Local-global principles over semi-global fields

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Local-global principles: an overview

Given a property of an algebraic object over a field F, does it hold globally if it holds locally? "Locally" = at each completion F_v of F.

Classically: consider objects over a global field F.
In this talk: *semi-global fields*, e.g. p-adic function fields.

Examples of LGP over global fields:

1) Object = central simple algebra; property = split (isomorphic to a matrix algebra). LGP holds by the theorem of Albert-Brauer-Hasse-Noether.

2) Object = quadratic form; property = isotropic (non-trivial 0). LGP holds by the Hasse-Minkowski theorem.

3) Object = a variety; property = has a rational point. LGP can fail, but often holds, esp. for many homogeneous spaces under algebraic groups – in particular, for torsors (PHS).

LGP for torsors over global fields

Given an algebraic group G over a field F:

$$G\text{-torsors}/F \leftrightarrow H^1(F,G)$$

$$\text{the trivial torsor } (G \text{ itself}) \leftrightarrow \text{the distinguished element}$$

A G-torsor over F has an F-point iff it is the trivial torsor. So the obstruction to LGP is (the Tate-Shafarevich set)

$$Ш(F,G) = \ker\left[H^1(F,G) \to \prod_v H^1(F_v,G)\right].$$

So LGP holds for all G-torsors$/F \Leftrightarrow Ш(F,G)$ is trivial.

For F *global*, & G a lin. alg. gp. (smooth subgp of GL_n):

1) $Ш(F,G)$ is finite (Borel-Serre; B. Conrad);

2) LGP holds if G is connected and rational as an F-variety (Sansuc-Chernousov);

3) LGP holds if G is a semisimple simply connected group (Kneser-Harder-Chernousov).

Applications of LGP's for torsors over global fields

Using LGP's for torsors, can get other LGP's:

Central simple algebras over F of degree n (i.e., dim n^2 over F) are classified by $H^1(F, \mathrm{PGL}_n)$, and PGL_n is a rational connected group. So LGP for torsors $\Rightarrow$ LGP for central simple algebras (Albert-Brauer-Hasse-Noether).

Quadratic forms over F of dimension n and discriminant 1 are classified by $H^1(F, \mathrm{SO}_n)$, and SO_n is a rational connected group. Using LGP for torsors, can get LGP for isotropy of quadratic forms (Hasse-Minkowski).

Can use these LGP's to get structural information:

ABHN $\Rightarrow \forall$ csa A/F, $\mathrm{period}(A) = \mathrm{index}(A)$.

HM $\Rightarrow$ Every quadratic form$/F$ of dimension > 4 (and indefinite, if F has a real embedding) is isotropic.

Semi-global fields

A *semi-global field* is a one-variable function field F over a complete discretely valued field K; i.e., the function field of a curve over K.

Examples: $F = \mathbb{Q}_p(x)$, $F = k((t))(x)$, any finite extension of these.

Let $T = \mathcal{O}_K$ (e.g. $\mathbb{Z}_p$ or $k[[t]]$), and $t \in T$ a uniformizer. Then there is a *regular model* $\mathcal{X}$ of F over T; i.e., a flat projective regular T-curve $\mathcal{X} \to \mathrm{Spec}(T)$ with function field F.

$F = K(x)$, $\mathcal{X} = \mathbb{P}^1_T$ $\quad$ $\mathrm{Spec}(T)$

Local in one direction, global in the other: "semi-global".
Want to carry over LGPs and applications to this situation.

LGP over semi-global fields

Several possible LGP's and obstructions to consider here:

$Ш(F, G) = \ker \left[H^1(F, G) \to \prod_v H^1(F_v, G)\right]$; v ranging over discrete val's on F corresp. to codim 1 points on models of F.

$Ш_X(F, G) = \ker \left[H^1(F, G) \to \prod_P H^1(F_P, G)\right]$; X the closed fiber of a model $\mathcal{X}$ of F; P the points of X; $F_P = \mathrm{frac}\,\widehat{\mathcal{O}}_{\mathcal{X},P}$.

Are these finite? trivial? related? Implications for alg. structures?

HHK (via patching methods): • $Ш_X(F, G) \subseteq Ш(F, G)$.

• G a *rational* connected lin. alg. group $/F \Rightarrow Ш_X(F, G)$ is trivial. (Analog of Sansuc-Chernousov over global fields.)

• $Ш_X(F, G)$ trivial $\Rightarrow$ LGP holds $\forall$ G-homogeneous spaces $/F$.

Applications of LGP's for torsors over semi-global fields

As for global fields, LGP for torsors implies other LGP's over semi-global fields:

Since csa's are classified by $H^1(F, \mathrm{PGL}_n)$, get: A is split over F iff it is split over every F_P (HHK); or if it is split over every F_v (CPS) – – analog of Albert-Brauer-Hasse-Noether.

Since the projective quadric hypersurface defined by a quadratic form q is a homogeneous space under $\mathrm{SO}(q)$ if $\dim(q) > 2$, get: if $\dim(q) > 2$ then q is isotropic$/F$ iff it is isotropic over every F_P (HHK); or every F_v (CPS) – analog of Hasse-Minkowski.

We then get structural information. E.g., say $F = \mathbb{Q}_p(x)$. Then

• $\forall$ csa A/F, index(A) divides period$(A)^2$. (HHK; also Lieblich)

• Every quadratic form$/F$ of dimension > 8 is isotropic. (HHK, also PS, for p odd; Leep)

LGP via patching

To prove the above results (and other recent results by HHK+CPS), we use *another* form of LGP, wrt another set of overfields of F. This set is *finite*, making it easier to study.

Pick a finite subset $\mathcal{P} \subset X$, including all points where irreducible components of X meet. Let $\mathcal{U}$ be the set of connected components U of $X - \mathcal{P}$; these are affine curves. Each $P \in \mathcal{P}$ and each $U \in \mathcal{U}$ is defined over a finite extension of $k =$ res. field of $T = O_K$.

For each $P \in \mathcal{P}$, have the fraction field F_P of $\widehat{R}_P := \widehat{\mathcal{O}}_{\mathcal{X},P}$. For each $U \in \mathcal{U}$, take the fraction field F_U of $\widehat{R}_U$, the completion of the subring of F consisting of functions regular along U.

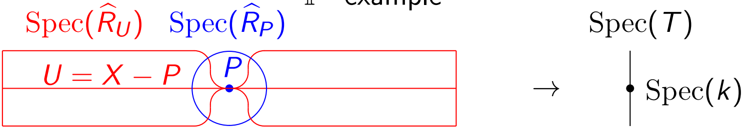
These fields F_P, F_U provide a finite set of overfields of F.

We can then consider a third LGP for G-torsors over F: if it is trivial over each F_P ($P \in \mathcal{P}$) and each F_U ($U \in \mathcal{U}$), must it be trivial over F? The obstruction:

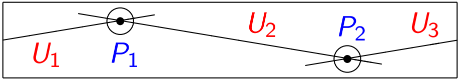
$$Ш_{\mathcal{P}}(F, G) = \ker \left[H^1(F, G) \to \prod_{P \in \mathcal{P}} H^1(F_P, G) \times \prod_{U \in \mathcal{U}} H^1(F_U, G) \right].$$

Here $Ш_{\mathcal{P}}(F, G) \subseteq Ш_X(F, G)$, and $Ш_X(F, G) = \bigcup_{\mathcal{P}} Ш_{\mathcal{P}}(F, G)$.

$\mathbb{P}^1$ example



Example with reducible closed fiber



The closed fiber can have > 1 branch at a point $P \in \mathcal{P}$. Branches $\leftrightarrow$ height 1 primes $\wp \in \widehat{R}_P$ lying on X (i.e., containing $t \in T$). Let $\mathcal{B}$ be the set of branches at all $P \in \mathcal{P}$.

Each branch $\wp$ at P lies on an irred. comp. of X, so on a $U \in \mathcal{U}$. The local ring of $\widehat{R}_P$ at $\wp$ is a dvr. Let $\widehat{R}_\wp$ be the completion, with fraction field $F_\wp$. Then $\widehat{R}_P, \widehat{R}_U \subset \widehat{R}_\wp$, and we regard $\mathrm{Spec}(\widehat{R}_\wp)$ as the overlap of $\mathrm{Spec}(\widehat{R}_P)$ and $\mathrm{Spec}(\widehat{R}_U)$.

For a G-torsor $Z \in Ш_{\mathcal{P}}(F, G)$, pick $z_U \in Z(F_U)$, $z_P \in Z(F_P)$ ($\forall U \in \mathcal{U}$, $P \in \mathcal{P}$). Then $z_P, z_U \in Z(F_\wp)$ for $\wp$ a branch on U at P, so $\exists! g_\wp \in G(F_\wp)$ taking z_U to z_P. Get $g = (g_\wp) \in \prod_{\wp \in \mathcal{P}} G(F_\wp)$.

New choices z_U, z_P multiply $g_\wp$ on the left by an element of $G(F_U)$, and on the right by an element of $G(F_P)$. So

$$Ш_{\mathcal{P}}(F, G) \leftrightarrow \prod_{U \in \mathcal{U}} G(F_U) \backslash \prod_{\wp \in \mathcal{B}} G(F_\wp) / \prod_{P \in \mathcal{P}} G(F_P),$$

a double coset space. (Analogous to classical double coset space.)

LGP via double cosets

To prove a LGP for G-torsors over a semi-global field F, we want to show that $Ш_{\mathcal{P}}(F, G)$ is trivial. Via double coset description this is equivalent to proving simultaneous factorization of elements $g_\wp$:

$\forall (g_\wp) \in \prod_{\wp \in \mathcal{P}} G(F_\wp)\ \exists (g_U) \in \prod_{U \in \mathcal{U}} G(F_U),\ (g_P) \in \prod_{P \in \mathcal{P}} G(F_P)$ s.t. $g_\wp = g_U g_P^{-1}$ whenever $\wp$ is a branch at P on U.

In the case that G is a *rational connected* linear algebraic group over F, such a simult. factorization holds; so $Ш_{\mathcal{P}}$ is trivial and so is $Ш_X$. (HHK '09, '15; a generalization of Cartan's Lemma on GL_n)

The proof finds elements in each $G(\widehat{R}_U), G(\widehat{R}_P)$ by using that G is rational to work locally in affine space near the origin, and finding these elements modulo successive powers of $t \in T$.

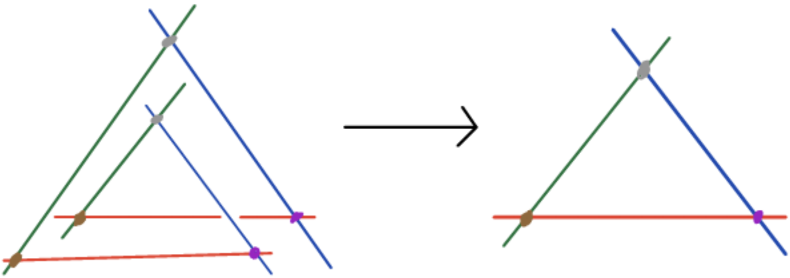
This gives the asserted local-global principle for rational groups, and the applications to quadratic forms (via SO_n) and csa (via PGL_n).

What about *other* types of linear algebraic groups, e.g. O_n?
O_n is disconnected, but (each component) is rational.

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{SO}_n & \to & \mathrm{O}_n & \to & C_2 & \to & 1 \\ & & \text{rat'l conn} & & \text{disconn} & & \text{const.fin.} & & \end{array}$$

Want to relate $Ш_{\mathcal{P}}(F, \mathrm{O}_n)$ to $Ш_{\mathcal{P}}(F, \mathrm{SO}_n)$ and $Ш_{\mathcal{P}}(F, C_2)$.

$Ш_{\mathcal{P}}(F, C_2)$ classifies locally trivial degree 2 covers. This need not be trivial; e.g. on the closed fiber can have:



Reduction graph

This example suggests LGP can fail if $\exists$ loops in the closed fiber.

More precisely: Given a regular model $\mathcal{X}$ of a semi-global field F, and a choice of $\mathcal{P} \subset X$ (yielding $\mathcal{U}$ and $\mathcal{B}$), the *reduction graph* is the bipartite graph Γ whose vertices are $\mathcal{P} \cup \mathcal{U}$ and whose edges between vertices P and U are branches $\wp \in \mathcal{B}$ at P lying on U.

(If X is a normal crossings divisor on $\mathcal{X}$, then Γ is homotopy equivalent to the dual graph of Deligne-Mumford.)

Viewing Γ as a topological space, we can take $\pi_1(\Gamma)$ (which is independent of $\mathcal{P}$). If G is a *constant finite* group (e.g., C_2), then:

$$Ш(F,G) = Ш_X(F,G) = Ш_{\mathcal{P}}(F,G) \;\cong\; \mathrm{Hom}(\pi_1(\Gamma), G)/\!\sim \;\cong\; G^m/\!\sim,$$

where Γ has m loops (i.e. $\pi_1(\Gamma) = F_m$) and $\sim$ is conjugation by G. (Here $Ш(F,G) = Ш_{\mathcal{P}}(F,G)$ by Purity of Branch Locus.)

So LGP holds iff Γ is a tree or G is trivial.

Disconnected rational groups

More generally, say G is a linear algebraic group over F that's rational but possibly disconnected (e.g., O_n). Then

$$\begin{array}{ccccccccc} 1 & \to & G^0 & \to & G & \to & \bar{G} & \to & 1 \\ & & \text{rat'l conn} & & \text{disconn} & & \text{const.fin.} & & \end{array}$$

with $Ш_X(F, G^0) = 1$ and $Ш_X(F, \bar{G}) \cong \mathrm{Hom}(\pi_1(\Gamma), \bar{G})/\sim$. By a factorization argument, it can then be shown (HHK'15) that in fact

$$Ш_X(F, G) \cong \mathrm{Hom}(\pi_1(\Gamma), \bar{G})/\sim \cong \bar{G}^m/\sim.$$

So LGP holds iff Γ is a tree or G is connected (i.e. $\bar{G}$ is trivial).

This explains the exception to Hasse-Minkowski for binary quadratic forms q over F: The quadric hypersurface Q defined by q is a homogeneous space under $\mathrm{O}(q)$ but not $\mathrm{SO}(q)$ since it has two geometric points; so LGP for q fails if Γ is not a tree (e.g. if F is the function field of a Tate curve) but holds otherwise (e.g. $F = K(x)$).

Non-rational groups

What if the group is connected but not rational? Must LGP hold?

In a 2016 paper of CPS, there is an example of a semi-global field F and a non-rational torus G over F such that $Ш_X(F, G)$ is non-trivial (and hence also $Ш(F, G)$). In this example, the closed fiber X consists of three copies of $\mathbb{P}^1_k$ meeting at k-points, forming a triangle (so the reduction graph has a loop).

This suggests that the reduction graph is relevant even for groups that are *connected*, if they're not rational.

Questions: (1) For non-rational groups G, under what circumstances is there a LGP, and what is the obstruction?

(2) In analogy with the case of groups over *global* fields, must the obstruction be finite, and must it be trivial for sssc groups?

Non-rational groups over T

In ongoing work of HHK+CPS, we prove results related to these questions for groups G that are defined over the cdvr $T = O_K$. In this situation, we have both positive and negative results concerning these two questions, with answers in terms of the reduction graph. Again, the tree condition is relevant.

Given a regular model $\mathcal{X}$ of a semi-global field over K, the closed fiber X is a (possibly reducible) curve over $k = T/\mathfrak{m}$. If the reduction graph Γ associated to $X_{k'}$ is a tree for every algebraic extension of k, we call it a *geometric tree*.

Under appropriate hypotheses, we show that LGP holds if Γ is a geometric tree, but can fail if Γ has loops or is a non-geometric tree.

LGP for reductive groups in characteristic 0

As above, let G be a linear algebraic group over the cdvr T, and $\mathcal{X}$ a regular model of a semi-global field F over $K = \mathrm{frac}(T)$.

For simplicity, we first assume $T = k[[t]]$, where char(k)=0.

Theorem. (CHHKPS) Assume that the closed fiber X of $\mathcal{X}$ is reduced, and that the reduction graph is a geometric tree. If G is connected and reductive over T, then LGP holds (in all senses).

Here, recall that a connected group G over an algebraically closed field is *reductive* if it has no non-trivial unipotent normal subgroups. Equivalently, the unipotent radical (maximal unipotent normal subgroup) is trivial. A group over a ring R is *reductive* if it is so over every geometric point of Spec(R).

Here $Ш(F, G) = Ш_X(F, G)$ via a result of Nisnevich (for G is reductive over a cdvr A with $E = \mathrm{frac}(A)$, $H^1(A, G) \to H^1(E, G)$ is injective). So STS the theorem for $Ш_X$; or equivalently $\forall\ Ш_{\mathcal{P}}$.

That is, given $\mathcal{P}$ (and hence $\mathcal{U}$), we want to show that a G-torsor over F that's trivial over each F_P and F_U is trivial over F.

By a recent theorem of Gille-Parimala-Suresh, a G-torsor over F is trivial if it is the generic fiber of a G-torsor over $\mathcal{X}$ whose restriction to X is trivial. So to prove the theorem it suffices to show that a locally trivial torsor over F is of that form.

This is done in two steps, each involving a matrix factorization result (extending Cartan's Lemma).

Step 1: Local factorization of G over F_P:

Using the structure of reductive groups (Bruhat decomposition), we find rational connected subgroups $S, \mathscr{U} \subseteq G$ and $w_0 \in G$ s.t. every element of G in a dense open subset can be written as suw_0u' for $s \in S$ and $u, u' \in \mathscr{U}$. (S is a split torus $\mathbb{G}_m^n$, and $\mathscr{U}$ is unipotent.)

Step 2: Global factorization of G over F:

By hypothesis, Γ is a geometric tree. In char 0, this is equivalent to Γ being a *monotonic tree*: $\exists$ vertex v_0 (the "root") s.t. for any path in Γ away from v_0, the field of definition of each vertex (a $P \in \mathcal{P}$ or a $U \in \mathcal{U}$) contains the field of definition of the previous vertex. Working inductively along Γ, starting at v_0, Step 1 yields a (simultaneous) factorization, yielding the desired globalization. $\square$

LGP for linear algebraic groups in char 0

So we have the above theorem, that LGP holds for connected reductive groups over T if the closed fiber is reduced and Γ is a geometric tree, in equal char 0. Building on this, we get a result about groups that need not be connected or reductive (CHHKPS):

Cor. Let G be a lin. alg. group over a field k of char 0, and let F be a semi-global field / $K := k((t))$. Assume the closed fiber of a regular model is reduced, and the reduction graph is a geometric tree. Then $Ш_X(F, G)$ is trivial; so is $Ш(F, G)$ if G is connected.

To prove this for connected groups, we let $\mathscr{U}$ be the unipotent radical of G. So $G/\mathscr{U}$ is reductive (and connected), so $Ш(F, G/\mathscr{U})$ is trivial by the above theorem. But a unipotent group in char 0 is built of successive extensions of $\mathbb{G}_a$, so $\mathscr{U}$-torsors are trivial by Hilbert 90. So $H^1(F, G) \to H^1(F, G/\mathscr{U})$ is injective, and $Ш(F, G)$ is trivial.

To handle the disconnected case, we combine the connected case with the case of a finite group scheme G (not nec. constant):

Prop. Let G be a smooth finite group scheme over a s.g.f. F, and F'/F a finite Galois extension that splits G. Suppose the reduction graph Γ' associated to F' is a tree. Then LGP holds for G over F.

The proof of this proposition combines Galois cohomology with a theorem of Serre about group actions on graphs: If no element of the group interchanges two adjacent vertices, then some vertex is fixed by every group element. That theorem is applied to the group $\mathrm{Gal}(F'/F)$ acting on the graph Γ', using that Γ' is bipartite to satisfy Serre's hypothesis.

(In the conn. reductive and finite cases, $\text{Ш}_X(F, G) = \text{Ш}(F, G)$, but in the general case we don't know this.)

Case of char(k)=p

The LGP for connected reductive groups G over a semi-global field F carries over to the case where F is a function field over a cdvf K whose residue field k has char p, and G is a reductive group defined over $T = O_K$, with extra hypotheses:

• We assume that p is not a “bad prime” for G – roughly, this avoids inseparable homomorphisms $H \twoheadrightarrow G$. This is needed to invoke the theorem of Gille-Parimala-Suresh.

• We assume that the reduction graph Γ is a monotonic tree. This is strictly stronger than being a geometric tree in char p, and it is needed for the induction argument on the graph.

We can't then pass from the reductive case to the general case, because the unipotent radical might not be built from copies of $\mathbb{G}_a$.

Counterexamples to LGP for semi-global fields

What if we have a semi-global field F whose reduction graph is not a geometric tree? As with finite groups, can LGP then fail for a connected group defined over T (where $T = O_K$ and F is over K)?

Ex. 1. Suppose the closed fiber X of a model $\mathcal{X}$ of F consists of copies of $\mathbb{P}^1_k$ meeting at k-points. Let m be the number of loops in Γ (i.e. $\pi_1(\Gamma) = F_m$). Then using matrix factorization, it follows for a reductive group G over T that

$$Ш(F, G) \cong \mathrm{Hom}(\pi_1(\Gamma), G(k)/\mathrm{R})/\sim \;\cong\; (G(k)/\mathrm{R})^m/\sim,$$

analogously to the case of (possibly disconnected) rational groups. As before, $\sim$ denotes conjugation. Also, $G(k)/\mathrm{R}$ denotes the group of R-equivalence classes in $G(k)$, where $g_0, g_1 \in G(k)$ are R-*equivalent* they are connected by an open subset of $\mathbb{A}^1_k$ contained in G_k. (For rational groups, $G(k)/\mathrm{R} = G/G^0$.)

To use this to get a counterexample to LGP for a connected reductive group G over a semi-global field:

Take a reductive group G over a field k of char 0 with non-trivial $G(k)/\mathrm{R}$. E.g. $k = \mathbb{Q}(\sqrt{17})(u, v)$ and $G = \mathrm{SL}_1(D)$, with D the biquaternion algebra $(-1, u) \otimes_k (2, v)$. Then $G(k)/\mathrm{R} = C_2$.

Next, take a semi-global field over $K = k((t))$ having a regular model $\mathcal{X}$ whose closed fiber consists of copies of $\mathbb{P}^1_k$ meeting at k-points with $m \geq 1$ loops in Γ. E.g. take $T = k[[t]]$ and $\mathcal{X} = \mathrm{Proj}\, T[x, y, z]/(xyz - t(x + y + z)^3)$. Here X is a triangle of $\mathbb{P}^1_k$'s meeting at k-points.

With this choice of k, G, F, the graph Γ is not a tree, and $G(k)/\mathrm{R} \neq 1$; so $\text{Ш}(F, G)$ is non-trivial, and LGP fails.

In fact, by enlarging k we can enlarge $G(k)/\mathrm{R}$, and even find examples with $\text{Ш}(F, G)$ infinite: replacing k by $k(G)$ enlarges $G(k)/\mathrm{R}$; repeat infinitely often, getting a field k of infinite transcendence degree and infinite $\text{Ш}(F, G)$.

Ex. 2. Suppose the closed fiber X of a model $\mathcal{X}$ of F consists of two copies of $\mathbb{P}^1_k$ meeting at a single closed point, where the residue field is k' (with k, k' of char 0, and k' strictly containing k). So Γ is a tree, but *not* a geometric tree.

If $G(k)/\mathrm{R}$ is trivial and $G(k')/\mathrm{R}$ is non-trivial, then a factorization argument shows that $\text{Ш}(F, G)$ is non-trivial.

For example, let $k = \mathbb{Q}(u, v)$ and $k' = \mathbb{Q}(\sqrt{17})(u, v)$. Take $T = k[[t]]$ and $\mathcal{X} = \operatorname{Proj} T[x, y, z]/((y - x)(xy - 17z^2) + tz^3)$. Then X is a union of two k-lines, meeting at a k'-point. Moreover with G as in the previous example, $G(k)/\mathrm{R} = 1$ and $G(k')/\mathrm{R} = C_2$. So $\text{Ш}(F, G) \neq 1$ and LGP fails there.

The above examples are in particular counterexamples to LGP for *semi-simple simply connected* groups — unlike the situation over global fields.

Moreover they show that that $Ш(F, G)$ can be infinite – again unlike the situation over global fields.

Question. If one restricts to semi-global fields over a *local* field (e.g. taking F to be a p-adic function field), then is $Ш(F, G)$ finite for all connected reductive groups G, and is it trivial for all sssc groups?

These have both been conjectured, and there is evidence (Y. Hu, Preeti, Parimala, Suresh, Y. Tian).

In addition:

Question. In general, for groups G over semi-global fields F, what is a description of $Ш(F, G)$, and precisely when does LGP hold?