Moments and non-vanishing of L-functions

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Moments of the Riemann Zeta Function

Let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \qquad \Re(s) > 1.$$

It was shown by Riemann that $\zeta(s)$ satisfies the functional equation

$$\Lambda(s) = \Lambda(1-s), \quad \text{where } \Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

and then has meromorphic continuation to the complex plane, with a simple pole at $s = 1$.

Moments of the Riemann Zeta Function

Some of the deepest questions of analytic number theory are concerned with the size and the zeroes of $\zeta(s)$ inside the critical strip $0 < \Re(s) < 1$.

Riemann Hypothesis:
If $\zeta(s) = 0$ and $0 < \Re(s) < 1$, then $\Re(s) = \frac{1}{2}$.

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Moments of $\zeta(\frac{1}{2} + it)$: There exists g_k such that

$$\lim_{T \to \infty} \frac{1}{(\log T)^{k^2}} \frac{1}{T} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt = a_k g_k$$

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- $k = 1$: Hardy and Littlewood with $g_1 = 1$ (1918)
- $k = 2$: Ingham, with $g_2 = 1/12$ (1928)

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What is the combinatorial constant g_k?

- $k = 3$ Conrey and Ghosh (1992) conjectured that $g_3 = \frac{42}{9!}$
- $k = 4$ Conrey and Gonek (1998) conjectured that $g_4 = \frac{24024}{16!}$

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Random Matrix Model: Following the work of Montgomery (1973), and then Katz and Sarnak (1999), we believe that statistics on the zeroes of L-functions match the statistics on the eigenvalues of random matrices in certain symmetry groups associated to the family.

Moments of unitary matrices

Let $Z(U,\theta) = \det(I - Ue^{-i\theta})$ be the characteristic polynomial of a of $N \times N$ unitary matrix $U \in U(N)$. The set $U(N)$ is a probability space with respect to the Haar measure.

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Keating and Snaith (2000) showed

$$\lim_{N\to\infty} \frac{1}{N^{k^2}} \int_{U(N)} |Z(U,\theta)|^{2k}\, dU = g_k, \quad g_k = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

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Keating and Snaith Conjecture:

$$\int_{-T}^{T} \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \sim a_k g_k\ T\,(\log T)^{k^2}, \quad g_k = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

Families of L-functions

From the work of Katz and Sarnak, we are lead to believe that statistics for families of L-functions are similar to statistics for $\zeta(\frac{1}{2} + it), \; t \in [0, T]$, and are also modelled by random matrix theory. Each family has a symmetry type (unitary, symplectic or orthogonal) which governs the statistics.

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Let χ be a Dirichlet character of modulus q, i.e. a multiplicative function

$$\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*,$$

extended to the integers by periodicity and by setting $\chi(a) = 0$ if $(a, q) \neq 1$. The Dirichlet L-function associated to χ is

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \qquad \Re(s) > 1.$$

Functional equation for Dirichlet L-functions

If χ is a primitive character of conductor q (and χ is even), then

$$\Lambda(s,\chi) = \frac{\tau(\chi)}{q^{1/2}}\,\Lambda(1-s,\overline{\chi}), \quad \Lambda(s,\chi) = \left(\frac{\pi}{q}\right)^{-s/2}\Gamma(s/2)L(s,\chi)$$

where the *sign of the functional equation* involves the Gauss sum

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m) e^{2\pi i m/q}.$$

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It is easy to show that $|\tau(\chi)| = q^{1/2}$ for all primitive characters.

Quadratic Dirichlet L-functions

For quadratic characters $\chi^2 = \chi_0$, it was showed by Gauss that for $\chi(\cdot) = \left(\frac{\cdot}{q}\right)$, q prime, we have

$$\tau(\chi) = \begin{cases} q^{1/2} & q \equiv 1 \bmod 4 \\ iq^{1/2} & q \equiv 3 \bmod 4, \end{cases}$$

and for any primitive quadratic character of conductor q

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• If $L(\frac{1}{2},\chi) = 0$ for a quadratic character, then the zero has order at least 2.

Moments of quadratic Dirichlet L-functions

Using Random Matrix Theory, Keating and Snaith (2000) conjectured that

$$\sum^{*}_{\substack{\chi^2=\chi_0 \\ \mathrm{cond}(\chi)\le X}} L(\tfrac{1}{2},\chi)^k \sim a'_k g'_k \, X \, (\log X)^{\frac{k(k+1)}{2}}$$

with an explicit formula for the combinatorial constant g'_k coming from the random matrix model.

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$$\sum_{\substack{\chi^2=\chi_0 \\ \mathrm{cond}(\chi)\leq X}}^{*} L(\tfrac{1}{2},\chi)^k \sim a_k' g_k' \, X \, (\log X)^{\frac{k(k+1)}{2}}$$

with an explicit formula for the combinatorial constant g_k' coming from the random matrix model.

Comparing with the moments of $\zeta(\frac{1}{2}+it)$,

$$\int_{-T}^{T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \sim a_k g_k \, T \, (\log T)^{k^2}$$

we notice that the formula are different (symplectic versus unitary).

Moments of quadratic Dirichlet L-functions

Conjecture:

$$\sideset{}{^*}\sum_{\substack{\chi^2=\chi_0 \\ \mathrm{cond}(\chi)\leq X}} L(\tfrac{1}{2},\chi)^k \sim a'_k g'_k \, X \, (\log X)^{\frac{k(k+1)}{2}}.$$

- $k=1$ Jutila (1981)
- $k=2$ Jutila (1981), Soundararajan (secondary terms, 2000)
- $k=3$ Soundararajan (2000), Diaconu, Goldfeld, Hoffstein (2003)
- $k=4$ Shen (2019, under GRH)

Non-vanishing of Dirichlet L-functions

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The conjectures of Katz and Sarnak on the one-level density in families of L-functions imply that $L(\frac{1}{2}, \chi) \neq 0$ for all L-functions in a family $\mathscr{F}$ except a set of density 0.
The conjectures of Katz and Sarnak cover number fields and function fields. Over function fields, we will see that Chowla's conjecture is not true (Li, 2018).

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For a given family $\mathscr{F}$, can we prove:

- Infinitely many $\chi \in \mathscr{F}$ such that $L(\frac{1}{2}, \chi) \neq 0$?
- A positive density of $\chi \in \mathscr{F}$ such that $L(\frac{1}{2}, \chi) \neq 0$?
- $L(\frac{1}{2}, \chi) \neq 0$ for all $\chi \in \mathscr{F}$ except a set of density 0?
- $L(\frac{1}{2}, \chi) \neq 0$ for all $\chi \in \mathscr{F}$?

Non-vanishing of quadratic Dirichlet L-functions

By Cauchy–Schwarz,

$$\sideset{}{^*}\sum_{\substack{\chi^2=\chi_0\\ \mathrm{cond}(\chi)\le X}} L(\tfrac{1}{2},\chi) \le \left(\sideset{}{^*}\sum_{\substack{\chi^2=\chi_0\\ \mathrm{cond}(\chi)\le X}} L\left(\tfrac{1}{2},\chi\right)^2\right)^{1/2} \left(\sideset{}{^*}\sum_{\substack{\chi^2=\chi_0\\ \mathrm{cond}(\chi)\le X\\ L\left(\frac{1}{2},\chi\right)\neq 0}} 1\right)^{1/2}$$

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$$\implies N_2(X) \ge \frac{\left(\displaystyle\sideset{}{^*}\sum_{\substack{\chi^2=\chi_0\\ \mathrm{cond}(\chi)\le X}} L\left(\tfrac12,\chi\right)\right)^2}{\displaystyle\sideset{}{^*}\sum_{\substack{\chi^2=\chi_0\\ \mathrm{cond}(\chi)\le X}} L\left(\tfrac12,\chi\right)^2}$$

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$$\Longrightarrow N_2(X) \ge \frac{\left(\sideset{}{^*}\sum_{\substack{\chi^2=\chi_0\\ \mathrm{cond}(\chi)\le X}} L\left(\tfrac{1}{2},\chi\right)\right)^2}{\sideset{}{^*}\sum_{\substack{\chi^2=\chi_0\\ \mathrm{cond}(\chi)\le X}} L\left(\tfrac{1}{2},\chi\right)^2} \gg \frac{(X\log X)^2}{X\log^3 X} = \frac{X}{\log X}.$$

using the first 2 moments.

Non-vanishing of quadratic Dirichlet L-functions

Using the first 2 moments (an upper bound is sufficient for the second moment), we get that

$$N_2(X) := \sum_{\substack{\chi^2=\chi_0 \\ \mathrm{cond}(\chi)\le X \\ L(\frac{1}{2},\chi)\neq 0}}^{*} 1 \gg \frac{X}{\log X}, \quad \text{where} \sum_{\substack{\chi^2=\chi_0 \\ \mathrm{cond}(\chi)\le X}}^{*} 1 \sim c_2 X$$

which is infinitely many, but not a positive density.
The moments grow too fast to get a positive density.

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The moments grow too fast to get a positive density.

We need to use a mollifier to control the growth of the moments.
The idea of a mollifier goes back to Bohr and Landau (1914) to study the zeroes of $\zeta(s)$. Selberg (1946) used it to show that a positive proportion of the non-trivial zeroes of $\zeta(s)$ satisfy RH.

Mollified moments and non-vanishing

For the moments, we want a mollifier $M(\chi) \approx L(\frac{1}{2}, \chi)^{-1}$ such that

$$\sum_{\substack{\chi^2=\chi_0 \\ \mathrm{cond}(\chi)\le X}}^{*} L(\tfrac{1}{2}, \chi)^k M(\chi)^k \sim c_k\, X,$$

compared to the non-mollified moments

$$\sum_{\substack{\chi^2=\chi_0 \\ \mathrm{cond}(\chi)\le X}}^{*} L(\tfrac{1}{2}, \chi)^k \sim a_k' g_k'\, X\, (\log X)^{\frac{k(k+1)}{2}}.$$

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Soundararajan (2000) computed the first two mollified moments and showed that $L(\frac{1}{2}, \chi) \neq 0$ for at least 87.5% of quadratic χ.

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Under GRH, Ozluk and Snyder (1999) showed that $L(\frac{1}{2}, \chi) \neq 0$ for at least 93.75% of quadratic χ, using the one-level density.

Cubic Dirichlet characters

Let χ be a cubic character of modulus p

$$\chi : (\mathbb{Z}/p\mathbb{Z})^* \to \{1, \omega, \omega^2\} \subseteq \mathbb{C}^*, \quad \omega = e^{2\pi i/3},$$

with $p \equiv 1 \bmod 3$.

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with $p \equiv 1 \bmod 3$.

For each prime $p \equiv 1 \bmod 3$, we have the two cubic residue symbols

$$\chi_p(a) \equiv a^{(p-1)/3} \bmod p, \quad \text{and the conjugate } \overline{\chi}_p,$$

which extend by multiplicativity to all (primitive) cubic characters.

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which extend by multiplicativity to all (primitive) cubic characters.

The functional equation for a primitive cubic character χ of modulus q is

$$\Lambda(s, \chi) = \frac{\tau(\chi)}{\sqrt{q}}\, \Lambda(1 - s, \overline{\chi}), \quad \text{where } \tau(\chi) \text{ is the cubic Gauss sum.}$$

Moments for cubic characters

Conjecture:

$$\sum_{\substack{\chi^3=\chi_0 \\ \mathrm{cond}(\chi)\le X}}^{*} |L(\tfrac{1}{2},\chi)|^{2k} \sim a_k'' \, g_k \, X \, (\log X)^{k^2}$$

$$\sum_{\substack{\chi^3=\chi_0 \\ \mathrm{cond}(\chi)\le X}}^{*} L(\tfrac{1}{2},\chi)^{\ell} \sim c_\ell X.$$

The family of cubic characters has unitary symmetry.

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- over $\mathbb{Q}$ (non-Kummer case): $\ell = 1$ Baier & Young (2010)
- over $\mathbb{Q}(\omega)$ (Kummer case, Hecke L-functions): $\ell = 1$ Luo (2004) and Diaconu (2004) for a thin subfamily of the cubic characters (taking χ_π and not $\overline{\chi}_\pi$)

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Moments and non-vanishing: Using Cauchy-Schwarz and bounding the second moment, this gives infinitively many cubic χ such that $L(\frac{1}{2},\chi) \neq 0$ over $\mathbb{Q}$ and $\mathbb{Q}(\omega)$.

Kummer, non-Kummer and the thin subfamily

The family of cubic characters is different depending if the ground field contains the cube roots of unity (the Kummer case) or not (the non-Kummer case).

$$\begin{aligned} \#\left\{\chi/\mathbb{Q} \; : \; \chi^3 = \chi_0, \; \mathrm{cond}(\chi) \le X\right\} &\sim aX \\ \#\left\{\chi/\mathbb{Q}[\omega] \; : \; \chi^3 = \chi_0, \; \mathrm{N}\,\mathrm{cond}(\chi) \le X\right\} &\sim bX \log X \\ \#\left\{\chi/\mathbb{Q}[\omega] \; : \; \chi = \left(\frac{\cdot}{c}\right)_3, c \text{ SF}, \; \mathrm{N}\,\mathrm{cond}(\chi) \le X\right\} &\sim cX \end{aligned}$$

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Over $\mathbb{Q}$, there are 2 characters of conductor p for $p \equiv 1 \bmod 3$.

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Over $\mathbb{Q}[\omega]$, there are 2 characters of conductor π for each prime π.

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Over $\mathbb{Q}[\omega]$, there are 2 characters of conductor π for each prime π.

For the thin subfamily, there is 1 character of conductor π for each prime π, picking χ_π but not $\overline{\chi}_\pi$.

Number fields and Function fields

Let q power of a prime, $\mathbb{F}_q$ finite field with q elements.

Number Fields		Function Fields
$\mathbb{Q}, \mathbb{Z}$	$\leftrightarrow$	$\mathbb{F}_q(T), \mathbb{F}_q[T]$
p positive prime	$\leftrightarrow$	$P(T)$ monic irreducible polynomial
$\lvert n\rvert = \lvert\mathbb{Z}/n\mathbb{Z}\rvert = n \in \mathbb{N}$	$\leftrightarrow$	$\lvert F(T)\rvert = \lvert\mathbb{F}_q[T]/(F(T))\rvert = q^{\deg F}$
$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$	$\leftrightarrow$	$\zeta_q(s) = \sum_{\substack{F \in \mathbb{F}_q[T] \\ F \text{ monic}}} \frac{1}{\lvert F\rvert^s}$

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Moments of quadratic L-functions over function fields

Andrade and Keating (2014) conjectured

$$\sideset{}{^*}\sum_{\substack{\chi^2=\chi_0 \\ \deg \operatorname{cond}(\chi)=d}} L(\tfrac{1}{2},\chi)^k \sim c_k g_k' \, q^d \, d^{\frac{k(k+1)}{2}}$$

and proved this for $k = 1$.

- Florea (2017, several papers): second order term for $k = 1$ and cases $k = 2, 3, 4$.
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- Li (2018): vanishing for $\gg (q^d)^{\frac{1}{3}-\varepsilon}$ of the quadratic characters with conductor of degree d.

Cubic characters over $\mathbb{F}_q[T]$

- Kummer case $q \equiv 1 \bmod 3$: There are 2 cubic characters of conductor P for every prime $P \in \mathbb{F}_q[T]$.
- Non-Kummer case $q \equiv 2 \bmod 3$: There are 2 characters of conductor P for every prime P of even degree $P \in \mathbb{F}_q[T]$.

Then,

$$\sideset{}{^*}\sum_{\substack{\chi^3=\chi_0 \\ \deg \operatorname{cond}(\chi)=d}} 1 \sim \begin{cases} C_1 q^d & q \equiv 2 \bmod 3 \\ C_2 d q^d & q \equiv 1 \bmod 3 \end{cases}$$

First moments of cubic L-functions over function fields

Theorem (D., Florea, Lalin (2019))

Let q be an odd prime power such that $q \equiv 2 \pmod 3$. Then

$$\sideset{}{^*}\sum_{\substack{\chi^3=\chi_0 \\ \deg \mathit{cond}(\chi)=d}} L(\tfrac{1}{2},\chi) = Cq^d + O\left(q^{d\left(\frac{7}{8}+\varepsilon\right)}\right).$$

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Nonvanishing for cubic L-functions - Non Kummer case

- Bounding the second moment and using Cauchy-Schwartz (DFL, 2019): nonvanishing for $\gg (q^d)^{1-\varepsilon}$.
- Ellenberg-Li-Shusterman (2020) using algebraic geometry methods to prove nonvanishing for $\gg q^d/d^{1/2}$.
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Theorem (D., Florea, Lalin (2020))

Let q be an odd prime power such that $q \equiv 2 \pmod 3$.

$$\# \left\{\chi^3 = \chi_0,\ \deg cond(\chi) = d,\ L\left(\tfrac{1}{2}, \chi\right) \neq 0\right\}^* \geq cq^d,$$

where $c > 0$ is an explicit constant.

This is the first result about a positive proportion for a cubic family. It uses mollified moments.

The results would be the same over number fields, under GRH.

Parenthesis: Nonvanishing for cubic (Kummer)

In the Kummer case, it is possible to find a a positive proportion of non-vanishing inside inside the thin subfamily, using the one-level density. This was done over number fields and then under GRH.

Theorem (D., Güloğlu (2020))

Assume GRH. Then

$$\#\left\{\chi = \left(\frac{\cdot}{c}\right)_3, c \in \mathbb{Q}[\omega], c\ SF, N(c) \leq X, L\left(\tfrac{1}{2}, \chi\right) \neq 0\right\} \geq \frac{2}{13} X.$$

This is a positive proportion in the thin subfamily of cubic residue symbols $\left(\frac{\cdot}{c}\right)_3$, where $c \in \mathbb{Z}[\omega]$ is square-free.

Technique of mollified moments (non-Kummer)

- We compute the first mollified moment, following the techniques of DFL for the first moment:

$$\sideset{}{^*}\sum_{\substack{\chi^3=\chi_0 \\ \deg \operatorname{cond}(\chi)=d}} L(\tfrac{1}{2},\chi)M(\chi) \sim B_1 q^d,$$

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- We compute a sharp upper bound for the second mollified moment, using the recent techniques of Soundararajan (2000), Harper (2013) and Lester-Radziwill (2019):

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Then, by Cauchy–Schwarz,

$$\#\left\{\chi^3=\chi_0,\ \deg \operatorname{cond}(\chi)=d,\ L\left(\tfrac{1}{2},\chi\right)\neq 0\right\}^* \geq \frac{B_1^2}{B_2} q^d.$$

The first mollified moment

$$M(\chi) \approx \sum_{\deg h \leq N} \frac{\mu(h) w(h; N) \chi(h)}{|h|^{1/2}} \text{ behaves "like } L(\tfrac{1}{2}, \chi)^{-1}\text{"}.$$

Reversing the sums, we first evaluate

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- Approximate functional equation for $L(\frac{1}{2}, \chi)$ gives a principal sum and a dual sum.
- The main term comes from the cubes in the principal sum.
- The non-cubes from the principal sum can be bounded by Lindelöf.
- The dual sum contains Gauss sums, and averages of cubic Gauss sums can be evaluated/bounded from the deep work of Kubota and Patterson.

Bounding the mollified second moment

We need to show that

$$\sum_{\substack{\chi^3=\chi_0 \\ \deg \operatorname{cond}(\chi)=d}}^{*} |L(\tfrac{1}{2},\chi)M(\chi)|^2 \leq B_2 q^d,$$

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- Soundararajan (2009): almost sharp bounds for all moments of $\zeta(s)$, under RH: $\ll_k T(\log T)^{k^2+\varepsilon}$.
- Harper (2013): sharp bounds for all moments of $\zeta(s)$, under RH: $\ll_k T(\log T)^{k^2}$.
- Lester & Radziwiłł (2019): sharp bounds for all mollified moments of quadratic twists of modular forms, under GRH.

Bounding $|L(\frac{1}{2}, \chi)|$

Under GRH, we can bound $\log |L(\frac{1}{2}, \chi)|$ by an extremely short Dirichlet polynomial. (Soundararajan (2009), adapted by Bui, Florea, Keating, Roditty-Gershon (2019) to function fields):

$$\log |L(\tfrac{1}{2}, \chi)| \leq \sum_{\deg(P) \leq N} \frac{\Re(\chi(P)) a(P; N)}{|P|^{\frac{1}{2}}} + \frac{d}{N} + O(1),$$

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where $d = \deg \operatorname{cond}(\chi)$.

Taking $N \approx \theta d$ for some θ small enough, we have

$$|L(\tfrac{1}{2}, \chi)|^k \leq c_k \exp\left(\frac{k}{\theta}\right) \exp\left(k \sum_{\deg P \leq \theta d} \frac{\Re(\chi(P)) a(P, N)}{|P|^{\frac{1}{2}}}\right).$$

Bounding the exponential

We need to bound

$$\exp\left(k \sum_{\deg P \le N} \frac{\Re(\chi(P))a(P,N)}{|P|^{\frac{1}{2}}}\right) = \exp\left(k \Re P_N(\chi)\right),$$

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Lester & Radziwiłł (2019): Use the bound

$$e^t \le (1 + e^{-\ell/2}) \sum_{s \le \ell} \frac{t^s}{s!}$$

for $t \le \ell/e^2$ and ℓ even.

Bounding the mollified moment with the exponential

Using $N = \theta d$, we get the bound

$$\sideset{}{^*}\sum_{\substack{\chi^3=\chi_0 \\ \deg \operatorname{cond}(\chi)=d}} (\Re P_N(\chi))^s |M(\chi)|^k \ll q^d d^{O(1)} (C(\log d)^{1/2}\theta)^{1/\theta}.$$

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Taking $N = \frac{d}{\log d}$, we get the bound

$$\ll \frac{q^d}{(\log d)^{(\log d)/2}} \ll \frac{q^d}{d^{1000000000}}.$$

But the short approximation is too short.

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Following Lester & Radziwiłł (2019), we separate $I = (0, \theta d]$ in:

$$I_0 = (0, d\theta_0], \quad I_1 = (d\theta_0, d\theta_1], \quad \dots, \quad I_J = (d\theta_{J-1}, d\theta_J],$$

$$\theta_j = \frac{e^j}{(\log d)^{1000}}, \quad \theta_J = \theta, \quad J \approx \log\log d$$

A positive proportion

Collecting the results from all the intervals, we get a sharp upper bound of the type

$$\sum_{\substack{\chi^3=\chi_0 \\ \deg \mathrm{cond}(\chi)=d}}^{*} |L(\tfrac{1}{2},\chi)M(\chi)|^{2k} \leq e^{e^{O(k)}} q^d.$$

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Let q be an odd prime power such that $q \equiv 2 \pmod 3$. Then

$$\frac{\#\left\{\chi^3=\chi_0,\ \deg \mathit{cond}(\chi)=d,\ L(\tfrac{1}{2},\chi)\neq 0\right\}^*}{\#\{\chi^3=\chi_0,\ \deg \mathit{cond}(\chi)=d\}^*} \geq 0.47e^{-e^{182}}.$$