

- If $p \not\equiv 1 \mod d$, the slope sequence is much more complicated, even for $n = 1$. But we have

- Davis-Xiao-Wan (Math Ann., 2016). For $q = p^a$ and all

$$n \geq n_0 := \lceil 1 + \log_p \frac{(d-1)^2 a}{8d} \rceil, \ (n_0 = 2, \text{if } p \geq \frac{da}{8})$$

the slope sequence for $L(\chi_n, s)$ is

$$\bigcup_{i=0}^{p^{n-n_0}-1} \{\frac{i}{p^{n-n_0}}, \frac{i+s_1}{p^{n-n_0}}, \frac{i+s_2}{p^{n-n_0}}, \cdots, \frac{i+s_{dp^{n_0-1}-1}}{p^{n-n_0}}\} - \{0\},$$

a truncation of dp^{n_0-1} arithmetic progressions!

- $\{s_1, s_2, \cdots, s_{dp^{n_0-1}-1}\}$ is the slope sequence for $L(\chi_{n_0}, s)$. Thus, knowing the slopes at level n_0 gives the slopes at all level $n \geq n_0$, a finiteness property called **slope stability**.

Outline of Proof.

- For all $0 < |t|_p < 1$, $t^{a(p-1)}$-adic NP$(D(t,s))$ touches $\mathrm{HP}(D)$, at all points when $x = dk$ and $dk+1$ $(k = 0, 1, \cdots,)$ by considering its specialization at the classical point $t_1 = \zeta_p - 1$. This gives a rather tight upper bound for Newton polygon.

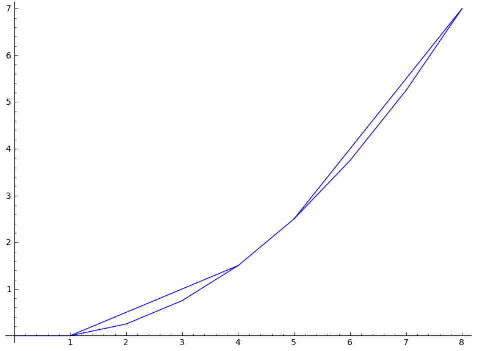


Figure: The upper and lower bounds for the Newton polygon over the interval $[0, 8]$ for $d = 4$.

- Lemma. This and $\dim(\mathbb{Z}_p) = 1$ imply $t^{a(p-1)}$-adic NP for $D(t, s)$ is **independent** of t in the annulus $\epsilon < |t|_p < 1$.

- Lemma. This and $\dim(\mathbb{Z}_p) = 1$ imply $t^{a(p-1)}$-adic NP for $D(t, s)$ is **independent** of t in the annulus $\epsilon < |t|_p < 1$.

- For any large n_0, this common NP is the $t_{n_0}^{a(p-1)}$-adic NP of

$$D(t_{n_0}, s) = \prod_{i=0}^{\infty} L(\chi_{n_0}/\mathbb{G}_m, q^i s).$$

Its $t_{n_0}^{a(p-1)}$-adic slope sequence is (as $v_q(t_{n_0}^{a(p-1)}) = 1/p^{n_0}$):

$$\bigcup_{i=0}^{\infty} \{p^{n_0} i, p^{n_0}(i + s_1), \cdots, p^{n_0}(i + s_{dp^{n_0-1}-1})\},$$

where $\{0, s_1, \cdots, s_{dp^{n_0-1}-1}\}$ is the q-adic slopes of the polynomial $L(\chi_{n_0}/\mathbb{G}_m, s)$.

- For all $n \geq n_0$, $t_n^{a(p-1)}$-adic slope sequence for $D(t_n, s)$ is

$$\bigcup_{i=0}^{\infty}\{p^{n_0}i, p^{n_0}(i+s_1), \cdots, p^{n_0}(i+s_{dp^{n_0-1}-1})\}.$$

- For all $n \geq n_0$, q-adic slope sequence for $D(t_n, s)$ is

$$\bigcup_{i=0}^{\infty}\{p^{n_0-n}i, p^{n_0-n}(i+s_1), \cdots, p^{n_0-n}(i+s_{dp^{n_0-1}-1})\}.$$

- For all $n \geq n_0$, the q-adic slope sequence for

$$L(\chi_n, s) = D(t_n, s)/(1-s)D(t_n, qs)$$

is the truncation

$$\bigcup_{i=0}^{p^{n-n_0}-1}\{p^{n_0-n}i, p^{n_0-n}(i+s_1), \cdots, p^{n_0-n}(i+s_{dp^{n_0-1}-1})\}-\{0\}.$$

- These ideas played a crucial role in later papers by
  Wan-Xiao-Zhang (Math Ann., 2017)
  R. Liu-Wan-Xiao (Duke Math., 2017)
  on slopes of p-adic modular forms and the spectral halo conjecture for eigencurves near the boundary.

- These ideas played a crucial role in later papers by
  Wan-Xiao-Zhang (Math Ann., 2017)
  R. Liu-Wan-Xiao (Duke Math., 2017)
  on slopes of p-adic modular forms and the spectral halo conjecture for eigencurves near the boundary.

- What is the optimal condition to guarantee slope stability?

- These ideas played a crucial role in later papers by
  Wan-Xiao-Zhang (Math Ann., 2017)
  R. Liu-Wan-Xiao (Duke Math., 2017)
  on slopes of p-adic modular forms and the spectral halo conjecture for eigencurves near the boundary.

- What is the optimal condition to guarantee slope stability?

  Slope stability $\Rightarrow$ strong genus stability. Is the converse true?

- These ideas played a crucial role in later papers by
  Wan-Xiao-Zhang (Math Ann., 2017)
  R. Liu-Wan-Xiao (Duke Math., 2017)
  on slopes of p-adic modular forms and the spectral halo conjecture for eigencurves near the boundary.

- What is the optimal condition to guarantee slope stability?

  Slope stability $\Rightarrow$ strong genus stability. Is the converse true?

- **Slope Question**. Strong genus stablility $\Rightarrow$ slope stability?

  i.e., the slope sequence for $L(\chi_n, s)$ is given by a truncation of a finite number of arithmetic progressions for all large n?

- X. Li (JNT, 2017) extended the slope stability to any polynomial $f(x)$ using (p, T)-adic topology.

- X. Li (JNT, 2017) extended the slope stability to any polynomial $f(x)$ using (p, T)-adic topology.

- Kosters-Zhu (JNT, 2018) almost settled the slope question, by working with $(p, T_1, T_2, \cdots)$-adic topology in **A** and deforming to polynomial $f(x)$ case in **B**.

A . If strongly genus stable, approximate slope stability holds.

B. If strongly genus stable and for some explicit $h > 0$,

$$\frac{n}{p^{v_p(c_n)}} \leq \max_{(i,p)=1}\{\frac{i}{p^{v_p(c_i)}}\} - \frac{h}{p^{v_p(c_n)}}, \ n \gg 0,$$

then full slope stability holds.

- X. Li (JNT, 2017) extended the slope stability to any polynomial $f(x)$ using (p, T)-adic topology.

- Kosters-Zhu (JNT, 2018) almost settled the slope question, by working with $(p, T_1, T_2, \cdots)$-adic topology in **A** and deforming to polynomial $f(x)$ case in **B**.

  A . If strongly genus stable, approximate slope stability holds.

  B. If strongly genus stable and for some explicit $h > 0$,

$$\frac{n}{p^{v_p(c_n)}} \leq \max_{(i,p)=1}\{\frac{i}{p^{v_p(c_i)}}\} - \frac{h}{p^{v_p(c_n)}}, \ n \gg 0,$$

  then full slope stability holds.

- Can one take $h = 0$ in **B** and thus settle the slope question?

- Slopes for $L(\mathrm{Sym}^k \rho_E, s)$, where ρ_E is the rank two F-crystal from the universal elliptic curve over $\mathbb{F}_p$.

  (slopes of modular forms of weight $k+2$).

- Slopes for $L(\mathrm{Sym}^k \rho_E, s)$, where ρ_E is the rank two F-crystal from the universal elliptic curve over $\mathbb{F}_p$.

  (slopes of modular forms of weight $k+2$).

- There are many slope conjectures by Gouvea, Gouvea-Mazur, Buzzard, Bergdall-Pollack.

- Slopes for $L(\mathrm{Sym}^k \rho_E, s)$, where ρ_E is the rank two F-crystal from the universal elliptic curve over $\mathbb{F}_p$.

  (slopes of modular forms of weight $k+2$).

- There are many slope conjectures by Gouvea, Gouvea-Mazur, Buzzard, Bergdall-Pollack.

- A uniform quadratic lower bound for NP and a quadratic upper bound for GM conjecture (W, Invent Math. 1998).

- Slopes for $L(\mathrm{Sym}^k \mathrm{Kl}_2, s)$, where Kl_2 is the rank two Kloosterman F-crystal over $\mathbb{F}_p$.
- Similar slope conjectures are expected.

- Slopes for $L(\mathrm{Sym}^k \mathrm{Kl}_2, s)$, where Kl_2 is the rank two Kloosterman F-crystal over $\mathbb{F}_p$.
- Similar slope conjectures are expected.
- A uniform quadratic lower bound for NP by Doug Haessig (Math. Ann. 2017). Studied the first example of p-adic cohomology of infinite symmetric power!

- Slopes for $L(\mathrm{Sym}^k \mathrm{Kl}_2, s)$, where Kl_2 is the rank two Kloosterman F-crystal over $\mathbb{F}_p$.
- Similar slope conjectures are expected.
- A uniform quadratic lower bound for NP by Doug Haessig (Math. Ann. 2017). Studied the first example of p-adic cohomology of infinite symmetric power!
- An improved quadratic lower bound by Fresán-Sabbah-Yu (arXiv 2018) who also established the potential modularity of the motive $L(\mathrm{Sym}^k \mathrm{Kl}_2, s)$ for all $k > 0$, using different method (exponential motives and irregular Hodge theory).

Thank You!