Tate conjectures in function field arithmetic

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The setting

- $\mathbb{F}_q$, a field of finite cardinality q.
- $A = \mathbb{F}_q[t]$, the ring of coefficients.
  $F = \mathbb{F}_q(t)$, the fraction field of A.
- K, a field over $\mathbb{F}_q$.

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Usual motives have coefficient ring $\mathbb{Z}$: the category is $\mathbb{Z}$-linear.
E.g. abelian varieties, algebraic tori.

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$E \mapsto M(E)$ is a *contravariant* functor.

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A *Drinfeld A-module* is an Anderson A-module of dimension 1.

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Conclusion: E is an Anderson module of dimension 1 and rank 2.

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In the example: a prime $\mathfrak{p} = (f)$ is special if and only if $f(\alpha) = 0$.
If α is transcendental over $\mathbb{F}_q$ then every prime is generic.

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Assume that K *is finitely generated.* Then the functor $E \mapsto T_{\mathfrak{p}}E$ is

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NB: not every motive arises from E.

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Dieudonné-Manin classification theorem

Assume that K is *algebraically closed*. Then

- The category of $\mathcal{E}_K$-isocrystals is semi-simple.
- Simple objects M_λ are classified by *slope* $\lambda \in \mathbb{Q}$.

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Filtration theorem (for arbitrary K)

Every $\mathcal{E}_K$-isocrystal M carries a unique filtration

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such that:

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This is called the *Harder-Narasimhan filtration*.
Splits if K is perfect (and does not split otherwise).

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The functor $M \mapsto T(M)$ is an equivalence of

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Can extend this to pure modules of any slope!
The Weil group W_K appears instead of G_K.
The target category is more complicated.

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Theorem (Drinfeld '77)

A motive of rank $r > 0$ arises from a Drinfeld module if and only if it is pure of weight $\frac{1}{r}$.

Tate conjectures

Tate conjectures for A-motives over K

Assume that K is *finitely generated*. Then the functor $M \mapsto M_{\mathfrak{p}}$ is

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Folklore theorem

Assume that K is finite. Then the Tate conjecture (FF) holds for all motives M and places $\mathfrak{p}$.

Tate conjectures

Tate conjectures for A-motives over K

Assume that K is *finitely generated*. Then the functor $M \mapsto M_{\mathfrak{p}}$ is

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Assume that K is finite. Then the Tate conjecture (FF) holds for all motives M and places $\mathfrak{p}$.

Reason: $\mathcal{E}_{K,F_{\mathfrak{p}}} = F_{\mathfrak{p}} \otimes_A K[t]$. Implies injectivity for arbitrary K.

Results

- Taguchi '91, '93: SS for Drinfeld modules, $\mathfrak{p} \neq \infty$.
- Taguchi '95: FF for generic $\mathfrak{p}$ and tr deg $K = 1$.
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- A. Tamagawa '94, '95, '96: FF+SS for generic $\mathfrak{p}$.
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M.'20: counterexample to FF for mixed motives, $\mathfrak{p} = \infty$.
The work still continues: $\mathrm{FF}_{\mathfrak{p}=\infty}$ is true for many mixed motives.

Full faithfulness: the algebraic part

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Goal: understand what is $F_{\mathfrak{p}} \otimes_A \operatorname{Hom}(M, N)$.

Gardeyn's theory

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A *left* $\mathcal{O}_{X\times C}\{\tau\}$*-module*: a pair $(\mathcal{F}, \tau)$ where $\mathcal{F}$ is an $\mathcal{O}_{X\times C}$-module, $\tau\colon \mathcal{F} \to \sigma_*\mathcal{F}$ is a morphism.

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Gardeyn's maximal model

There is a unique left $\mathcal{O}_{X\times C}\{\tau\}$-submodule $\mathcal{M} \subset \iota_* \tilde{M}$ which is

- locally free of finite type over $\mathcal{O}_{X\times C}$,
- maximal with respect to the inclusion relation.

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Theorem

$$F_{\mathfrak{p}} \otimes_A \mathrm{Hom}(M, N) = \mathrm{Hom}(\mathcal{M}_{\mathfrak{p}}, \mathcal{N}_{\mathfrak{p}})$$

Instant consequence of proper base change.

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We know that $\mathcal{M}_\eta$ is an isocrystal.

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When the functor $\mathcal{M} \mapsto \mathcal{M}_\eta$ is fully faithful?

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To prove full faithfulness it is enough to show that every morphism $\mathcal{M}_\eta \to \mathcal{N}_\eta$ maps $\mathcal{M}_x$ to $\mathcal{N}_x$ for all $x \in X$.

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$$\mathrm{Hom}(\mathcal{M}|_{\mathcal{U}}, \mathcal{N}|_{\mathcal{U}}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{M}_\eta, \mathcal{N}_\eta)$$

is an isomorphism.

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The z-adic analog of the p-adic overconvergent ring ($\hat{F} = \mathbb{Q}_p$).

NB: $\mathcal{E}_R \subset \mathcal{E}_R^\dagger$. Furthermore $\mathcal{E}_R^\dagger$ is a field.

An overconvergent isocrystal is

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For each x the module $\mathcal{M}_x^\dagger = \mathcal{E}_R^\dagger \otimes_{\mathcal{E}_R} \mathcal{M}_x$ is an overconvergent isocrystal.

We shall study the inclusion $\mathrm{Hom}(\mathcal{M}_x, \mathcal{N}_x) \subset \mathrm{Hom}(\mathcal{M}_x^\dagger, \mathcal{N}_x^\dagger)$.

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NB: $\mathcal{M}_x$ is a local maximal model for every x.
This follows from the fact that $\mathcal{M}$ is a Gardeyn model.

theorem (M., in progress)

The base change functor $\mathcal{E}_R^\dagger \otimes_{\mathcal{E}_R} -$ is fully faithful on local maximal models.

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Corollary

For all A-motives M, N over K and all places $\mathfrak{p}$ of F we have

$$F_\mathfrak{p} \otimes_A \mathrm{Hom}(M, N) = \{f\colon M_\mathfrak{p} \to N_\mathfrak{p} \mid \forall x \;\; f(M^\dagger_{\mathfrak{p},x}) \subset N^\dagger_{\mathfrak{p},x}\}.$$

Here $M^\dagger_{\mathfrak{p},x} = \mathcal{E}^\dagger_{R,F_\mathfrak{p}} \otimes_{K[t]} M$. Note that $K(t) \subset \mathcal{E}^\dagger_{R,F_\mathfrak{p}}$ for all R, $\mathfrak{p}$.

By Watson the condition holds for almost all x.

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In the p-adic setting the $\mathcal{E}_K$-isocrystals carry extra data: a *connection* ∇.

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Theorem (Kedlaya '03)

In the p-adic setting the base change functor $\mathcal{E}_K \otimes_{\mathcal{E}_R^\dagger} -$ is fully faithful.

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The Robba ring for the valuation v

$\mathcal{R}_v = \{\sum_{n\in\mathbb{Z}} x_n z^n \mid \text{converges on a punctured open disk w.r.t. } v\}$

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This leads to a counterexample to FF for $\mathfrak{p} = \infty$.

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Theorem (folklore)

The base change functor $\mathcal{E} \otimes_{\mathcal{E}_R^\dagger} -$ is fully faithful on pure isocrystals.

Yields FF for generic $\mathfrak{p}$, and $\mathfrak{p} = \infty$ for pure motives.

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Theorem (Ambrus Pal - M. '20)

The base change functor $\mathcal{E} \otimes_{\mathcal{E}_R^\dagger} -$ is fully faithful on isocrystals with "good" monodromy.

"good" $=$ the result of the p-adic monodromy theorem translated to the z-adic setting.

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Yields Watson's base change theorem, and FF for Drinfeld modules, special $\mathfrak{p}$. Also applies to $\mathfrak{p} = \infty$ when the motive has potential good reduction everywhere.