Equivariant birational types

joint with Kontsevich-Pestun, Kresch, and Hassett

- $H_p(x) := \max(1, |x|_p)$, $x \in \mathbb{Q}_p$

Calc II: p-adic Integration

- $H_p(x) := \max(1, |x|_p)$, $x \in \mathbb{Q}_p$
- $$U(0) := \{x \,|\, |x|_p \leq 1\}, \quad \mathrm{vol}(U(0)) = 1$$
$$U(j) := \{x \,|\, |x|_p = p^j\}, \quad \mathrm{vol}(U(j)) = p^j(1 - \frac{1}{p})$$

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$$\int_{\mathbb{Q}_p} H_p(x_p)^{-s} \mathrm{d}x_p = \int_{U(0)} H_p(x_p)^{-s} \mathrm{d}x_p + \sum_{j \geq 1} \int_{U(j)} H_p(x_p)^{-s} \mathrm{d}x_p$$

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$$= 1 + \sum_{j\geq 1} p^{-js}\mathrm{vol}(U(j))$$

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We interpret this as a volume with respect to a natural measure.

$X = X_\Sigma$ - projective equivariant compactification of $T = \mathbb{G}_m^d$.

- $N \simeq \mathbb{Z}^d$, $M = \mathrm{Hom}(N, \mathbb{Z})$, $\Sigma = \{\sigma\}$ - fan
- $e_1, \ldots, e_n$ - 1-dimensional cones in Σ
- $$0 \to M \to \mathrm{PL}(\Sigma) \to \mathrm{Pic}(X_\Sigma) \to 0,$$

Toric varieties

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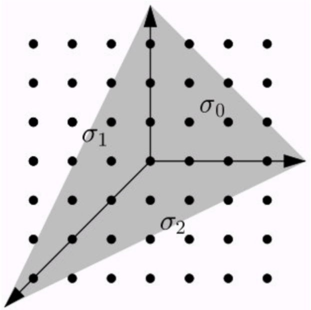
  $\varphi = \varphi_{\mathbf{s}} \in \mathrm{PL}(\Sigma)$ is defined by its values on e_j: $\varphi_{\mathbf{s}}(e_j) = s_j \in \mathbb{C}$
- $T(\mathbb{Q}_p)/T(\mathbb{Z}_p) = N$


σ_0
σ_1
σ_2

Height integrals (Batyrev–T. 1995)

- Local heights:
$$H_p(\varphi_{\mathbf{s}}, t_p) := p^{\varphi_{\mathbf{s}}(\bar{t}_p)}$$

- $$\int_{T(\mathbb{Q}_p)} H_p(\varphi_{\mathbf{s}}, t_p)^{-1} \, \mathrm{d}t_p = \left(\sum_{k=1}^{d} \sum_{\sigma \in \Sigma(k)} (-1)^k \left(\sum_{n \in \sigma \cap N} p^{-\varphi_{\mathbf{s}}(n)} \right) \right)$$
$$= \sum_{k=1}^{d} \sum_{\sigma \in \Sigma(k)} (-1)^k \prod_{e_j \in \sigma} \frac{1}{1 - p^{-s_j}}$$

Toric modular forms (Lev Borisov–Gunnels 2000)

Fix φ such that $\varphi(e_j) \notin \mathbb{Z}$, for all j. For $q \in \mathfrak{H}$, put

$$f_{N,\varphi}(q) := \sum_{m \in M} \left(\sum_{k=1}^{d} \sum_{\sigma \in \Sigma(k)} (-1)^k \left(\sum_{n \in \sigma \cap N} q^{m \cdot n} e^{2\pi i \varphi(n)} \right) \right),$$

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Let p be a prime, and assume that $\varphi(e_j) \in \frac{1}{p}\mathbb{Z}$, for all j. Then

$$f_{N,\varphi}(q)$$

is a modular form for $\Gamma_1(p)$ of weight d.

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For every p, the ring of toric modular forms for $\Gamma_1(p)$ coincides with the ring of modular forms, for weights ≥ 3.

- X – smooth projective Fano variety, $\dim(X) = d$, over a number field F
- $-K_X$ is equipped with an adelic metrization.

For $x \in X(F_v)$ choose local analytic coordinates $x_1, \ldots, x_d$, in a neighborhood U_x. In U_x, a section of the canonical line bundle has the form $\mathsf{s} := \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$. Put

$$\tau_v = \tau_{X,v} := \|\mathsf{s}\|_v \mathrm{d}x_1 \cdots \mathrm{d}x_d,$$

where $\mathrm{d}x_1 \cdots \mathrm{d}x_d$ is the standard normalized Haar measure on F_v^d. It globalizes to $X(F_v)$.

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where $\mathrm{d}x_1 \cdots \mathrm{d}x_d$ is the standard normalized Haar measure on F_v^d. It globalizes to $X(F_v)$. For almost all v, and Zariski open $U \subset X$,

$$\int_{U(F_v)} \tau_v = \int_{X(F_v)} \tau_v = \int_{X(\mathfrak{o}_v)} \tau_v = \sum_{\tilde{x} \in X(\mathbb{F}_q)} \int_{\pi^{-1}(\tilde{x})} \tau_v = \frac{\# X(\mathbb{F}_q)}{q^d}.$$

Birational Calabi-Yau (Batyrev 1997)

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- A canonical bundle of a Calabi-Yau variety has a canonical metrization
- $$\int_{X(F_v)} \tau_v = \frac{\#X(\mathbb{F}_q)}{q^n} \quad \forall' v$$
- If $X \supset U \subset Y$, then

$$\frac{\#X(\mathbb{F}_q)}{q^n} = \int_{X(F_v)} \tau_v = \int_{U(F_v)} \tau_v = \int_{Y(F_v)} \tau_v = \frac{\#Y(\mathbb{F}_q)}{q^n}, \quad \forall q$$

Let $U := X \setminus D$, with

$$D = \cup_{\alpha \in \mathcal{A}} D_\alpha, \quad -K_X = \sum \rho_\alpha D_\alpha,$$

where D_α are geometrically irreducible, smooth, and intersecting transversally.

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$D_A \subset X$ is smooth, of codimension $\#A$ (or empty).

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Local heights and height integrals

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$$Z_v(\mathbf{s}) := \int_{U(F_v)} \prod_{\alpha \in \mathcal{A}} H_\alpha(x)^{-s_\alpha} \mathrm{d}\tau_v$$

In charts, via partition of unity: in a neighborhood of $x \in D_A^\circ(F)$ it takes the form

$$\int \prod_{\alpha \in A} |x_\alpha|_v^{s_\alpha - \rho_\alpha} \, \mathrm{d}\tau_v$$

Local computations

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Essentially, this is a product of integrals of the form

$$\int_{|x|_v \leq 1} |x|_v^{s-1} \mathrm{d}x_v.$$

Denef's formula

For almost all v one has:

$$Z_v(\mathbf{s}) = \sum_{A} \frac{\# D_A^{\circ}(\mathbb{F}_q)}{q^{\dim(X)}} \prod_{\alpha \in A} \frac{q-1}{q^{s_\alpha - \rho_\alpha + 1} - 1}.$$

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Specialize to $s_\alpha = \rho_\alpha$, for all $\alpha \in \mathcal{A}$:

$$Z_v(\rho) = \sum_A \frac{\#D_A^\circ(\mathbb{F}_q)}{q^{\dim(X)}} = \frac{\#X(\mathbb{F}_q)}{q^{\dim(X)}}.$$

The integral

- is an invariant under blowups,
- encodes information about singularities of X,
- plays a central role in analytic/spectral approches to Manin's conjectures, volume asymptotics, etc.

Basic questions

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- How much geometry can be read off from arithmetic?

Rationality

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(U) unirational: if $\mathbb{P}^n \dashrightarrow X$, for some n

Specialization of (stable) rationality

- Voisin (2013): integral decomposition of Δ (Bloch-Srinivas)

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- Kontsevich–T. (2017): $\mathrm{Burn}(k)$, $\mathrm{char}(k) = 0$

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- **Kontsevich–T. (2017):** Same formula for

$$\mathrm{Burn}(K) \to \mathrm{Burn}(k),$$

  the free abelian group spanned by classes of varieties over the corresponding field, modulo rationality.

Specialization (Kontsevich-T. 2017)

- Let $\mathfrak{o} \simeq k[[t]]$, $K \simeq k((t))$, $\mathrm{char}(k) = 0$.
- Let X/K be a smooth proper (or projective) variety, with function field $L = K(X)$.
- Choose a regular model

$$\pi : \mathcal{X} \to \mathrm{Spec}(\mathfrak{o}),$$

such that π is proper and the special fiber $\mathcal{X}_0$ over $\mathrm{Spec}(k)$ is a simple normal crossings (snc) divisor:

$$\mathcal{X}_0 = \cup_{\alpha \in \mathcal{A}}\, d_\alpha D_\alpha, \quad d_\alpha \in \mathbb{Z}_{\geq 1}.$$

- Put

$$\rho([L/K]) := \sum_{\emptyset \neq A \subseteq \mathcal{A}} (-1)^{\#A-1} [D_A \times \mathbb{A}^{\#A-1}/k] \in \mathrm{Burn}(k),$$

From birational types to equivariant birational types

There are close similarities between the study of birational properties of varieties over nonclosed fields and the study of birational group actions on varieties over algebraically closed fields.

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This motivated the search for ways to "integrate in presence of group actions".

$\mathbb{P}^1$ with $\mathbb{Z}/5\mathbb{Z}$-action

Consider two actions

$$(x_0, x_1) \mapsto (\zeta_5 x_0, \zeta_5^4 x_1), \qquad \text{or} \;\; \mapsto (\zeta_5^2 x_0, \zeta_5^3 x_1).$$

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- weights at 0 and ∞ differ, in both cases, by ± 1,
- these two actions are not equivariantly birational.

Reichstein–Youssin (2002)

Let V and W be d-dimensional faithful representations of an abelian group G of rank $r \leq d$, and

$$\chi_1, \ldots, \chi_d, \quad \text{respectively} \quad \eta_1, \ldots, \eta_d,$$

the characters of G appearing in V, respectively W. Then V and W are G-equivariantly birational if and only if

$$\chi_1 \wedge \cdots \wedge \chi_d = \pm\, \eta_1 \wedge \cdots \wedge \eta_d$$

(This condition is meaningful only when $r = d$.)

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- There exist actions of $(\mathbb{Z}/p\mathbb{Z})^2$ on $\mathbb{P}^2$ that are not equivariantly rational.
- Note that representations of a group G are equivariantly stably birational.

Equivariant bir. types (Kontsevich-T. 2019)

- G - finite abelian group, $A = G^\vee$ its group of characters,
- X - smooth projective, of dimension n, with regular G-action,
- $X^G = \sqcup\, F_\alpha$,

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- $X^G = \sqcup F_\alpha$,
- β_α - (equivalence class of) representation of G, acting in the tangent space $\mathcal{T}_{x_\alpha}X$, for $x_\alpha \in F_\alpha$, i.e.,

$$\beta_\alpha := [a_{1,\alpha}, \dots, a_{n,\alpha}],$$

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-
$$\beta : X \mapsto \sum_\alpha \beta_\alpha.$$

First examples: $\mathbb{P}^2$

Consider an action of $\mathbb{Z}/N\mathbb{Z}$ on $X = \mathbb{P}^2$ given by

$$(x : y : z) \mapsto (\zeta^a x : \zeta^b y : z),$$

$$\zeta = \zeta_N, \quad a, b \in \mathbb{Z}/N\mathbb{Z}, \quad \gcd(a, b, N) = 1, \quad a \neq b.$$

Fixed points are

$$(0 : 0 : 1), \quad (0 : 1 : 0), \quad (1 : 0 : 0).$$

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Then

$$\beta(X) = [a, b] + [a - b, -b] + [b - a, -a].$$

First examples: $\mathbb{P}^2$

All such actions are equivalent. Declare $\beta(X) = 0$, i.e.,

$$[a, b] = -[b - a, -a] - [a - b, -b]$$

Allowing

$$[a, b] = -[a, -b]$$

we find

$$[a, b] = [a, b - a] + [a - b, b].$$

BIRATIONAL TYPES $\mathcal{B}_2(\mathbb{Z}/N\mathbb{Z})$

Generators: $[a, b]$, $a, b \in \mathbb{Z}/N\mathbb{Z}$, $\gcd(a, b, N) = 1$

Relations:

- $[a, b] = [b, a]$
- $[a, b] = [a, b - a] + [a - b, b]$ if $a \neq b$
- $[a, a] = [a, 0]$

This gives $\binom{p}{2}$ linear equations in the same number of variables.

Birational types

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$$\mathrm{rk}_{\mathbb{Q}}(\mathcal{B}_2(G)) = \frac{p^2+23}{24} = \frac{p^2-1}{24} + 1$$

Let $\tilde{X} \to X$ be a G-equivariant blowup. Consider relations

$$\beta(\tilde{X}) - \beta(X) = 0.$$

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It turns out, that these can be formalized as follows.

Birational types $\mathcal{B}_n(G)$

Consider the $\mathbb{Z}$-module

$$\mathcal{B}_n(G)$$

generated by unordered tupels $[a_1, \dots, a_n]$, $\quad a_i \in A$, such that

(G) $\sum_i \mathbb{Z} a_i = A$, and

(B) for all $a_1, a_2, b_1, \dots, b_{n-2} \in A$ we have

$$[a_1, a_2, b_1, \dots b_{n-2}] =$$

$$[a_1 - a_2, a_2, b_1, \dots, b_{n-2}] + [a_1, a_2 - a_1, b_1, \dots, b_{n-2}] \text{ if } a_1 \neq a_2,$$

$$[a_1, 0, b_1, \dots, b_{n-2}] \qquad \text{if } a_1 = a_2.$$

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Jumps at

$$p = 43, 59, 67, 83, \ldots$$

Birational types

Consider $X^G = \sqcup F_\alpha$ and record eigenvalues of G

$$[a_{1,\alpha}, \dots, a_{n,\alpha}]$$

in the tangent space $\mathcal{T}_{x_\alpha} X$, at some $x_\alpha \in F_\alpha$. Put

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Kontsevich-T. 2019

The class

$$\beta(X) \in \mathcal{B}_n(G)$$

is a well-defined G-equivariant birational invariant.

Variant: introduce the quotient

$$\mu^- : \mathcal{B}_n(G) \to \mathcal{B}_n^-(G)$$

by an additional relation

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Variant: introduce the quotient

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The class of $\mathbb{P}^n$, $n \geq 2$, with linear action of $G := \mathbb{Z}/N\mathbb{Z}$ is

- torsion in $\mathcal{B}_n(G)$ and
- trivial in $\mathcal{B}_n^-(G)$.

Cyclic action on $\mathbb{P}^n$, $n \geq 2$

Since all such actions are birationally equivalent, it suffices to consider one, with $G = \mathbb{Z}/N\mathbb{Z}$ acting by

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This action fixes the point $(1, 0, \ldots, 0)$ and the hyperplane $x_0 = 0$. We have

$$\beta(\mathbb{P}^n) = [1, 0, \ldots, 0] + [0, -1, \ldots, -1] = [1, 0, \ldots] + [-1, 0, \ldots].$$

Equivariant Burnside group (Kresch-T. 2020)

Let G be a finite abelian group. Let

$$\mathrm{Burn}_n(G)$$

be the quotient of the free abelian group generated by symbols

$$(H, G/H \circlearrowright K, \beta),$$

where

- $H \subseteq G$ is a subgroup
- K is a G/H-Galois algebra over a field of transcendence degree $d \leq n$ over k, up to isomorphism, and β is a faithful $(n-d)$-dimensional representation of H,

modulo somewhat complicated blowup relations.

The class of a G-variety is computed on a standard model X:

- X is smooth projective,
- there exists a Zariski open $U \subset X$ such that G acts freely on U,
- the complement $X \setminus U$ is a normal crossings divisor,
- for every $g \in G$ and every irreducible component D of $X \setminus U$, either $g(D) = D$ or $g(D) \cap D = \emptyset$.

Passing to a standard model X, define:

$$[X \circlearrowleft G] := \sum_{H \subseteq G} \sum_{F} (H, G/H \circlearrowright k(F), \beta_F(X)) \in \mathrm{Burn}_n(G),$$

where

- the sum is over all strata $F \subset X$ with generic stabilizer H,
- the symbols records the eigenvalues of H in the tangent space at $x \in F$, as before, as well as the G/H-action on the function field of F, respectively the orbit of F.

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This is a G-birational invariant.

Equivariant birational geometry: tools

- Equivariant MMP (classification of links, ...)

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- Equivariant birational rigidity (analysis of singularities, ...)

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In particular, $\mathcal{B}_2(G)$ does not give anything new in dimension 2. However, it enters as coefficient group in higher dimensions, and can contribute nontrivially.

Abelian actions

Abelian actions in dimension 3 are not fully settled, but should be, in principle, accessible.

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The following examples focus on dimension 4, where we currently do not know how to systematically factor birational maps, and in particular, do not understand the (failure of) rationality of cubic fourfolds.

There is an extensive literature on their automorphisms (and on automorphisms of their variety of lines) , e.g., Laza, Zheng, Fu, Mongardi, Mayanskiy, Ouchi, ...

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Here are $N > 1$, with $\mathbb{Z}/N\mathbb{Z}$ acting on a smooth cubic fourfold:

$$N = 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 15, 16, 18, 21, 24, 30, 32, 33, 36, 48.$$

Note that

$$d_{\mathbb{Q}} := \dim \mathcal{B}_4(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} = 0, \qquad \text{for all } N < 27,\ N = 30, 32,$$

but

N	33	36	48
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Cubic fourfolds

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These are rigid. Are they rational?

One can also work with finite coefficients. Let

$$d_p = d_p(N) := \dim \mathcal{B}_4(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{F}_p.$$

We have $d_2, d_3 = 0$, for all $N \leq 15$, and $N = 18, 21$.

N	16	24	30	32	33	36	48
d_2	1	5	10	12	3	19	50
d_3	0	0	0	0	2	3	7

Birational types: using $\mathcal{B}_n(G)$

Consider the cubic fourfold $X \subset \mathbb{P}^5$ given by

$$x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1 + x_4^2 x_5 + x_5^2 x_0 + x_0^3 = 0.$$

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(Solving a system of 443557 linear equations in 82251 variables,...)

$$\beta(X) \neq \beta(\mathbb{P}^4) = 0 \in \mathcal{B}_4(\mathbb{Z}/36\mathbb{Z}) \otimes \mathbb{F}_2 = \mathbb{F}_2^{19}$$

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Thus X is not G-equivariantly birational to $\mathbb{P}^4$ (with linear action).

Birational types: using $\mathrm{Burn}_n(G)$

Consider the cubic fourfold $X \subset \mathbb{P}^5$, given by

$$x_0x_1^2 + x_0^2x_2 - x_0x_2^2 - 4x_0x_4^2 + x_1^2x_2 + x_3^2x_5 - x_2x_4^2 - x_5^3 = 0.$$

$G = \mathbb{Z}/6\mathbb{Z}$ acts with weights $(0,0,0,1,3,4)$. This X is rational, since it contains the disjoint planes

$$x_0 = x_1 - x_4 = x_3 - x_5 = 0 \quad \text{and} \quad x_2 = x_1 - 2x_4 = x_3 + x_5 = 0,$$

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but not G-equivariantly birational to $\mathbb{P}^4$ with linear action. There is a cubic surface $S \subset X$, with $\mathbb{Z}/3\mathbb{Z}$-stabilizer, $\mathbb{Z}/2\mathbb{Z}$ fixes an elliptic curve, and this S is not stably $\mathbb{Z}/2\mathbb{Z}$-equivariantly rational; the corresponding symbol

$$[\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \circlearrowright k(S), \beta] \neq 0 \in \mathrm{Burn}_4(\mathbb{Z}/6\mathbb{Z}),$$

does not interact with any other symbols in $[X \circlearrowleft G]$.

Let G be a finite nonabelian group. Let

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be the quotient of the free abelian group generated by symbols

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where

- $H \subseteq G$ is an abelian subgroup
- K is a $N_G(H)/H$-Galois algebra over a field of transcendence degree $d \leq n$ over k, up to isomorphism, and β is a faithful $(n-d)$-dimensional representation of H,

modulo somewhat complicated blowup relations.

As before, passing to a standard G-equivariant smooth projective model X, which, in particular, has only abelian stabilizers, define:

$$[X \circlearrowleft G] := \sum_{H \subseteq G} \sum_{F} (H, G/H \circlearrowright k(F), \beta_F(X)) \in \mathrm{Burn}_n(G),$$

where the sum is over abelian subgroups $H \subset G$, and all strata $F \subset X$ with generic stabilizer H, ...

Nonabelian actions on surfaces

Consider the action of $G = C_2 \times \mathfrak{S}_3 = W(\mathsf{G}_2)$ on the corresponding torus T and its Lie algebra $\mathfrak{t}$.

- These are stably equivariantly birational (Lemire-Popov-Reichstein 2005)

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- These are stably equivariantly birational (Lemire-Popov-Reichstein 2005)
- They are not equivariantly birational (Iskovskikh 2005)

These actions can be realized via:

- the action on $y_1y_2y_3 = 1$ via permutation of variables and taking inverses, with model DP6

Nonabelian actions on surfaces

These actions can be realized via:

- the action on $y_1 y_2 y_3 = 1$ via permutation of variables and taking inverses, with model DP6
- the action on $x_1 + x_2 + x_3$ via permutation and reversing signs, with model $\mathbb{P}^2$

The action on $\mathbb{P}^2 = \mathbb{P}(I \oplus V)$, with coordinates $(u_0 : u_1 : u_2)$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \quad \iota := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

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There is one fixed point, $(1 : 0 : 0)$; after blowing up, the exceptional curve is stabilized by the central involution ι, and comes with a nontrivial $\mathfrak{S}_3$-action, contributing the symbol

$$(C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (1)) \in [X \circlearrowleft G].$$

Additionally, the line $\ell_0 := \{u_0 = 0\}$ has as stabilizer the central C_2, contributing the same symbol.

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Additionally, the line $\ell_0 := \{u_0 = 0\}$ has as stabilizer the central C_2, contributing the same symbol. ... There are also other terms.

A better model for the second action is the quadric

$$v_0v_1 + v_1v_2 + v_2v_0 = 3w^2,$$

where $\mathfrak{S}_3$ permutes the coordinates $(v_0 : v_1 : v_2)$ and the central involution exchanges the sign on w. There are no G-fixed points, but a conic $R_0 := \{w = 0\}$ with stabilizer the central C_2 and a nontrivial action of $\mathfrak{S}_3$, A better model for the second action is the quadric

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The crucial difference is that the summand

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appears twice in the $\mathbb{P}^2$ model, and only once in the quadric model.

The crucial difference is that the summand

$$(C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (1))$$

appears twice in the $\mathbb{P}^2$ model, and only once in the quadric model. No relations can eliminate this symbol.

This $\mathbb{P}^1$, with $\mathfrak{S}_3$-action, should be viewed as an analog of a curve of genus ≥ 1 in the fixed locus – it will appear on every equivariantly birational model.

Nonabelian actions on $\mathbb{P}^2$

Iskovskikh/Dolgachev: *"Are there embeddings of a finite group G into PGL_3 that are not conjugate in Cr_2?"*

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Let $G = C_5 \times \mathfrak{S}_3$, and V be the standard 2-dimensional representation of $\mathfrak{S}_3$. Let χ be a nontrivial character of C_5.

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Kresch-T. 2021

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KRESCH-T. 2021

The class

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is nontrivial. Moreover, if $\chi \neq \pm\chi'$ then the corresponding classes are distinct.

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Specialization of equivariant birational types

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Theorem (Kresch-T. 2020)

Let X and X' be smooth projective varieties over K with generically free G-actions, admitting regular models $\mathcal{X}$, respectively $\mathcal{X}'$, smooth and projective over $\mathfrak{o}$, to which the G-action extends. If X and X' are G-equivariantly birational over K then so are the special fibers of $\mathcal{X}$ and $\mathcal{X}'$.

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There is also a notion of BG-rational singularities, allowing to understand the equivariant birational type of special fibers with mild singularities.

Modular/motivic types $\mathcal{M}_n(G)$

Fix an integer $n \geq 2$. Consider the $\mathbb{Z}$-module

$$\mathcal{M}_n(G) \quad \text{generated by} \quad \langle a_1, \dots, a_n \rangle, \quad a_i \in A, \quad \sum_i \mathbb{Z} a_i = A,$$

(S) for all $\sigma \in \mathfrak{S}_n$, $a_1, \dots, a_n \in A$ we have

$$\langle a_{\sigma(1)}, \dots, a_{\sigma(n)} \rangle = \langle a_1, \dots, a_n \rangle,$$

(M) for all $2 \leq k \leq n$, all $a_1, \dots, a_k \in A$, $b_1, \dots, b_{n-k} \in A$ such that

$$\sum_i \mathbb{Z} a_i + \sum_j \mathbb{Z} b_j = A$$

we have

$$\langle a_1, \dots, a_k, b_1, \dots b_{n-k} \rangle =$$

$$= \sum_{1 \leq i \leq k} \langle a_1 - a_i, \dots, a_i, \dots, a_k - a_i, b_1, \dots, b_{n-k} \rangle$$

Modular/motivic types $\mathcal{M}_n(G)$, $n \geq 2$

Let G be an abelian group. Consider the $\mathbb{Z}$-module

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generated by unordered tupels $\langle a_1, \dots, a_n \rangle$, $\quad a_i \in A$, such that

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The only difference with $\mathcal{B}_n(G)$: $[a, a] = [a, 0], \quad \langle a, a \rangle = 2\langle a, 0 \rangle.$

Consider the map

$$\mu : \mathcal{B}_n(G) \to \mathcal{M}_n(G)$$

$$
\begin{aligned}
&(\boldsymbol{\mu}_0) \qquad [a_1, \dots, a_n] \mapsto \langle a_1, \dots, a_n \rangle, \quad \text{if all } a_1, \dots, a_n \neq 0,\\
&(\boldsymbol{\mu}_1) \qquad [0, a_2, \dots, a_n] \mapsto 2\langle 0, a_2, \dots, a_n \rangle, \quad \text{if all } a_2, \dots, a_n \neq 0,\\
&(\boldsymbol{\mu}_2) \qquad [0, 0, a_3, \dots, a_n] \mapsto 0, \quad \text{for all } a_3, \dots, a_n,
\end{aligned}
$$

and extended by $\mathbb{Z}$-linearity.

Theorem

- μ is a well-defined homomorphism; surjective, modulo 2-torsion (Kontsevich-Pestun-T. 2019)

Birational types $\to$ modular types

Theorem

- μ is a well-defined homomorphism; surjective, modulo 2-torsion (Kontsevich-Pestun-T. 2019)
- μ is an isomorphism, $\otimes \mathbb{Q}$ (Hassett-Kresch-T. 2020)

Modular types – lattice theory

Consider the free abelian group $\mathcal{S}_n(G)$, generated by symbols

$$\beta = [a_1, \dots, a_n] = [a_{\sigma(1)}, \dots, a_{\sigma(n)}], \quad \forall \sigma \in \mathfrak{S}_n,$$

where β is an n-dimensional faithful representation of G, i.e., a collection of characters $a_1, \dots, a_n$ of G, up to permutation, spanning $G^\vee$.

Consider the free abelian group $\mathcal{S}_n(G)$, generated by symbols

$$\beta = [a_1, \ldots, a_n] = [a_{\sigma(1)}, \ldots, a_{\sigma(n)}], \quad \forall \sigma \in \mathfrak{S}_n,$$

where β is an n-dimensional faithful representation of G, i.e., a collection of characters $a_1, \ldots, a_n$ of G, up to permutation, spanning $G^\vee$.

We have a diagram

$$\begin{tikzcd} \mathcal{S}_n(G) \arrow[r, "\mathsf{b}"] & \mathcal{B}_n(G) \arrow[d, "\mu"] \\ \mathcal{S}_n(G) \arrow[r, "\mathsf{m}"] & \mathcal{M}_n(G) \end{tikzcd}$$

Consider the free abelian group on triples

$$(\mathbf{L}, \chi, \Lambda),$$

where

- $\mathbf{L} \simeq \mathbb{Z}^n$ is an n-dimensional lattice,
- $\chi \in \mathbf{L} \otimes A$ is an element inducing, by duality, a surjection $\mathbf{L}^\vee \to A$,
- Λ is a basic cone, i.e., a simplicial cone spanned by a basis of $\mathbf{L}$.

Modular types – lattice theory

Let $\mathbf{T}$ be the quotient by $\mathrm{GL}_n(\mathbb{Z})$-equivalence. There is a natural map

$$\begin{array}{rcl} \mathbf{T} & \to & \mathcal{S}_n(G), \\ (\mathbf{L}, \chi, \Lambda) & \mapsto & [a_1, \ldots, a_n], \end{array}$$

defined by decomposing

$$\chi = \sum_{i=1}^{n} e_i \otimes a_i, \quad a_i \in A,$$

where $\{e_1, \ldots, e_n\}$ is a basis of Λ.

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where $\{e_1, \ldots, e_n\}$ is a basis of Λ.

The symmetry property is precisely the ambiguity in the order of generating elements of Λ.

Imposing scissor-type relations on $\mathbf{T}$, via subdivision of cones, we obtain a diagram

$$\begin{tikzcd} \mathbf{T} \arrow[d, two heads, "\mathsf{s}"'] \arrow[rr, "\psi"] & & \mathcal{M}_n(G) \\ \mathbf{T}/(\text{scissor-type relations}) \arrow[urr, "\sim"] & & \end{tikzcd}$$

There is a similar group $\widetilde{\mathbf{T}}$, based on triples

$$(\mathbf{L}, \chi, \Lambda'),$$

where now Λ' is a smooth cone of arbitrary dimension (i.e., one spanned by part of a basis of $\mathbf{L}$), such that

- $$\chi \in \mathrm{Im}(\mathbf{L}' \otimes A \to \mathbf{L} \otimes A),$$

  where $\mathbf{L}' \subseteq \mathbf{L}$ is the sublattice spanned by Λ'.

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Again, impose relations coming from the $\mathrm{GL}_n(\mathbb{Z})$-action.

There is a natural map

$$\begin{array}{rcl} \widetilde{\mathbf{T}} & \to & \mathcal{S}_n(G), \\ (\mathbf{L}, \chi, \Lambda') & \mapsto & [a_1, \ldots, a_n]. \end{array}$$

For a face Λ'' of Λ' of dimension at least 2,

$$\Lambda'' = \mathbb{R}_{\geq 0}\langle e_1, \dots, e_r \rangle \subset \Lambda' = \mathbb{R}_{\geq 0}\langle e_1, \dots, e_s \rangle,$$

consider the star subdivision

$$\Sigma^*_{\Lambda'}(\Lambda''),$$

consisting of the $2^r - 1$ cones spanned by

$$e_1 + \cdots + e_r, e_{r+1}, \dots, e_s,$$

and all proper subsets of $\{e_1, \dots, e_r\}$.

We introduce **Subdivision relations** on $\widetilde{\mathbf{T}}$:

(S) Put

$$(\mathbf{L}, \chi, \Lambda') = \sum_{\substack{\widetilde{\Lambda}' \in \Sigma^*_{\Lambda'}(\Lambda'') \\ \chi \in \mathrm{Im}(\widetilde{\mathbf{L}}' \otimes A \to \mathbf{L} \otimes A)}} (-1)^{\dim(\Lambda') - \dim(\widetilde{\Lambda}')} (\mathbf{L}, \chi, \widetilde{\Lambda}'),$$

respectively,

$(\mathbf{L}, \chi, \Lambda') = (\mathbf{L}, \chi, \Lambda)$, for a basic cone Λ, having Λ' as a face.

Birational types – lattice theory

We have:

$$\begin{tikzcd}
\widetilde{\mathbf{T}} \arrow[d, two heads, "\tilde{\mathsf{s}}"'] \arrow[rr, "\tilde{\psi}"] & & \mathcal{B}_n(G) \\
\widetilde{\mathbf{T}}/(\text{subdivision relations}) \arrow[urr, "\sim"] & &
\end{tikzcd}$$

The definition of

$$\tilde{\psi}(\mathbf{L}, \chi, \Lambda')$$

extends to the case of a simplicial cone Λ' (satisfying the condition), with $\mathbf{L}' = \mathbf{L} \cap \Lambda'_{\mathbb{R}}$.

The definition of

$$\tilde{\psi}(\mathbf{L}, \chi, \Lambda')$$

extends to the case of a simplicial cone Λ' (satisfying the condition), with $\mathbf{L}' = \mathbf{L} \cap \Lambda'_{\mathbb{R}}$.

We can define Hecke operators on $\mathcal{B}_n(G)$

$$T_{\ell,r} : \mathcal{B}_n(G) \to \mathcal{B}_n(G),$$

where $\ell \nmid |G|$ and $1 \le r \le n-1$, as a sum over certain overlattices:

$$T_{\ell,r}(\tilde{\psi}(\mathbf{L}, \chi, \Lambda')) := \sum_{\substack{\mathbf{L} \subset \widehat{\mathbf{L}} \subset \mathbf{L} \otimes \mathbb{Q} \\ \widehat{\mathbf{L}}/\mathbf{L} \simeq (\mathbb{Z}/\ell\mathbb{Z})^r}} \tilde{\psi}(\widehat{\mathbf{L}}, \chi, \Lambda').$$

Hecke operators on $\mathcal{M}_n(G)$

The Hecke operators:

$$T_{\ell,r} : \mathcal{M}_n(G) \to \mathcal{M}_n(G) \quad 1 \le r \le n-1$$

are well-defined and commute.

Hecke operators on $\mathcal{M}_n(G)$

The Hecke operators:

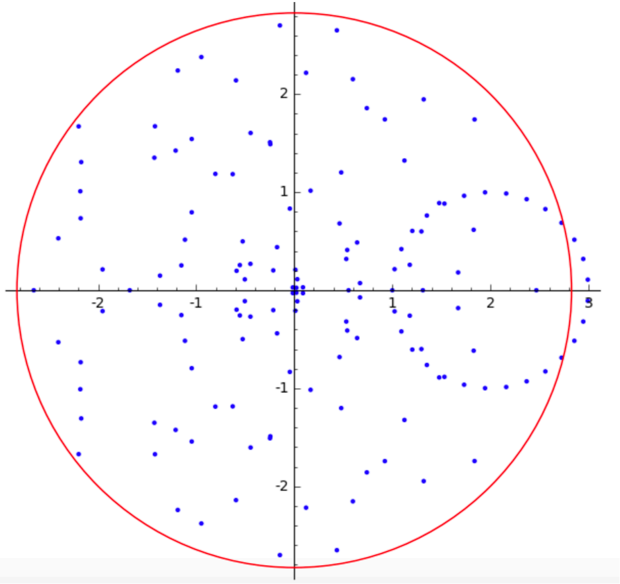
$$T_{\ell,r} : \mathcal{M}_n(G) \to \mathcal{M}_n(G) \quad 1 \le r \le n-1$$

are well-defined and commute.

Example:

$$T_2(\langle a_1, a_2 \rangle) = \langle 2a_2, a_2 \rangle + \big(\langle a_1 - a_2, 2a_2 \rangle + \langle 2a_1, a_2 - a_1 \rangle\big) + \langle a_1, 2a_2 \rangle.$$

Eigenvalues of T_2 on $\mathcal{M}_2(\mathbb{Z}/59\mathbb{Z})$



Birational types: summary

- Construction of groups related to $\mathcal{B}_n(G)$
- Nonabelian versions
- Refined G-equivariant birational invariants
- Unexpected connection between the Cremona group and automorphic forms (cohomology of congruence subgroups), Hecke operators