Bounding ramification with covers and curves

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Lefschetz theorem: topology

X sm q-proj var over $\mathbb{C}$, $\pi_1^{\rm top}(X, x)$ top fund gr based at $x \in X(\mathbb{C})$.

Theorem (Lefschetz)

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$\mathbb{C} \rightsquigarrow k$ alg. cl. of char. 0, $\pi_1^{\rm top}(X,x) \rightsquigarrow \pi_1(X,x)$ Grothendieck's étale fundamental gr $\rightsquigarrow$ same thm (and tiny rmk).

No Lefschetz $/k$ of char. $p > 0$

No Lefschetz thm: eg $X = \mathbb{A}^2$, Artin-Schreier cover $t^p - t = f, f \in \mathcal{O}(\mathbb{A}^2)$ splits on curve $C : f = 0$. So $\nexists C$ with $\pi_1(C, x) \twoheadrightarrow \pi_1(X, x)$.

Tameness: Kerz-Schmidt's definition

Recall: R complete (or henselian) dvr, finite Galois ext $R \subset S$ of such, perfect res fields, Galois gr G, then $G = \exists G_0 \supset G_1 \supset \ldots G_{\exists N \geq 1} = 0$ with $G_0/G_1 \subset \mathrm{Frac}(S)^\times$ cyclic of order prime to p, G_i/G_{i+1} = fin pr of cyclic gr of order p.

Definition

1) Sw $(S/R) \leq n$ iff $N \leq n+1$; Sw $(S/R) = 0$ iff S/R *tame*.
2) [Kerz-Schmidt] X/k sm, k perfect, $Y \to X$ fin étale is tame if $\forall$ sm curve $C \to X$, $Y \times_X C \to C$ is tame .
3) $\rightsquigarrow \pi_1(X,x) \twoheadrightarrow \pi_1^t(X,x)$ tame quotient.

- tame allows non-perfect res fields: res field ext should be sep and ram index prime to p
- if has good comp $X \hookrightarrow \bar{X}$, defn agrees with Grothendieck's defn: tame at the codim 1 points in $\bar{X} \setminus X$

Tame Lefschetz $/k$ of char. $p > 0$

Theorem (Drinfeld)

X/k sm quasi-proj, $\exists C \to X,\ x \in C$ sm curve st $\pi_1^t(C,x) \twoheadrightarrow \pi_1^t(X,x)$.

if $X \hookrightarrow \bar{X}$ good compactification, then any ci of sm ample divisors in good position wrt $\bar{X} \setminus X$ does it (E-Kindler)

Ramification in geometry: definition

Definition

Given $X \hookrightarrow \bar{X}$ normal comp $/k$ perfect, D eff div supp in $\bar{X} \setminus X$, then
1) $Y \to X$ finite étale has *ramification bounded by* D if $\forall C \to X$ sm curve, $\mathrm{Sw}\,(Y \times_X C/C) \le D \times_{\bar{X}} \bar{C}$;
2) $\bar{\mathbb{Q}}_\ell$- loc sys $\mathcal{V}_\rho$ defined by $\rho : \pi_1(X, x) \to GL_r(\bar{\mathbb{Z}}_\ell) \subset GL_r(\bar{\mathbb{Q}}_\ell)$ has ramification bounded by D iff Galois cover $\pi : X_{\bar{\rho}} \to X$ defined by $\bar{\rho} : \pi_1(X, x) \to GL_r(\bar{\mathbb{F}}_\ell)$ has ramification bounded by D $\big(_{\text{depends only on}}$ $(\bar{\rho})^{ss}\big)$.
3) $\pi^* \mathcal{V}_\rho$ tame: say π *tamifies* ρ.
4) A sm curve $C \to X$ is a *Lefschetz curve for* a family $\mathcal{S} = \{\mathcal{V}\}$ if $\mathcal{V}|_C$ keeps the same monodromy $\forall \mathcal{V} \in \mathcal{S}$.

$k = \mathbb{F}_q$ finite field

Theorem (L. Lafforgue dim 1, Deligne in higher dim, cor Langlands corr)

X sm q-proj$/k$, then $\exists$ only finitely many $\bar{\mathbb{Q}}_\ell$- simple loc sys $\mathcal{V}$ with (r, D) bounded, up to twist by a char. of k.

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Corollary

1) (r, D) bounded, then $\exists \pi : Y \to X$ finite étale which tamifies all $\mathcal{V}$ with (r, D) bounded ('covers' from title).

2) Given (r, D), $\exists$ Lefschetz curve for all $\mathcal{V}$ with bound (r, D) ('curves' from title).

on Proof of Corollary

1) take cover $\pi : X_{\oplus_{\rm fin}\bar\rho} \to X$

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2)a) top gr th: $\pi_1(C,x) \to \pi_1(X,x) \twoheadrightarrow I \subset GL_r(\mathcal{O}_E)$ surj ($E/\mathbb{Q}_\ell$ finite) iff $\pi_1(C,x) \to \pi_1(X,x) \twoheadrightarrow \bar{I} \subset GL_r(\mathcal{O}_E/\mathfrak{m}_E^2)$ surj ($\mathcal{O}_E/\mathfrak{m}^4$for $\ell = 2$);

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2)b) Hilbert irreducibility (or Bertini if we allow ext $\mathbb{F}_{q^m} \supset \mathbb{F}_q$) $\Rightarrow \exists C$.

How to bound the ramification if $k = \bar{k}$?

The notion of ramification bounded by D is purely geometric, i.e. depends only on cover $(Y \to X)_{\bar{k}}$ or $\mathcal{V}|_{\pi_1(X_{\bar{k}}, x)}$.

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To 1) 'covers': $/k = \bar{k}$, (r, D) bounded, then $\nexists \pi : Y \to X$ finite étale which tamifies all simple $\mathcal{V}$ with (r, D) bounded: given Sw, Witt-Artin-Schreier covers with Galois gr $\mathbb{Z}/p^n$ $\forall n \geq 1$ with this Sw exist (Brylinski-Kato).

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To 2) 'curves' (Deligne): $/k = \bar{k}$, $X, (r, D)$, $\exists$ Lefschetz curve for all $\mathcal{V}$ with bounded (r, D)?

To 'covers': Tamifying up to codim 2

Definition (E-S)

$\pi : Y \to X$ finite connected *tamifies* $\mathcal{V}$ *outside of codim* 2 if there is a normal compactification $Y \hookrightarrow \bar{Y}$ st $\pi^*\mathcal{V}$ is tame at codim 1 points of $\bar{Y}$.

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Theorem (E-S)

X sm q-proj $/k = \bar{k}$, given (r, D), $\exists n \in \mathbb{N}$, $\forall \mathcal{V}$ with rank $\leq r$ and ramification bounded by D, $\exists \pi_{\mathcal{V}} : Y_{\mathcal{V}} \to X$ of deg $\leq n$ which tamifies $\mathcal{V}$ up to codim 2.

For $X \rightsquigarrow R$, R complete dvr with res field k, (E-Kindler-S)

On Proof of Thm

1) reduce to X affine; via $X \to \mathbb{A}^d$ finite étale, reduce to

$$X = \mathbb{A}^d \hookrightarrow \bar{X} = \mathbb{P}^d;$$

2) prove local thm on $k(Z)[[t]]$ using (E-K-S), $Z = \mathbb{P}^d \setminus \mathbb{A}^d$ to produce a finite étale extension of $k(Z)((t))$ tamifying $\mathcal{V}|_{k(Z)((t))}$;
3) use Harbater-Katz-Gabber to extend to a finite étale cover of $\mathbb{G}_m/k(Z)$;
4) close it up to get the normal finite cover of $\mathbb{P}^d$, then of X.

To ‘curves’: rank 1 case

Theorem (Kerz-S.Saito if $X \hookrightarrow \bar{X}$ good compactification, E-S in general)

$/k = \bar{k}$, $X \hookrightarrow \bar{X}$ *normal compactification,* D, $\exists$ *Lefschetz curve for* $(1, D)$.

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Theorem (Kerz-S.Saito if $X \hookrightarrow \bar{X}$ good compactification, E-S in general)

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On Proof.

1) reduce to Artin-Schreier;
2) $\{\mathcal{V}\}$ with $(r, D) \subset \{\mathcal{W}\}$ with $(r, D \cap X^{\mathrm{reg}})$ (less curves to test).
3) use coh description (Kerz-S.Saito) on X^{reg} and finiteness of Frobenius invariant $\mathcal{O}$-modules of local coh gr along $\bar{X} \setminus X^{\mathrm{reg}}$ to prove: $\exists N \geq 1$ so $\{\mathcal{W}\}$ with $(r, D \cap X^{\mathrm{reg}}) \subset \{\mathcal{V}\}$ with (r, ND). □

Application of classical Bertini theorem

Theorem (E-S)

$\exists K/k$ alg. cl. of purely tr. fin. gen. field $/k$, $C_K \to X$ curve $/K$ st $\pi_1(C_K, x) \twoheadrightarrow \pi_1(X, x)$.

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It is an illustration of the fact that if C is not proper,
1) $\pi_1(C, x)$ does not satisfy base change;
2) there is no specialization map $\pi_1(C, x) \to \pi_1(C_k, x_k)$ for a specialization $K \rightsquigarrow k$.