Serre-type conjecture for projective representations

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GOAL:
We consider automorphy of many representations of the form $\overline{\rho} : G_K \to \mathrm{PGL}_2(k)$ with K a CM field and $k = \mathbf{F}_3, \mathbf{F}_5$. In particular we prove (under some mild conditions) that for F totally real, a surjective representation $\overline{\rho} : G_F \to \mathrm{PGL}_2(\mathbf{F}_5)$ with totally odd sign character arises from a Hilbert modular form of weight $(2, \ldots, 2)$.

This is an apparently new case of the Buzzard-Diamond-Jarvis conjectures which extend Serre's modularity conjecture to totally real fields.

The result uses 2-adic automorphy lifting results over CM fields proved in:

[AKT] Allen, Khare, Thorne: Modularity of $GL_2(\mathbf{F}_p)$ representations over CM fields, available at `arXiv:1910.12986`.

We first describe in somewhat qualitative terms the results of this paper.

Part I: Modularity of $GL_2(\mathbf{F}_p)$ representations over CM fields

In [AKT] we take the first steps towards proving the modularity (as opposed to potential modularity) of elliptic curves over CM fields (for example, imaginary quadratic fields). We say that an elliptic curve E over a number field K is modular if either E has complex multiplication, or there exists a cuspidal, regular algebraic automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_K)$ such that E and π have the same L-function. This implies, for example, that the L-function $L(E, s)$ has an analytic continuation to the whole complex plane, and therefore that the Birch–Swinnerton-Dyer conjecture for E can be formulated unconditionally.

The following theorem is an illustration of what we prove.

Theorem

Let K be an imaginary quadratic field. Then a positive proposition of elliptic curves over K are modular.

In fact, we give precise conditions that imply the modularity of an elliptic curve over any CM field not containing a primitive 5th root of unity. It suffices to impose local conditions at finitely many places, leading to the theorem stated above.

The first general results concerning modularity of elliptic curves over number fields were proved in the case $K = \mathbb{Q}$ with methods pioneered by Wiles in the mid 1990's. In these works, the modularity of all semistable elliptic curves over $\mathbb{Q}$ is established broadly following two steps.

Step 1

In the first step, modularity lifting theorems are proved for 2-dimensional Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_p)$. These theorems state that if ρ satisfies a list of conditions, chief among them that the residual representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_p)$ is already known to arise from modular forms (say holomorphic of weight 2), then ρ also arises from modular forms. Since modularity of an elliptic curve E is equivalent to modularity of any one of its associated p-adic Galois representations $\rho_{E,p}$, this largely reduces the problem of showing modularity of E to proving modularity of some $\overline{\rho}_{E,p}$.

Step 2

In the second step, modularity of $\overline{\rho}_{E,p}$ is proved to hold if $p = 3$. The first main observation is that the Langlands–Tunnell theorem (which asserts the modularity of certain Artin representations with soluble image) implies that $\overline{\rho}_{E,3}$ arises from a holomorphic modular form of weight 1. Since any such form is congruent to a form of weight 2, one can apply the previously established modularity lifting theorem to deduce the modularity of E, provided that $\overline{\rho}_{E,3}$ satisfies the technical conditions of the modularity lifting theorem. In particular, it should be irreducible. For elliptic curves which do not satisfy this irreducibility condition, a clever trick (the '3–5 switch') is used to show that $p = 5$ works instead.

This two-pronged approach has proved very fruitful for studying the modularity of elliptic curves over totally real number fields, leading to a number of impressive results. For example, the modularity of all elliptic curves over real quadratic fields was proved by N. Freitas, Bao Le Hung, Samir Siksek using a similar strategy.

What about imaginary quadratic fields K? The problem has a very different flavor because the analogs of modular curves for K – 3 dimensional hyperbolic manifolds – do not have complex structures.

Recent progress in our understanding of the Galois representations attached to torsion classes in the cohomology of arithmetic locally symmetric spaces has led to the possibility of studying the modularity of elliptic curves over CM fields. We recall that a CM field is, by definition, a totally imaginary quadratic extension of a totally real field.

The simplest class of CM fields is that of the imaginary quadratic fields. Two groups of authors (Boxer-Calegari-Gee-Pilloni and the ten author paper) have now proved modularity lifting theorems that have the potential to be applied to proving the modularity of elliptic curves over CM fields. Combined with Taylor's technique for verifying the potential modularity of a given residual representation, this leads to the potential modularity of all elliptic curves over CM fields and even over an arbitrary quadratic extension of a totally real field.

This naturally leads to the question of whether, for an elliptic curve E over a CM field K, one can follow the same lines as Wiles by establishing e.g. the residual modularity of the modulo 3 representation $\overline{\rho}_{E,3}$, and then use this to prove modularity of E over its original field of definition. The following theorem, which is a special case of one of the main theorems of [AKT], affirms that this is the case:

Theorem

Let K be an imaginary quadratic field, and let $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F}_3)$ be a continuous homomorphism of cyclotomic determinant. Suppose that for each place $v|5$ of K, there exists a Tate elliptic curve E_v over K_v such that $\overline{\rho}|_{G_{K_v}} \cong \overline{\rho}_{E_v,3}$. Then there exists a modular elliptic curve E over K such that $\overline{\rho} \cong \overline{\rho}_{E,3}$.
In particular, $\overline{\rho}$ is modular, in the sense that it arises from a regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$ of weight 2.

In fact, we prove a similar theorem where K is allowed to be any imaginary CM field such that $\zeta_5 \notin K$, and for mod 2 and mod 5 representations as well as mod 3 representations. We use “Tate elliptic curve” as a synonym for “elliptic curve with split multiplicative reduction” and “weight 2” as a synonym for “cohomological”. We note in particular that the local condition at the places $v|5$ of K in theorem above is always satisfied after passage to a soluble CM extension. One therefore expects that, combined with base change, this theorem will have many applications to proving modularity of elliptic curves.

As mentioned above, the modularity of a residual representation $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F}_3)$ can be proved in the case that K is a totally real field using the Langlands–Tunnell theorem. The same chain of reasoning no longer applies when K is a CM field.

Difficulty of applying Langlands-Tunnell

Indeed, a $\mathrm{PGL}_2(\mathbb{F}_3)$-representation can still be lifted to an Artin representation in characteristic 0, and the automorphy of this lift proved using the Langlands–Tunnell theorem. However, there is no known method to construct congruences between the resulting automorphic representation and one which is cohomological. More informally, one does not know how to go from weight 1 to weight 2. Note that such a method would presumably also imply the existence of Galois representations attached to algebraic Maass forms for $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$! We must therefore find a different route.

Avoiding Langlands-Tunnell

Our argument is inspired by the '3-5' switch of Wiles, and in particular makes crucial use of the fact that certain modular curves of low level can have infinitely many rational points (because they have genus 0 or 1). A first approximation to the argument is that instead of using $p = 3$, we want to use $p = 2$. Indeed, the mod 2 representation of an elliptic curve is always dihedral, if it is irreducible, and one knows that automorphic induction can lead to cohomological automorphic representations, in contrast to the automorphic representations asserted to exist by the Langlands–Tunnell theorem.

2-3 switch

We therefore first prove a 2-adic automorphy lifting theorem for ordinary 2-adic Galois representations. Unfortunately, we must impose a local condition at the places $v|2$ of K which precludes the existence of an ordinary, cohomological, dihedral lift of the given dihedral residual representation. A weaker version of this condition is necessary to force the dihedral locus in the ordinary deformation space to be sufficiently small, following a strategy of Skinner–Wiles.

The stronger version that we impose further ensures that the ordinary Galois deformation problems at the 2-adic places are formally smooth, a condition that we need in order to control the relevant global Selmer groups. We therefore impose the condition that $\overline{\rho}$ extends to a representation of the Galois group of the maximal totally real subfield K^+ of K.

Under this condition, we can verify the residual automorphy by using automorphic induction in weight 1 over K^+, passing to weight 2 over K^+ (using Hida theory), and then using base change to get the residual automorphy over K. This leads to a modularity theorem for ordinary 2-adic Galois representations of $\mathrm{Gal}(\overline{K}/K)$ that does not have any assumption of residual modularity (but does have an assumption that the residual representation extends to $\mathrm{Gal}(\overline{K}/K^+)$).

Diophantine trouble

This seems at first to be catastrophic for e.g. the proof of our main theorem Theorem 2, since the condition that $\overline{\rho}_{E,2}$ extends to $\mathrm{Gal}(\overline{K}/K^+)$ is a Diophantine condition on elliptic curves E over K. Note that the modular curve parameterizing elliptic curves E with fixed mod 2 and mod 3 representations (which is a twist of $X(6)$) has genus 1, and often has no rational points at all. The key to getting around this is a result which shows that the main obstruction to the existence of rational points is the image of the discriminant of the elliptic curve E in $K^\times/(K^\times)^6$; this can be read off from the action of the Galois group on $E[6] = E[2] \times E[3]$.

In order to prove Theorem 2, we carefully construct a soluble CM extension L/K over which this obstruction can be shown to vanish. This implies the modularity of $\overline{\rho}|_{G_L}$, and then a further argument using soluble base change and the rationality of $X(3)$ gives the modularity of $\overline{\rho}$ itself. Similar arguments can then be used to prove the analogue of Theorem 2 for the primes $p = 2$ and $p = 5$.

INTERLUDE

Langlands-Tunnell avoidance again: totally real fields

As a warm-up to our application of the theorems from the modularity theorems [AKT] over CM fields, to prove new cases of automorphy of mod p representations (primarily over totally real fields):

Let F be a totally real field. We explain how to avoid using the Langlands-Tunnell theorem (which gives modularity of Artin representations $\rho : G_F \to \mathrm{GL}_2(\mathbf{C})$ with image $\tilde{S}_4$), when proving that representations $\rho : G_F \to \mathrm{GL}_2(\mathbf{F}_3)$ arises from HMF's of weight $(2, \ldots, 2)$.

Of course over totally real F, unlike the CM case, Langlands-Tunnell does give a proof of the mod 3 modularity result, but as an illustration of using the 2-3 switch of [AKT] over totally real fields let's show how one can avoid it.

As we are over a totally real field, we make use of earlier 2-adic automorphy lifting results of P. Allen. As the dihedral case is classical (Hecke), we focus on the case that $\bar{\rho}$ has image that contains $\mathrm{SL}_2(\mathbf{F}_3)$.

Proposition

Let F be a totally real field. A totally odd representation $\rho : G_F \to \mathrm{GL}_2(\mathbf{F}_3)$ with image containing $\mathrm{SL}_2(\mathbf{F}_3)$ arises from a weight 0 cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$.

Proof: The idea of the proof is to show that $\overline{\rho}$ (after solvable totally real base change if necessary) arises from an elliptic curve E over F whose mod 2 representation is “benign” (in particular irreducible and hence dihedral image). This allows one to prove that E, and hence $\overline{\rho}$, is modular using Allen’s 2-adic modularity lifting results proved in his UCLA 2012 thesis.

There is a totally real solvable extension L/F such that:

- $\rho : G_L \to \mathrm{GL}_2(\mathbf{F}_3)$ has cyclotomic determinant character, projectively trivial at places above 2, and arises from an elliptic curve E over L that is ordinary at places above 2 and 3, has bad semistable reduction at places above 2, and such that the mod 2 representation τ arising from E satisfies the conditions 2-adic lifting theorems of Allen:
    - τ has image $\mathrm{SL}_2(\mathbf{F}_2)$;
    - letting L'/L denote the unique quadratic extension such that $\tau|_{G_{L'}}$ is abelian, then there is some place $v|2$ in L that does not split in L'.

Thus we get from Allen's 2-adic lifting results that E is modular, and as $\overline{\rho}|_{G_L}$ is not dihedral, we conclude by mod 3 solvable descent that $\overline{\rho}$ arises from a weight 0 cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$.

Part II: Modularity of projective representations over CM fields

In the second part of the talk, we explain how to use the 2-adic lifting theorems of [AKT] and its 2-3 switch to prove automorphy of many representations of the form $\overline{\rho} : G_K \to \mathrm{PGL}_2(k)$ with K a CM field and $k = \mathbf{F}_3, \mathbf{F}_5$. In particular we prove (under some mild conditions) that for F totally real, a surjective representation $\overline{\rho} : G_F \to \mathrm{PGL}_2(\mathbf{F}_5)$ with totally odd sign character arises from a Hilbert modular form of weight $(2, \ldots, 2)$.

Serre's conjecture: review

We begin by recalling Serre's conjecture. Serre conjectured, starting in 1973 and culminating with the general and very precise conjectures in his Duke paper in 1986, that irreducible odd representations $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ with k a finite field arise as reductions of representations $r_{f,\iota} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O})$ arising from newforms f which are cohomological (i.e. of weight $k \geq 2$). Before the proof of the general conjecture, several results towards it were known when the cardinality of k was small.

For $k = \mathbf{F}_2$, the conjecture follows from results of Hecke. One shows more generally that representations whose projective image is dihedral arise from cohomological Θ series when the projective representation lifts to a representation of the form $\mathrm{Ind}_K^{\mathbf{Q}}(\chi)$ where $\chi : G_K \to \overline{k}^\times$ is a character and K an imaginary quadratic field.

When it does not arise in this manner, and only lifts to an induced representation of the form $\mathrm{Ind}_F^{\mathbf{Q}}(\chi)$ where $\chi : G_F \to \overline{k}^\times$ is a character and F a real quadratic field, there are no cohomological Θ-series which give rise to ρ. In this case ρ arises from a (non-cohomological) Θ series of weight one. Multiplying such a Θ by a weight one Eisenstein series that is congruent to 1 modulo 2 shows that nonetheless ρ does arise from a cohomological cusp form.

Serre's conjecture for odd representations $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{PGL}_2(\mathbf{F}_3)$ played a special role in Wiles' approach to modularity of elliptic curves. Wiles deduced it as a consequence of the Langlands–Tunnell theorem, and the group theoretic accident, that the homomorphism $\mathrm{PGL}_2(\mathbf{Z}_3) \to \mathrm{PGL}_2(\mathbf{F}_3)$ splits.

The Langlands–Tunnell theorem yields that the characteristic 0 lift of $\overline{\rho}$ to a representation valued in $\mathrm{PGL}_2(\mathbf{C})$ (in fact valued in $\mathrm{PGL}_2(\mathbf{Z}[\sqrt{-2}])$), which exists because of the splitting above, arises from a holomorphic weight one form. To make $\overline{\rho}$ arise from a cohomological form one shifts weights by mutiplying by an Eisenstein series that is congruent to 1 modulo 3.

Thereafter Serre's conjecture, using in an essential way the methods Wiles introduced to go beyond the solvable case, was proven for odd representations of $G_{\mathbf{Q}}$ with image in $\mathrm{GL}_2(\mathbf{F}_4)$, $\mathrm{GL}_2(\mathbf{F}_5)$, $\mathrm{GL}_2(\mathbf{F}_7)$ and $\mathrm{GL}_2(\mathbf{F}_9)$. This gave cases of Serre's conjecture for representations with non-solvable, although still small, image. Such results are now available even when $\mathbf{Q}$ is replaced by a totally real field F and we consider totally odd representations $\rho : G_F \to \mathrm{GL}_2(k)$ with $k = \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5, \mathbf{F}_7, \mathbf{F}_9$.

The measure of complexity of ρ for proving automorphy in these works was the size of the image of ρ, and was related to the success in being able to find rational points defined over totally real solvable extensions of the base field of a Diophantine problem that was encountered in the method. Later approaches that proved the general conjecture measured the difficulty in proving Serre's conjecture for ρ in terms of the ramification of ρ.

The image of the projective representation $\overline{\rho}$ associated to a (linear) representation $\rho : G_F \to \mathrm{GL}_2(k)$ is arguably a better measure of its (group theoretic) complexity, rather than the image of ρ itself. If one thinks from the point of view of theory of equations, it is natural to consider projective representations $\overline{\rho}$ of small image, as these arise from splitting fields of polynomials of low degrees via the exceptional isomophisms $\mathrm{PSL}_2(\mathbf{F}_9) = A_6$, $\mathrm{PGL}_2(\mathbf{F}_5) = S_5$, $\mathrm{PGL}_2(\mathbf{F}_3) = S_4$, $\mathrm{PGL}_2(\mathbf{F}_2) = S_3$.

Projective representations

We could formulate an apparently more general version of Serre's conjecture for projective representations $\overline{\rho} : G_F \to \mathrm{PGL}_2(k)$ that are totally odd. As long as we do not work with a fixed finite field k, this generality is only apparent as one can lift $\overline{\rho}$ to $\overline{k}^{\times}\mathrm{GL}_2(k)$ using the vanishing of $H^2(G_F, \overline{k}^{\times})$ proved by Tate (the Galois action on the coefficients is the trivial action). Indeed Tate proved that $H^2(G_K, \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ is trivial for a number field K and any prime ℓ which implies $H^2(G_F, \overline{k}^{\times}) = 0$

But the representation may not lift to a representation to $\mathrm{GL}_2(k)$ and thus if we fix a finite field k, it is not formal to deduce Serre's conjecture for $\mathrm{PGL}_2(k)$ representations from Serre's conjecture for $\mathrm{GL}_2(k)$ representations. All the lifts to $\mathrm{GL}_2(\overline{k})$ of $\overline{\rho}$ are twists of each other, but the exact nature of the minimal k' for which there exists a lift is a little complicated.

Proving automorphy of 2-dimensional projective representations rational over small fields as opposed to 2-dimensional representations rational over small fields is one of the main themes of the second part of the talk. We use the modularity lifting theorems of [AKT] and hence are able to prove automorphy results for representations $\overline{\rho} : G_K \to \mathrm{PGL}_2(k)$ when K is totally real or imaginary CM.

We describe the main result of this talk towards cases of Serre-type conjecture for projective representations over CM fields K which rely on the "2-3 switch" of [AKT] and the 2-adic automorphy lifting results proved there. We say that a representation $\overline{\rho}: G_K \to \mathrm{PGL}_2(k)$ is modular if it arises from reducing $r_\iota(\pi)$ for π a cohomological automorphic representation of $\mathrm{GL}_2(\mathbf{A}_K)$.

Theorem

Let K be an imaginary CM field, let $p \in \{3,5\}$, and let $\overline{\rho} : G_K \to \mathrm{PGL}_2(\mathbf{F}_p)$ a representation whose image contains $\mathrm{PSL}_2(\mathbf{F}_p)$. Let ε_p be the projective cyclotomic character and let $\det \overline{\rho}$ be the determinant of $\overline{\rho}$. We assume that the extension cut out by $(\det \overline{\rho})\varepsilon_p$ is CM and does not contain ζ_5.
Then there is a CM solvable extension L/K, such that $\overline{\rho}|_{G_L}$ has image containing $\mathrm{PSL}_2(\mathbf{F}_p)$ and arises from a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_L)$ of weight 0.

The new case of the Serre-type conjectures (as in Buzzard-Diamond-Jarvis) that we prove over totally real fields is as follows.

Theorem

Let F be a totally real field and suppose that $\overline{\rho} : G_F \to \mathrm{PGL}_2(\mathbf{F}_5)$ is a totally odd and non-exceptional, with image containing $\mathrm{PSL}_2(\mathbb{F}_5)$ and such that the sign character is either totally even or totally odd. Then $\overline{\rho}$ arises from a cuspidal Hilbert modular form of weight $(2, \ldots, 2)$.

This result is easily deduced from the literature in all cases except when $\overline{\rho}$ is surjective and the sign character is totally odd (see Corollary 8). Representations $\overline{\rho} : G_F \to \mathrm{PGL}_2(\mathbf{F}_5)$ with totally odd sign character cannot arise from elliptic curves after restricting to $G_{F'}$ for any totally real extension F' of F, but their restriction to G_K, for an imaginary CM extension K of F, may arise from an elliptic curve over K. The proofs of both the theorems use the 2-adic automorphy lifting results of [AKT] over *imaginary CM fields*. (It was an attempt to prove Theorem 5 that led to the paper [AKT]!)

GL_2-valued representations over totally real fields

The following theorem is due to Ellenberg ($\mathbf{F}_9$ case), Manoharmayum ($\mathbf{F}_7$ case), Taylor, Shepherd-Barron ($\mathbf{F}_4, \mathbf{F}_5$).

Theorem

Let F be a totally real field and let $\rho : G_F \to \mathrm{GL}_2(k)$ be totally odd and absolutely irreducible. If $|k| \in \{2, 3, 4, 5, 7, 9\}$, then ρ arises from a weight 0 cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$.

Lets sketch the strategy of proof of this theorem. The ingredients are modularity lifting, a Diophantine argument, and a mod p solvable descent result.

Diophantine argument

This result is proven by considering a certain twisted moduli space X_ρ over F parametrizing elliptic curves E over F with mod p (say $p = 7$) torsion which affords ρ. For example for $p = 7$ this is a curve of genus 3 (a twist of $X(7)$). One finds points of $X(\rho)$ over totally real solvable extensions L of F, and these correspond to elliptic curves E over L whose mod 3 torsion helps one prove that E is modular using the techniques of Wiles. Thus $\rho|_{G_L}$ is modular and then one uses mod 7 solvable descent.

Mod p solvable descent

Proposition

Let F be a totally real field and let $\rho : G_F \to \mathrm{GL}_2(k)$ be a totally odd representation with k a finite field of characteristic $p > 2$. Assume that there is a solvable totally real extension L/F satisfying the following hypotheses.

1. $\rho|_{G_L}$ *arises from a weight 0 cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbf{A}_L)$.
2. $\rho|_{G_{L(\zeta_p)}}$ *is absolutely irreducible.*

Then ρ arises from a weight 0 cuspidal automorphic representation π_F on $\mathrm{GL}_2(\mathbf{A}_F)$.

Proof of mod p solvable descent using lifting results

We lift ρ to a geometric representation $\tilde{\rho}$ using either a method due to Ramakrishna or a later method of KW. Then we show that $\tilde{\rho}|_{G_L}$ is modular using the assumption that $\rho|_L$ is modular and then modularity lifting method of Wiles.

Thus $\tilde{\rho}|_{G_L}$ arises from a a weight 0 cuspidal automorphic representation π_L on $\mathrm{GL}_2(\mathbf{A}_L)$. Then using the classical (characteristic 0) solvable descent results of Saito-Shintani, one shows that $\tilde{\rho}$, and hence ρ, arises from a weight 0 cuspidal automorphic representation π_F on $\mathrm{GL}_2(\mathbf{A}_F)$.

This is one of the early arguments which shows the utility of various types of lifting (Galois theoretic, automorphic) results in proving Serre-type conjectures. It reduces mod *p* solvable descent to the known cases of solvable descent for characteristic 0 forms. It is an interesting challenge to show mod *p* descent (even say for quadratic extensions!) for mod *p* Hilbert modular forms by working purely with mod *p* forms. So if $F/\mathbf{Q}$ is a real quadratic extension and π a Hilbert modular form such that π^σ and π are congruent mod *p* can one show without using lifting to char. 0 that π is congruent mod *p* to a form base changed from $\mathbf{Q}$?

Known projective cases

Corollary

Let F be a totally real field. Totally odd irreducible representations $\overline{\rho} : G_F \to \mathrm{PGL}_2(k)$ arise from weight 0 cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbf{A}_F)$ in the following cases:
(i) $k = \mathbf{F}_2, \mathbf{F}_4$.
(ii) $k = \mathbf{F}_3$ when the sign character of $\overline{\rho}$ is totally odd.
(iii) $k = \mathbf{F}_5$ when the sign character of $\overline{\rho}$ is totally even.
(iv) $k = \mathbf{F}_7$ when the sign character of $\overline{\rho}$ is totally odd.
(v) $k = \mathbf{F}_9$ when the sign character of $\overline{\rho}$ is totally even.

The idea of the proof of the corollary is to lift $\overline{\rho}$ to representations to $\mathrm{GL}_2(k)$ (after totally real solvable base change) and then apply previous results.

Modularity of mod 5 representations

We conclude with our main result of this second part of the talk which is about the modularity of totally odd representations $\overline{\rho} : G_K \to \mathrm{PGL}_2(\mathbf{F}_5)$ with the odd sign character. Note that this is not covered by the corollary. We again consider only the case when $\overline{\rho}$ has image containing $\mathrm{PSL}_2(\mathbf{F}_5)$, as the dihedral case is well-known.

Theorem

Let F be a totally real field and let $\overline{\rho} : G_F \to \mathrm{PGL}_2(\mathbf{F}_5)$ be a totally odd, non-exceptional representation with image containing $\mathrm{PSL}_2(\mathbf{F}_5)$, such that the sign character is either totally even or totally odd. Then $\overline{\rho}$ arises from weight 0 cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$.

By non-exceptional we mean that the splitting field of $\overline{\rho}$ does not contain ζ_5: this is no condition at all if $\sqrt{5}$ is not in F.

Sketch of proof

We use that under our assumptions there is a CM (non totally real) extension L/F, such $\overline{\rho}|_{G_L}$ lifts to a $\mathrm{GL}_2(\mathbf{F}_5)$ representation with cyclotomic determinant which arises from an elliptic curve E over L. Then we can play some prime switching games! Namely we can use the mod 3 representation of E which is known to be modular by the results of [AKT], and 3-adic lifting results to conclude that E is modular, and hence $\overline{\rho}|_{G_L}$ is modular. Then we conclude that $\overline{\rho}$ is modular by mod p descent results for solvable CM extensions of F.

The strategy we use does not work in the mixed signature case: in that case one notes that after any CM base change L/F, $\overline{\rho}|_{G_L}$ never lifts to a $\mathrm{GL}_2(\mathbf{F}_5)$ representation with cyclotomic determinant (although there is such a L for which $\overline{\rho}|_{G_L}$ lifts to $\mathrm{GL}_2(\mathbf{F}_5)$).

Thank you for your attention!